Chapter 1

The Complex Numbers

1.1 Definition and Simple Properties

The number system is a tool devised by humans to aid in the description of quantities of the various things humans have to deal with. It has evolved as human culture has evolved, beginning with something very primitive like: *1, 2, 3, many*, moving on to the natural numbers, then the integers, the rational numbers, the real numbers and then the complex numbers.

At each stage of development, the number system was expanded in response to the need to describe quantities that the old number system could not. For example, the negative numbers were introduced in order to be able to describe a loss as opposed to a gain, or moving backward rather than forward. The rational numbers were introduced because we don't always deal with whole numbers of things (we have 2/3 of a pie left). The real number system evolved from the rational number system out of a need to be able to describe such things as the length of the hypotenuse of a right triangle (this involves square roots) and the area or circumference of a circle (this involves π).

In this course, we will assume students are familiar with the real number system and its properties. We will define the complex number system as a needed extension of the real number system and develop its properties. We will then go on to study functions of a complex variable.

The Real Numbers are Insufficient

The complex number system was developed in response to the need for solutions to polynomial equations. The simplest polynomial equation that does not have a solution in the real number system is the equation

$$x^2 + 1 = 0,$$

which has no real solution because -1 has no real square root. More generally, a quadratic equation

$$ax^2 + bx + c, \tag{1.1.1}$$

where a, b and c are real numbers, formally has two solutions given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{1.1.2}$$

but these will not be real numbers if $b^2 - 4ac$ is negative. If we could take square roots of negative numbers, then the quadratic formula would give us solutions to (1.1.1) for all choices of real coefficients a, b, c. To make this possible, we expand the real number system in the following way, thus creating the complex number system $\mathbb{C}$.

Constructing $\mathbb{C}$

We begin by adjoining a single new number to our old number system $\mathbb{R}$. We will denote it by i and declare it to be a square root of -1. Thus,

$$i^2 = -1.$$

Our new number system is to contain both $\mathbb{R}$ and the new number i and it should be closed under addition and multiplication. If it is to be closed under multiplication, we need a number iy for every real number y. Likewise, if it is to be closed under addition, there should be a number $x + iy$ in our new number system for each pair of real numbers (x, y). It turns out that this is enough. If we define the set of complex numbers $\mathbb{C}$ to be the set of all symbols of the form $x + iy$ where (x, y) is a pair of real numbers, and if we define addition and multiplication appropriately, then the resulting number system is a field in which every polynomial equation has a root. We will be a long time proving the latter half of this statement, but it is not hard to prove the first part.

To define the operations of addition and multiplication in $\mathbb{C}$, we begin by noting that, as a set, $\mathbb{C}$ may be identified with $\mathbb{R}^2$ – the set of all pairs (x, y) of real numbers. Obviously, each pair (x, y) determines a symbol $x + iy$ and vice-verse. This identification makes $\mathbb{C}$ into a vector space over $\mathbb{R}$ and gives us operations of addition and scalar multiplication by reals which satisfy the usual associative and distributive rules. The resulting operation of addition is

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

It remains to define a product on $\mathbb{C}$.

We have already declared that $i^2 = -1$. If we also require that the associative and distributive laws of multiplication should hold and that the multiplication of real numbers should remain as before, then the product of two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ must be

$$(x_1+iy_1)(x_2+iy_2) = x_1x_2+ix_1y_2+iy_1x_2+i^2y_1y_2 = (x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2).$$

We formalize this conclusion in the following definition.

Definition 1.1.1. We define the system $\mathbb{C}$ of complex numbers to be the set of all symbols of the form $x+iy$ with $(x,y) \in \mathbb{R}^2$, with addition and multiplication defined by

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

A complex number of the form $x + i0$, with $x \in \mathbb{R}$ will be denoted simply as x. This identifies $\mathbb{R}$ as a subset of $\mathbb{C}$. Similarly, a complex number of the form $0 + iy$ with y real will be denoted simply as iy. The numbers of this form are traditionally called the *imaginary* numbers.

Note that, from the above definition, if $x, y \in \mathbb{R}$, then

$$yi = (y + i0)(0 + i) = 0 + iy = iy$$

and so $x + iy$ and $x + yi$ are the same complex number. Which form is used to describe this number is usually dictated by which looks best typographically. When specific numbers replace x and y the latter seems to look best. Thus, we usually write $2 + 3i$ rather than $2 + i3$.

Example 1.1.2. If $z_1 = 5 + 2i$ and $z_2 = 3 - 4i$, find $z_1 + z_2$ and z_1z_2.

Solution:

$$\begin{aligned} z_1 + z_2 &= 5 + 3 - (2 - 4)i = 8 - 2i \\ z_1z_2 &= (5 \cdot 3 - 2 \cdot (-4)) + (5 \cdot (-4) + 3 \cdot 2)i = 23 - 14i \end{aligned}$$

Example 1.1.3. Show that the quadratic equation (1.1.1) has solutions which are complex numbers.

Solution: If $b^2 - 4ac \geq 0$ The quadratic formula (1.1.2) tells us the solutions are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

On the other hand, if $b^2 - 4ac < 0$, then $4ac - b^2$ is positive and has real square roots. By squaring both sides, and using $i^2 = -1$, it is easy to see that

$$\pm\sqrt{b^2 - 4ac} = \pm i\sqrt{4ac - b^2}.$$

This suggests that the solutions to the quadratic equation in this case are the two complex numbers

$$-\frac{b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a} \quad \text{and} \quad -\frac{b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}.$$

That these two numbers are, indeed, solutions to the quadratic equation may be verified by directly substituting them in for x in (1.1.1) . We leave this as an exercise (Exercise 1.1.7).

Field Properties

In our definition of the product of two complex numbers, we were guided by the desire to have the usual rules of arithmetic hold – that is the commutative and associative laws for addition and for multiplication and the distributive law. Did we succeed? These are some of the properties of a *field*. Do these laws actually hold in $\mathbb{C}$ with the operations as defined above? The following theorem says they do.

Theorem 1.1.4. *If z_1, z_2, z_3 are complex numbers, then*

(a) $z_1 + z_2 = z_2 + z_1$ – *commutative law of addition;*

(b) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ – *associative law of addition;*

(c) $z_1 z_2 = z_2 z_1$ – *commutative law of multiplication;*

(d) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ – *associative law of multiplication;*

(e) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ – *distributive law.*

Proof. As far as the operation of addition is concerned, $\mathbb{C}$ is just $\mathbb{R}^2$, which is a vector space over $\mathbb{R}$. Parts (a) and (b) of the theorem follow directly from this. Part (c) is obvious from the definition of multiplication. We will prove part (d) and leave part (e) as an exercise (Exercise 1.1.8).

If $z_j = x_j + iy_j$ for j= 1, 2, 3, then

$$\begin{aligned}
(z_1 z_2) z_3 &= ((x_1 + iy_1)(x_2 + iy_2))(x_3 + iy_3) \\
&= (x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2))(x_3 + iy_3) \\
&= x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 \\
&\qquad + i(x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3),
\end{aligned}$$

while

$$\begin{aligned}
z_1 (z_2 z_3) &= (x_1 + iy_1)((x_2 + iy_2)(x_3 + iy_3)) \\
&= (x_1 + iy_1)(x_2 x_3 - y_2 y_3 + i(x_2 y_3 + y_2 x_3)) \\
&= x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3 \\
&\qquad + i(x_1 x_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3 - y_1 y_2 y_3).
\end{aligned}$$

Since the results are the same, the proof of (d) is complete. □

The properties described in the above theorem are some of the properties that must hold in a field. A field must also have additive and multiplicative identities - that is, elements 0 and 1 which satisfy

$$z + 0 = z \tag{1.1.3}$$

and

$$1 \cdot z = z \tag{1.1.4}$$

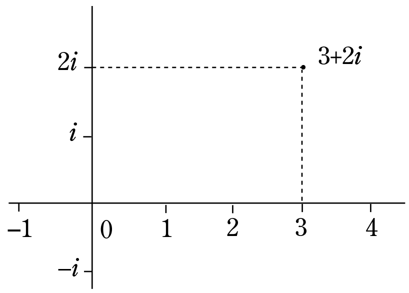

Figure 1.1: Plot of the complex number 3+2i.

for every element z in the field. That this holds for $\mathbb{C}$ follows immediately from the fact that $\mathbb{C}$ is a vector space over $\mathbb{R}$.

A field must also have the properties that every element z has an additive inverse, that is, an element $-z$ such that

$$z + (-z) = 0, \tag{1.1.5}$$

and every non-zero element z has a mutiplicative inverse, that is, an element z^{-1} such that

$$z \cdot z^{-1} = 1. \tag{1.1.6}$$

The first of these follows immediately from the fact that $\mathbb{C}$ is a vector space over $\mathbb{R}$. The second is nearly as easy. If $z = x + iy \neq 0$, then a direct calculation shows that

$$z^{-1} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i$$

satisfies 1.1.6. We conclude:

Theorem 1.1.5. *With addition and multiplication defined as in Definition 1.1.1, the complex numbers form a field.*

Complex Conjugation and Modulus

Definition 1.1.6. If $z = x+iy$ is a complex number, then its complex conjugate, denoted $\overline{z}$, is defined by

$$\overline{z} = x - iy,$$

while its modulus, denoted $|z|$ is defined by

$$|z| = \sqrt{x^2 + y^2}.$$

Note that the modulus, as defined above, is just the usual Euclidean norm in the vector space $\mathbb{R}^2$. Thus, if $z_1, z_2 \in \mathbb{C}$ then $|z_1 - z_2|$ is the Euclidean distance from z_1 to z_2. The term *modulus* is traditional, but the terms *norm* and *absolute value* are also commonly used to mean the same thing. We will use all three.

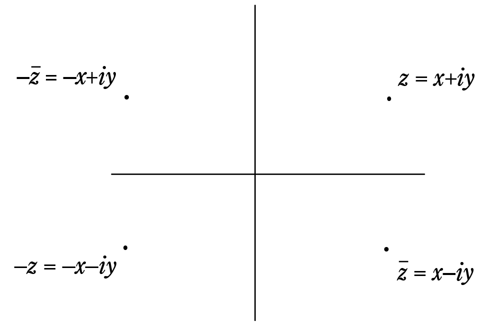

Figure 1.2: Plot of the complex numbers z, $\overline{z}$, $-z$, and $-\overline{z}$.

Note also that the two solutions of a quadratic equation with real coefficients given in Example 1.1.3 are complex conjugates of each other. Thus, the solutions to a quadratic equation with real coefficients occur in conjugate pairs. Quadratic equations with complex coefficients also have roots and they are also given by the quadratic formula. However, we can't prove this until we prove that every complex number has a square root. In fact, in Section 1.4 we will prove that every complex number has roots of all orders.

For a complex number $z = x + iy$, the real number x is called the *real part* of z and is denoted $\mathrm{Re}\,(z)$, while the number y is called the *imaginary part* of z and is denoted $\mathrm{Im}\,(z)$. In graphing complex numbers using a rectilinear coordinate system, x determines the coordinate on the horizontal axis, while y determines the coordinate on the vertical axis.

Note that a complex number z is real if and only if $\overline{z} = z$ and it is purely imaginary if and only if $\overline{z} = -z$. Note also, that if $z = x + iy$, then

$$\mathrm{Re}\,(z) = x = \frac{z + \overline{z}}{2} \quad \text{and} \quad \mathrm{Im}\,(z) = y = \frac{z - \overline{z}}{2i}.$$

The elementary properties of conjugation and modulus are gathered together in the next theorem.

Theorem 1.1.7. *If z and w are complex numbers, then*

(a) $\overline{\overline{z}} = z$

(b) $z\overline{z} = |z|^2$*;*

(c) $\overline{z + w} = \overline{z} + \overline{w}$*;*

(d) $\overline{zw} = \overline{z}\;\overline{w}$*;*

(e) $|zw| = |z||w|$ *and* $|\overline{z}| = |z|$*;*

(f) $|z|$ *is a non-negative real number and is* 0 *if and only if* $z = 0$*;*

(g) $|Re\,(z\overline{w})\,| \le |z||w|$*;*

(h) $|z+w| \le |z| + |w|$.

Proof. We will prove (g) and (h). The other parts are elementary computations or observations and will be left as exercises.

Parts (g) and (h) are the Cauchy-Schwarz inequality and the triangle inequality for the vector space $\mathbb{R}^2$. Versions of these inequlities hold in general Euclidian space $\mathbb{R}^n$. The proofs we give here are specializations to $\mathbb{C}$ of the standard proofs of these inequalities in $\mathbb{R}^n$.

To prove (g), we begin with the observation that (a) and (d) imply that

$$z\overline{w} + \overline{z}w = 2\mathrm{Re}\,(z\overline{w})\,.$$

We then let t be an arbitrary real number and note that, by Parts (c), (d), and (f),

$$0 \le |zt+w|^2 = (zt+w)(\overline{z}t+\overline{w}) = |z|^2t^2 + 2\mathrm{Re}\,(z\overline{w})\,t + |w|^2$$

for all values of t. This implies that the quadratic polynomial in t given by

$$|z|^2t^2 + 2\mathrm{Re}\,(z\overline{w})\,t + |w|^2$$

is never negative and, therefore, has at most one real root. This is only possible if the expression under the radical in the quadratic formula is negative or zero. Thus,

$$4(\mathrm{Re}\,(z\overline{w}))^2 - 4|z|^2|w|^2 \le 0.$$

Part (g) follows immediately from this.

Part (h) follows directly from Part (g) and the other parts of the theorem. In fact,

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\ &= |z|^2 + 2\mathrm{Re}\,(z\overline{w}) + |w|^2 \\ &\le |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2 \end{aligned}$$

On taking square roots, we conclude $|z+w| \le |z| + |w|$, which is Part (h). □

The inequalities in the following theorem are used extensively – particularly in the next section. The proofs are very simple and are left as an exercise (Exercise 1.1.12).

Theorem 1.1.8. *If* $z = x+iy$*, then* $\max\{|x|,|y|\} \le |z| \le |x| + |y|$.

Inversion and Division

Recall that the inverse of a non-zero complex number $z = x+iy$ is

$$z^{-1} = \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z\overline{z}}.$$

Stating this in the last form makes the identity $zz^{-1} = 1$ obvious.

This also suggests the right way to do complex division problems in general: to express w/z as a complex number in standard form (as a real number plus i times a real number), simply multiply both numerator and denominator by $\overline{z}$. That is,

$$\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}} = \frac{w\overline{z}}{|z|^2}.$$

The number $w\overline{z}$ is then easily put in standard form and the problem is finished by dividing by the real number $|z|^2$.

Example 1.1.9. Express $\dfrac{1}{2+3i}$ in the standard form $x + yi$.

Solution:

$$\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{|2+3i|^2} = \frac{2}{13} - \frac{3}{13}i.$$

Example 1.1.10. Express $\dfrac{3+4i}{3-4i}$ in standard form.

Solution:

$$\frac{3+4i}{3-4i} = \frac{(3+4i)^2}{|3+4i|^2} = -\frac{7}{25} + \frac{24}{25}i$$

Exercise Set 1.1

1. Express $(3 + i) + (2 - 7i)$ and $(3 + i)(2 - 7i)$ in the standard form $x + yi$.
2. Express $\dfrac{1}{3+5i}$ in the standard form $x + yi$.
3. Express $(1 + 2i)^2$ and $(1 + 2i)^{-2}$ in the standard form $x + yi$.
4. Express $\dfrac{2-3i}{3+2i}$ in the standard form $x + yi$.
5. Find a square root for i.
6. If $z = x + iy$, express z^3 in standard form.
7. By direct substitution, prove that the two solutions to the quadratic equation given in Example 1.1.3 really do satisfy Equation 1.1.1.
8. Prove Part (e) of Theorem 1.1.4.
9. Prove Parts (a), (b) and (c) of Theorem 1.1.7.
10. Prove Parts (d), (e) and (f) of Theorem 1.1.7.
11. For $z \in \mathbb{C}$ and $a \in \mathbb{R}$ prove the following:

(a) $\mathrm{Re}\,(az) = a\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(az) = a\mathrm{Im}\,(z)$;

(b) $\mathrm{Re}\,(iz) = -\mathrm{Im}\,(z)$ and $\mathrm{Im}\,(iz) = \mathrm{Re}\,(z)$.

12. Prove Theorem 1.1.8 – that is, prove that if $z = x + iy$, then
$$\max\{|x|, |y|\} \le |z| \le |x| + |y|.$$

13. Graph the set of points $z \in \mathbb{C}$ which satisfy the equation $|z - i| = 1$.

14. Graph the set of points $z \in \mathbb{C}$ which satisfy the equation $z^2 + \overline{z}^2 = 2$. Hint: If $z = x + iy$, rewrite this equation as an equation in x and y.

15. Prove that if z is a non-zero complex number, then $\overline{1/z} = 1/\overline{z}$ and $|1/z| = 1/|z|$.

16. If z is any non-zero complex number, prove that $z/\overline{z}$ has modulus one.

17. Prove that every complex number of modulus 1 has the form $\cos\theta + i\sin\theta$ for some angle θ.

18. Prove that every line or circle in $\mathbb{C}$ is the solution set of an equation of the form
$$a|z|^2 + \overline{w}z + w\overline{z} + b = 0,$$
where a and b are real numbers and w is a complex number. Conversely, show that every equation of this form has a line, circle, point, or the empty set as its solution set.

1.2 Convergence in $\mathbb{C}$

We assume the reader is familiar with the basics concerning convergent sequences and series of real numbers – particularly those results which follow from the completeness of the real number system, such as the fact that bounded monotone sequences converge and the various convergence tests for series. We also assume a familiarity with the basics of power series in a real variable. The purpose of this section is to extend these ideas and results to sequences and series of complex numbers and power series in a complex variable. This is an introductory section. A deeper study of complex power series will come later in the text.

There is no natural order relation on the complex numbers. The statement $z < w$ makes no sense for complex numbers z and w. However, if these numbers happen to be real, then the inequality does make sense, because $\mathbb{R}$ is an ordered field. We make heavy use of inequalities in this section, but note they are always inequalities between real numbers. Thus, if an inequality of the form $a < b$ occurs in this text the numbers a and b are assumed to be real numbers, even if there is no explicit statement to that effect.

Convergence of Sequences

A sequence of complex numbers converges if and only if it converges as a sequence of vectors in $\mathbb{R}^2$. The formal definition is the familiar one:

Definition 1.2.1. A sequence $\{z_n\}$ of complex numbers is said to converge to the number w if, for every $\epsilon > 0$, there exists an integer N such that

$$|z_n - w| < \epsilon \quad \text{whenever} \quad n \geq N.$$

In this case, we write $\lim_{n\to\infty} z_n = w$, or $\lim z_n = w$, or simply $z_n \to w$.

Remark 1.2.2. There are a couple of simple observations about convergence of sequences that will prove to be very useful.

1. If $\{z_n\}$ is a sequence of complex numbers, then $\lim z_n = w$ if and only if $\lim |z_n - w| = 0$.
2. If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers with $0 \leq a_n \leq b_n$ and if $b_n \to 0$, then also $a_n \to 0$.

These both follow immediately from the definition of limit of a sequence. The second one is one form of what is sometimes called the *squeeze principle*.

The first of these observations reduces the problem of showing a sequence of complex numbers converges to a given complex number to showing that a certain sequence of non-negative numbers converges to 0. This is useful because we have many tools at our disposal to show that a sequence of non-negative numbers converges to 0. One of the most useful of these tools is the second observation above. The next example and the proof of the following theorem are excellent examples of how this works.

Example 1.2.3. Prove that $\lim z^n = 0$ if $|z| < 1$.

Solution: Note that $|z^n| = |z|^n$ follows from Part (e) of Theorem 1.1.7. If $|z| < 1$ then $\lim |z|^n = 0$. That $\lim z^n = 0$, as well, follows from (1) of Remark 1.2.2.

Theorem 1.2.4. *A sequence of complex numbers $\{z_n\}$ converges to a complex number w if and only if $\{Re(z_n)\}$ converges to $Re(w)$ and $\{Im(z_n)\}$ converges to $Im(w)$.*

Proof. By Theorem 1.1.8 we know that

$$\begin{aligned}
&|\mathrm{Re}(z_n) - \mathrm{Re}(w)| \leq |z_n - w|,\\
&|\mathrm{Im}(z_n) - \mathrm{Im}(w)| \leq |z_n - w| \quad \text{and}\\
&|z_n - w| \leq |\mathrm{Re}(z_n) - \mathrm{Re}(w)| + |\mathrm{Im}(z_n) - \mathrm{Im}(w)|
\end{aligned}$$

for each n. The first two of these inequalities, together with Remark 1.2.2, imply that if $z_n \to w$, then $\mathrm{Re}(z_n) \to \mathrm{Re}(w)$ and $\mathrm{Im}(z_n) \to \mathrm{Im}(w)$. The third inequality, together with the fact that the sum of two sequences converging to zero also converges to zero, implies the converse – that is, if $\mathrm{Re}(z_n) \to \mathrm{Re}(w)$ and $\mathrm{Im}(z_n) \to \mathrm{Im}(w)$, then $z_n \to w$. ◻

Example 1.2.5. Show that the sequence $\{2^{-n} + in/(n+1)\}$ converges to i.

Solution: This follows from the previous theorem and the fact that $2^{-n} \to 0$ and $n/(n+1) \to 1$.

Example 1.2.6. Show that if $\{z_n\}$ and $\{w_n\}$ are two convergent sequences of complex numbers with $z_n \to z$ and $w_n \to w$, then $z_n + w_n \to z + w$ and $z_n w_n \to zw$

Solution: If $z_n = x_n + iy_n$, $z = x + iy$, $w_n = u_n + iv_n$ and $w = u + iv$, then

$$z_n + w_n = x_n + u_n + i(y_n + v_n),$$

$$z + w = x + u + i(y + v),$$

$$z_n w_n = x_n u_n - y_n v_n + (x_n v_n + y_n u_n)i$$

and

$$zw = xu - yv + (xv + yu)i.$$

We know that $x_n \to x$, $y_n \to y$, $u_n \to u$ and $v_n \to v$. We also know the rules about limits of products, sums and differences of sequences of real numbers. These rules imply $x_n + u_n \to x + u$, $y_n + v_n \to y + v$, $x_n u_n - y_n v_n \to xu - yv$ and $x_n v_n + y_n u_n \to xv + yu$. Since the real and imaginary parts of $z_n + w_n$ and $z_n w_n$ converge to the real and imaginary parts, respectively, of $z + w$ and zw, the previous theorem implies that $z_n + w_n \to z + w$ and $z_n w_n \to zw$.

Series of Complex Numbers

A series of complex numbers is a formal sum of the form

$$\sum_{k=0}^{\infty} z_k = z_0 + z_1 + z_2 \cdots + z_k + \cdots ,$$

with $z_k \in \mathbb{C}$. The series *converges* if its sequence of partial sums $\{s_n\}$ converges, where

$$s_n = \sum_{k=0}^{n} z_k. \tag{1.2.1}$$

In this case, if $s = \lim_{n\to\infty} s_n$, then we say s is the sum of the series and write

$$s = \sum_{k=0}^{\infty} z_k.$$

Just as with real series, if a series $\sum_{k=0}^{\infty} z_k$ converges, then its terms must tend to zero. Thus, if $\{z_k\}$ fails to have limit 0, then the series diverges. This test for divergence is called the *term test*. Its proof for complex series is the same as the proof for real series. We leave it as an exercise (Exercise 1.2.9).

Example 1.2.7. For what values of z does the complex geometric series

$$\sum_{k=0}^{\infty} z^k$$

converge and what does it converge to?

Solution: We first note that if $|z| \geq 1$, then $|z|^k \geq 1$ for all k and the series diverges by the term test. For $|z| < 1$ we use the same trick that is used to study the real geometric series. The $n\,th$ partial sum of the series is

$$s_n = \sum_{k=0}^{n} z^k.$$

If we multiply this by $(1 - z)$, a vast cancellation of terms occurs and we obtain

$$(1 - z)s_n = 1 - z^{n+1},$$

so that

$$s_n = \frac{1 - z^{n+1}}{1 - z}.$$

This sequence converges to $\dfrac{1}{1 - z}$ if $|z| < 1$, since

$$\left| s_n - \frac{1}{1 - z} \right| = \frac{|z|^n}{|1 - z|} \to 0$$

in this case.

A series $\sum_{k=0}^{\infty} z_k$ is said to *converge absolutely* if the series of positive terms $\sum_{k=0}^{\infty} |z_k|$ converges. From calculus, we know a great deal about convergence of positive termed series (comparison test, ratio test, root test, etc.) and so the following theorem will obviously play an important role.

Theorem 1.2.8. *If a series $\sum_{k=0}^{\infty} z_k$ of complex numbers converges absolutely, then it converges. Furthermore,*

$$\left| \sum_{k=0}^{\infty} z_k \right| \leq \sum_{k=0}^{\infty} |z_k|.$$

Proof. By hypothesis, the series $\sum_{k=0}^{\infty} |z_k|$ converges. If $z_k = x_k + iy_k$, then $|x_k| < |z_k|$ and $|y_k| < |z_k|$. By the comparison test, the two series

$$\sum_{k=0}^{\infty} |x_k| \quad \text{and} \quad \sum_{k=0}^{\infty} |y_k|$$

both converge. This, in turn, implies that

$$\sum_{k=0}^{\infty} x_k \quad \text{and} \quad \sum_{k=0}^{\infty} y_k$$

converge, since an absolutely convergent series of real numbers converges. This tells us that the real and imaginary parts of the sequence $\{s_n\}$ of partial sums of $\sum_{k=0}^{\infty} z_k$ are convergent and, by Theorem 1.2.4, so is the sequence $\{s_n\}$ itself. Thus, the series converges.

It follows from the triangle inequality that

$$|s_n| = \left| \sum_{k=0}^{n} z_k \right| \le \sum_{k=0}^{n} |z_k|.$$

The inequality of the theorem follows when we pass to the limit as $n \to \infty$ in this inequality and use the result of Exercise 1.2.7. □

Example 1.2.9. Prove that the complex series $\sum_{k=1}^{\infty} (k + k^2 i)^{-1}$ converges absolutely.

Solution: By the second form of the triangle inequality (see Exercise 1.2.2) we have

$$|k + k^2 i| \ge k^2 - k \ge \frac{k^2}{2} \quad \text{if} \quad k \ge 2.$$

Thus,

$$|k + k^2 i|^{-1} \le 2k^{-2} \quad \text{if} \quad k \ge 2.$$

Since the p-series $\sum_{k=1}^{\infty} k^{-2}$ converges, so does $\sum_{k=1}^{\infty} 2k^{-2}$ and, by comparison, $\sum_{k=1}^{\infty} |k + k^2 i|^{-1}$. Thus, $\sum_{k=1}^{\infty} (k + k^2 i)^{-1}$ converges absolutely.

Power Series

A complex power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{1.2.2}$$

where the coefficients a_n are complex numbers, z_0 is a complex number and z is a complex variable. A power series of this form is said to be *centered* at z_0. It defines a complex function of the complex variable z, with domain the set of those $z \in \mathbb{C}$ for which the series converges.

Remark 1.2.10. We know from calculus that the set on which a real power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges is an interval – *the interval of convergence* – consisting of an open interval centered at x_0 and possibly one or both of its endpoints. The power series converges absolutely at each point of the open interval. The radius of this interval is called the *radius of convergence* of the power series and is computed using the root test or the ratio test. As we shall see in Chapter 3, a similar result is true for complex power series, but the interval of convergence is replaced by a disc of convergence. If for $r > 0$ we set

$$D_r(z_0) = \{z \in \mathbb{C} : |z| < r\} \quad \text{and} \quad \overline{D}_r(z_0) = \{z \in \mathbb{C} : |z| \leq r\},$$

then $D_r(z_0)$ is called the open disc of radius r, centered at z_0, while $\overline{D}_r(z_0)$ is called the closed disc of radius r, centered at z_0. Given a power series, centered at z_0, there is a number $R \geq 0$, called the radius of convergence of the power series, such that the series converges absolutely for each $z \in D_R(z_0)$ and diverges for each $z \notin \overline{D}_R(z_0)$. When we study power series in detail we will prove this result and tell how to calculate R in general. However, for most of the series we will be studying, there are elementary ways to show that this result holds and to find R. One simply uses the standard convergence tests from calculus – particularly the ratio test.

Example 1.2.11. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}. \tag{1.2.3}$$

Solution: If we apply the ratio test to the series $\sum_{n=1}^{\infty} |z|^n/n$. we conclude that this series converges for $|z| < 1$ and diverges for $|z| > 1$. Hence, by Theorem 1.2.8, the series (1.2.3) also converges for $|z| < 1$. If $|z| > 1$, then the sequence of terms of this series fails to converge to zero (since $|z|^n/n \to +\infty$) and so the series diverges by the term test. Thus our series converges on $D_1(0)$ and diverges outside of $\overline{D}_1(0)$. We conclude that it has radius of convergence 1.

Example 1.2.12. Show that the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1.2.4}$$

is $+\infty$ – that is, the series converges for all z.

Solution: We apply the ratio test to the series $\sum_{n=0}^{\infty} |z|^n/n!$. We have

$$\left(\frac{|z|^{n+1}}{(n+1)!}\right)\left(\frac{|z|^n}{n!}\right)^{-1} = \frac{|z|}{n+1}$$

which converges to 0 for all z. Hence, by the ratio test, the power series (1.2.4) converges for all $z \in \mathbb{C}$. The radius of convergence is, thus, $+\infty$.

Example 1.2.13. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

where $a_n = 2^n$ if n is prime and $a_n = 0$ if n is not prime.

Solution: We can't use the ratio test on this one because most of the terms of the series are 0. However, if we compare the series $\sum_{n=0}^{\infty} a_n|z|^n$ with the series $\sum_{n=0}^{\infty} 2^n|z|^n$ – a series to which we can apply the ratio test – we conclude that the series converges for $|z| < 1/2$. If $|z| > 1/2$, then the terms of our series do not tend to 0 and so the series diverges by the term test. We conclude that the radius of convergence is $1/2$.

Exercise Set 1.2

1. Show that the sequence $(2+ni)^{-1}$ converges to 0.
2. Prove the second form of the triangle inequality for complex numbers: $|\,|z|-|w|\,| \le |z-w|$.
3. Show that $\left|\dfrac{1}{z+5}\right| \le \dfrac{1}{|z|-5}$.
4. Does the sequence $\left(1/\sqrt{2}+i/\sqrt{2}\right)^n$ converge?
5. For which values of z does the sequence $\{z^n\}$ converge?
6. Prove that if a sequence $\{z_n\}$ converges, then it is bounded – that is, there is a positive number M such that $|z_n| \le M$ for all n. Hint: show that there is an N such that $|z_n| \le |z|+1$ for all $n \ge N$. This gives an upper bound for $|z_n|$ for all but finitely many of the z_n.
7. Prove that if $\{z_n\}$ is a sequence with $\lim z_n = w$, then $\lim \overline{z_n} = \overline{w}$ and $\lim |z_n| = |w|$;
8. For the sequence of the previous exercise, prove that $\lim 1/z_n = 1/w$ provide $w \ne 0$;
9. Prove the term test for divergence of a series. That is, if the sequence of terms $\{z_k\}$ does not have limit 0, then the series $\sum_{k=0}^{\infty} z_k$ diverges.
10. Does the series $\sum_{n=0}^{\infty} n/(3+2ni)$ converge?
11. Does the series $\sum_{n=1}^{\infty} n/(n^3+2i)$ converge?
12. For which values of z does the series $\sum_{n=0}^{\infty} 1/(n^2+z^2)$ converge?
13. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} nz^n$;
14. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} z^n/3^n$;
15. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} z^n/(1+2^n)$;
16. Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(n!/n^n)z^n$;

1.3 The Exponential Function

There are many real valued functions of a real variable that have natural extensions to complex valued functions of a complex variable. In fact, this is true of all real functions that have convergent power series expansions. If $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, where this series converges on an interval of radius R about x_0, then the complex power series $f(z) = \sum_{n=0}^{\infty} a_n(z - x_0)^n$, converges in the open disc of radius R about x_0 (Exercise 1.3.10) and serves to extend f to a function $f(z)$ defined for z in the disc $D_R(x_0)$.

One of the most important examples of the use of this technique is provided by the exponential function. We know from calculus that

$$e^x = \exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

where the series converges to e^x on the entire real line. We saw in Example 1.2.12 that this same series, with x replaced by the complex variable z, converges absolutely on the entire complex plane. Thus, we get an extension of e^x to a function e^z defined on $\mathbb{C}$ as follows:

Definition 1.3.1. For each $z \in \mathbb{C}$, we define $\exp(z) = e^z$ by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This is the complex exponential function. It has many important properties, some expected and some surprising, as we shall see below.

The Law of Exponents

A property that we should expect of an exponential function is the following.

Theorem 1.3.2. *If $z, w \in \mathbb{C}$, then $e^{z+w} = e^z e^w$.*

Proof. The proof uses the complex form of the Binomial Theorem:

$$(z + w)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j}. \tag{1.3.1}$$

This is proved using induction on n. We leave it as an exercise (Exercise 1.3.11).

Proceeding with the proof of the theorem, and using (1.3.1), we have

$$\begin{aligned} e^{z+w} &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} z^j w^{n-j}. \end{aligned} \tag{1.3.2}$$

If we make a change of variables by setting $k = n - j$ in the inside summation, then (1.3.2) becomes

$$\sum_{n=0}^{\infty}\left(\sum_{j+k=n}\frac{z^j w^k}{j!k!}\right).$$

This is precisely what we get if we expand the product

$$\mathrm{e}^z\mathrm{e}^w = \left(\sum_{j=0}^{\infty}\frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{w^k}{k!}\right).$$

and collect terms of degree n. Provided this operation is valid, we conclude that $\mathrm{e}^{z+w} = \mathrm{e}^z\mathrm{e}^w$. It turns out that it is valid to expand the product of two infinite series in this fashion if they are both absolutely convergent. We prove this in the following lemma, which will complete the proof of the theorem. □

Lemma 1.3.3. *Let* $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{k=0}^{\infty} b_k$ *be two absolutely convergent series of complex numbers. Then,*

$$\left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\left(\sum_{j+k=n} a_j b_k\right) \qquad (1.3.3)$$

Proof. The partial sums of the series involved here are

$$s_J = \sum_{j=o}^{J} a_j, \quad t_K = \sum_{k=o}^{K} b_k, \quad u_N = \sum_{n=0}^{N}\left(\sum_{j+k=n} a_j b_k\right).$$

The left side of (1.3.3) is, by definition, $(\lim s_J)(\lim t_K)$, while the right side is $\lim u_N$. We know $\lim s_J$ and $\lim t_K$ both exist since the series defining them converge absolutely. We must prove that $\lim u_N$ exists and equals $(\lim s_J)(\lim t_K)$.

For a given pair J, K, we let $N = J + K$. Then u_N is the sum of all terms $a_j b_k$ for which $j + k \le N$, while $s_J t_K$ is the sum of those terms $a_j b_k$ for which $j \le J$ and $k \le K$. Thus,

$$\begin{aligned}|u_N - s_J t_K| &\le \sum\{|a_j b_k| : j + k \le N, \text{ and either } j > J \text{ or } k > K\} \\ &\le \sum_{j=J+1}^{\infty}|a_j|\sum_{k=0}^{\infty}|b_k| + \sum_{j=0}^{\infty}|a_j|\sum_{k=K+1}^{\infty}|b_k|. \qquad (1.3.4)\end{aligned}$$

The sum $\sum_{j=J+1}^{\infty}|a_j|$ converges to zero as $J \to \infty$ because the series $\sum_{j=0}^{\infty} a_j$ converges absolutely, while $\sum_{k=K+1}^{\infty}|b_k|$ converges to 0 as $K \to \infty$ because $\sum_{k=0}^{\infty} b_k$ converges absolutely.

To complete the proof, we choose for each non-negative integer N a J and K with $J+K = N$ and we do this in such a way that as $N \to \infty$ the corresponding J and K both also tend to ∞. For example, we could choose $J = K = N/2$ if N is even and $J = K+1 = N/2+1/2$ if N is odd. Then, from (1.3.4) it is clear that $\lim |u_N - s_J s_K| = 0$ which implies $\lim u_N = (\lim s_J)(\lim s_K)$. □

The Exponential of an Imaginary Number

If $z = x + iy$, then the law of exponents (Theorem 1.3.2) implies that

$$\mathrm{e}^z = e^x \mathrm{e}^{iy}.$$

We understand the behavior of e^x for real x from calculus. It is 1 at 0, is everywhere positive, increasing and concave upward; it rapidly approaches $+\infty$ as $x \to \infty$ and rapidly approaches 0 as $x \to -\infty$. What about e^{iy} – the exponential of a purely imaginary number? Here there are some surprises.

By the law of exponents, $\mathrm{e}^{iy}\mathrm{e}^{-iy} = \mathrm{e}^0 = 1$. Thus, $1/\mathrm{e}^{iy} = \mathrm{e}^{-iy}$. But also, for any $z \in \mathbb{C}$, $\overline{\mathrm{e}^z} = \mathrm{e}^{\overline{z}}$ (Exercise 1.3.1) and, in particular, $\overline{\mathrm{e}^{iy}} = \mathrm{e}^{-iy} = 1/\mathrm{e}^{iy}$. This implies that $|\mathrm{e}^{iy}|^2 = 1$ and, consequently, $|\mathrm{e}^{iy}| = 1$. Thus, e^{iy} is always a point on the circle of radius 1 centered at 0 (we call this the unit circle).

A more explicit description of the number e^{iy} comes from examination of its power series definition. If we group together the even numbered terms (terms with n of the form $n = 2k$) and the odd numbered terms (terms with n of the form $n = 2k + 1$) in this power series, we derive the identity

$$\begin{aligned}\mathrm{e}^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} &= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} \\ &= \cos y + i \sin y.\end{aligned}$$

Thus, we have proved the following theorem, which is known as *Euler's Identity.*

Theorem 1.3.4. *The identity $e^{iy} = \cos y + i \sin y$ holds for all $y \in \mathbb{R}$.*

This shows that, not only is e^{iy} a point on the unit circle, it is the point which is reached by rotating through an angle y (measured in radians) from the initial point $(1, 0)$.

Example 1.3.5. Express the complex numbers $\mathrm{e}^{2\pi i}$, $\mathrm{e}^{\pi i}$ and $\mathrm{e}^{\pi i/2}$ in standard form.

Solution: By Euler's identity, we have

$$\mathrm{e}^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1,$$

$$\mathrm{e}^{\pi i} = \cos \pi + i \sin \pi = -1,$$

while

$$\mathrm{e}^{\pi i/2} = \cos \pi/2 + i \sin \pi/2 = i.$$

Properties of the Exponential Function

The law of exponents and Eulers identity immediately imply:

Theorem 1.3.6. *If $z = x+iy$ is a complex number, then $e^z = e^x(\cos y + i \sin y)$.*

There are a number of properties of the exponential function which follow easily from this characterization of e^z. We collect them together in the following theorem whose proof is left to the exercises.

Theorem 1.3.7. *The exponential function has the following properties:*

(a) e^z *is never* 0*;*

(b) $|e^z| = e^{Re(z)}$*;*

(c) $|e^z| \le e^{|z|}$*;*

(d) e^z *is periodic of period* $2\pi i$*, meaning* $e^{z+2\pi i} = e^z$ *for every* $z \in \mathbb{C}$*;*

(e) $e^z = 1$ *if and only if* $z = 2\pi n i$ *for some integer* n*.*

Complex Trigonometric Functions

If we write out Euler's identity with y replaced by θ and then by $-\theta$, we obtain a pair of identies

$$\begin{aligned} \mathrm{e}^{i\theta} &= \cos\theta + i\sin\theta, \\ \mathrm{e}^{-i\theta} &= \cos\theta - i\sin\theta. \end{aligned}$$

If we solve this system of equations for $\cos\theta$ and $\sin\theta$, we obtain

$$\begin{aligned} \cos\theta &= \frac{\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}}{2}, \\ \sin\theta &= \frac{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}}{2i}. \end{aligned} \tag{1.3.5}$$

This suggests defining $\sin z$ and $\cos z$ for a complex variable z in the following way:

Definition 1.3.8. For each $z \in \mathbb{C}$, we set

$$\begin{aligned} \cos z &= \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2}, \\ \sin z &= \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i}. \end{aligned}$$

These are the same functions that one would get by replacing x by z in the power series expansions of $\sin x$ and $\cos x$.

With $\sin z$ and $\cos z$ defined, it is easy to define complex versions of the other trigonometric functions. For example,

$$\tan z = \frac{\sin z}{\cos z} = -i\frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}. \tag{1.3.6}$$

Example 1.3.9. How would you define a complex version of the arctan function?

Solution: The power series which converges to $\arctan x$ on $(-1, 1)$ is

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

If we replace x by the complex variable z, we obtain a series which converges on the disc $D_1(0)$. We then use it to define $\arctan z$ on this disc:

$$\arctan z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}.$$

.

Exercise Set 1.3

1. Using the power series for e^z, prove that $\overline{\mathrm{e}^z} = \mathrm{e}^{\overline{z}}$ for each $z \in \mathbb{C}$.
2. Express the complex numbers $\mathrm{e}^{\pi i/4}$, $\mathrm{e}^{3\pi i/2}$, and $\mathrm{e}^{13\pi i/6}$ in standard form.
3. Express $1/\sqrt{2} + i/\sqrt{2}$ in the form e^z for some complex number z.
4. Find all values of z for which $e^z = 1 + \sqrt{3}\, i$.
5. Prove that e^z is never 0 (Part (a) of Theorem 1.3.7).
6. Prove that $|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}(z)} \le \mathrm{e}^{|z|}$ for each $z \in \mathbb{C}$ (Parts (b)and (c) of Theorem 1.3.7).
7. Verify that $\mathrm{e}^{z+2\pi i} = \mathrm{e}^z$ for every $z \in \mathbb{C}$ (Part (d) of Theorem1.3.7).
8. Verify that $\mathrm{e}^z = 1$ if and only if $z = 2\pi n i$ for some interger n (Part (e) of Theorem1.3.7).
9. For which values of z is e^z a real number? For which values is it an imaginary number?
10. Show that if the real power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the corresponding complex power series $\sum_{n=0}^{\infty} a_n z^n$ also has radius of convergence R (you may assume radius of convergence for a complex power series makes sense and has the properties described in Remark 1.2.10).
11. Use induction on n to prove the complex binomial formula (1.3.1).
12. Derive the trigonometric identities

$$\sin(z + w) = \sin z \cos w + \cos z \sin w$$
$$\cos(z + w) = \cos z \cos w - \sin z \sin w$$

from the law of exponents (Theorem 1.3.2) and Euler's Identity (Theorem 1.3.4).

13. Show that $\cos ix = \cosh x$ and $\sin ix = i \sinh x$, where cosh and sinh are the hyperbolic cosine and hyperbolic sin from calculus.

14. Show that

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y, \quad \text{and}$$
$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

15. How would you define a complex version of the function $\log(1+x)$?

16. Verify that the power series defining arctan in Example 1.3.9 converges on the disc $D_1(0)$.

1.4 Polar Form for Complex Numbers

If u is a complex number of modulus 1, then $u = \cos\theta + i\sin\theta = e^{i\theta}$, where θ is an angle from the positive x axis to the ray from the origin through u. This angle is measured in radians and is the signed length of an arc on the unit circle joining $(1,0)$ to u. It is positive if the arc is traversed in the counterclockwise direction and negative if it is traversed in the clockwise direction.

If z is any non-zero complex number, then $z/|z|$ is a complex number of modulus 1 and so it has the form $\mathrm{e}^{i\theta}$ for some θ. If we set $r = |z|$, then

$$z = r\mathrm{e}^{i\theta}.$$

This is the *polar form* of the complex number z. The angle θ is called the *argument* of z and is denoted $\arg z$.

Comments:

1. By Euler's identity (Theorem 1.3.4), $re^{i\theta} = r(\cos\theta + i\sin\theta)$.

2. The number z must be non-zero for the argument to be defined. The polar form of $z = 0$ is just $z = 0$.

3. for $z \neq 0$, there are infinitely many angles θ for which $z = r\mathrm{e}^{i\theta}$. Given one such θ the others are all of the form $\theta + 2\pi n$, where n is an integer. Thus, $\arg(z)$ is not a single number, but a collection of numbers that differ from one another by multiples of 2π. We can specify a particular one of these by insisting it lie in a particular half open interval of length 2π - such as $[0, 2\pi)$ or $(-\pi, \pi]$.

4. The numbers (r, θ), where $r = |z|$ and θ is a value of $\arg(z)$, are polar coordinates for $z = x + iy$ considered as a point (x, y) in $\mathbb{R}^2$.

5. We now have two useful ways of expressing a complex number z, the *standard form* $x + iy$ and the *polar form* $re^{i\theta}$. It is important to be able to easily change from one to the other.

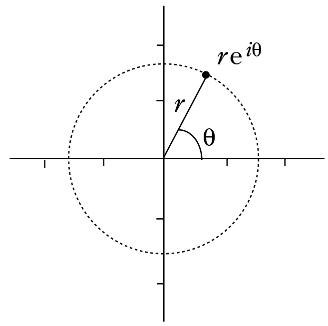

Figure 1.3: The Polar Form $re^{i\theta}$ of a Complex Number.

Example 1.4.1. Find the polar form for the complex number $z = 2 + 2i$.

Solution: We have

$$r = |z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

and

$$\theta = \arctan(2/2) = \arctan(1) = \pi/4.$$

Thus, $z = 2\sqrt{2}\mathrm{e}^{\pi i/4}$ is the polar form of z.

Example 1.4.2. Put the complex number $z = 2\mathrm{e}^{\pi i/6}$ in standard form $x + iy$.

Solution: We have

$$x = 2\cos\pi/6 = \sqrt{3} \quad \text{and} \quad y = 2\sin\pi/6 = 1$$

and so $z = \sqrt{3} + i$.

Products, Powers, and Roots

Polar form is particularly useful in dealing with products of complex numbers, since the product has such a simple expression if the numbers are given in polar form. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then the law of exponents implies that

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}. \tag{1.4.1}$$

Thus, $z_1 z_2$ is the complex number whose norm (distance from the origin) $|z_1 z_2|$ is the product of $|z_1|$ and $|z_2|$ and whose argument is the sum of the arguments of z_1 and z_2.

Similarly, the quotient of z_1 and z_2 is given by

$$z_1/z_2 = r_1/r_2 \mathrm{e}^{(\theta_1 - \theta_2)}. \tag{1.4.2}$$

Example 1.4.3. Find $z_1 z_2$ and z_1/z_2 if $z_1 = 2e^{\pi i/3}$ and $z_2 = 3e^{2\pi i/3}$.

Solution: By (1.4.1) and (1.4.2), we have

$$z_1 z_2 = 6e^{\pi i} = -6 \quad \text{and} \quad z_1/z_2 = (2/3)e^{-\pi i/3} = 1/3 - i\sqrt{3}/3.$$

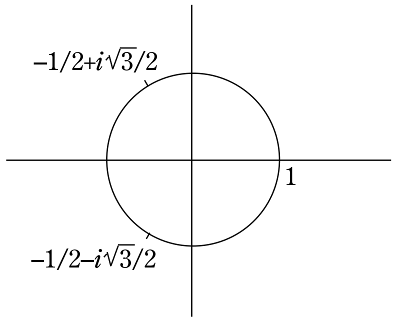

Figure 1.4: The Cube Roots of Unity.

It is evident from (1.4.1) that the nth power of a complex number $z = re^{i\theta}$ is

$$z^n = r^n e^{in\theta}. \tag{1.4.3}$$

From this we conclude that if, $z = \mathrm{r}e^{i\theta}$, and we choose $w = r^{1/n}\mathrm{e}^{i\theta/n}$, then $w^n = z$. Thus, w is an nth root of z. It is not the only one, however. Since z can also be written as $z = \mathrm{e}^{i(\theta+2\pi k)}$ for any integer k, each of the numbers

$$w_k = r^{1/n}\mathrm{e}^{i(\theta/n+2\pi k/n)},$$

where k is an integer, is also an nth root of z. Of course, these numbers are not all different. Those whose arguments differ by an integral multiple of 2π are the same. The numbers $w_0, w_1, w_2, \cdots, w_{n-1}$, are all distinct, but every other w_k is equal to one of these. This proves the following theorem.

Theorem 1.4.4. *If $z = re^{i\theta}$ is a non-zero complex number, then z has exactly n nth roots. They are the numbers*

$$r^{1/n}\,e^{i(\theta/n+2\pi k/n)} \quad \textit{for} \quad k = 0, 1, 2, \cdots, n-1.$$

Note that the n numbers $\mathrm{e}^{i(\theta/n+2\pi k/n)}$ that appear in this result are evenly spaced around the unit circle, with each successive pair separated by an an angle of $2\pi/n$.

The nth roots of the number 1 play a special role. They are called *the nth roots of unity.* If we apply the above theorem in the special case where $r = 1$ and $\theta = 0$, it tells us that the nth roots of unity are the numbers

$$\mathrm{e}^{2\pi ki/n} \quad \text{for} \quad k = 0, 1, 2, \cdots, n-1. \tag{1.4.4}$$

Example 1.4.5. Find the cube roots of unity.

Solution In the particular case where $n = 3$, (1.4.4) tells us that the roots of unity are

$$\begin{aligned} \mathrm{e}^0 &= 1, \\ \mathrm{e}^{2\pi i/3} &= -1/2 + (\sqrt{3}/2)i, \quad \text{and} \\ \mathrm{e}^{4\pi i/3} &= -1/2 - (\sqrt{3}/2)i. \end{aligned}$$

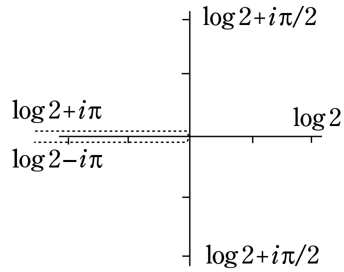

Figure 1.5: Principal Branch of Log at Points on the Circle of Radius 2.

Example 1.4.6. Find the cube roots of $2i$.

Solution Since $2i = 2\mathrm{e}^{\pi i/2}$, Theorem 1.4.4 implies that the cube roots of $2i$ are

$$\begin{aligned} 2^{1/3}\mathrm{e}^{\pi i/6} &= 2^{1/3}(\sqrt{3}/2 + i/2), \\ 2^{1/3}\mathrm{e}^{i(\pi/6+2\pi/3)} &= 2^{1/3}\mathrm{e}^{5\pi i/6} = 2^{1/3}(-\sqrt{3}/2 + i/2), \quad \text{and} \\ 2^{1/3}\mathrm{e}^{i(\pi/6+4\pi/3)} &= 2^{1/3}\mathrm{e}^{3\pi i/2} = -2^{1/3}i. \end{aligned}$$

The Logarithm

If $z = re^{i\theta}$, and $\log r$ is the natural logarithm of the positive number r, then the law of exponents implies that

$$z = \mathrm{e}^{\log r + i\theta}.$$

Thus, it would make sense to define $\log z$ to be $\log r + i\theta = \log|z| + i \arg z$. There is a problem with this, however. There are infinitely many possible choices for $\arg z$ and so $\log z$ is not well defined, just as $\arg z$ is not well defined.

The solution to this problem is to restrict $\theta = \arg z$ to lie in a specific half open interval of length 2π.

Definition 1.4.7. Given a half open interval I of length 2π on the line $\mathbb{R}$, let $\arg_I z$, for $z \neq 0$, be the value of $\arg z$ that lies in the interval I. Then the function defined for $z \neq 0$ by

$$\log z = \log|z| + i \arg_I z$$

will be called the *branch* of the log function defined by I. In the special case where $I = (-\pi, \pi]$, this function will be called the *principal branch* of the log function.

The following properties of the various branches of the log function follow easily from the definition and the properties of the exponential function. The proofs are left to the exercises.

Theorem 1.4.8. *If* log *is the branch of the the log function determined by an interval I, then*

(a) if $z \neq 0$, then $\mathrm{e}^{\log z} = z$;

(b) if $z \in \mathbb{C}$, then $\log \mathrm{e}^z = z + 2\pi k i$ for some integer k;

(c) if $z, w \in \mathbb{C}$, then $\log zw = \log z + \log w + 2\pi k i$ for some integer k;

(d) $\log 1 = 2\pi k i$ for some integer k;

(e) log *agrees with the ordinary natural log function on the positive real numbers if and only if the interval I contains 0.*

Suppose the interval I defining a branch of the log function has endpoints a and b with $a < b$. Then, since $b - a = 2\pi$, the polar coordinate equations $\theta = a$ and $\theta = b$ define the same ray. This ray is called the *cut line* for this branch of the logarithm. Observe that if z and w are two complex numbers with $|z| = |w| > 0$ which are close to each other, but on opposite sides of the cut line – say with $\arg_I z$ near a and $\arg_I w$ near b – then $\log w - \log z$ is nearly $2\pi i$. In other words, as we cross the cut line moving in the clockwise direction, the value of log jumps by $2\pi i$. Thus, log is not continuous at points on the cut line. Later we will show that it is continuous everywhere else.

Other Functions

If we fix a branch of the complex log function we can define a number of other complex functions which are extensions of familiar real functions. We mention some of these briefly.

An nth root function can be defined by setting $0^{1/n} = 0$ and

$$z^{1/n} = \mathrm{e}^{(1/n)\log z} \quad \text{if} \quad z \neq 0. \tag{1.4.5}$$

A special case of this is the square root function defined by $\sqrt{0} = 0$ and

$$\sqrt{z} = \mathrm{e}^{(1/2)\log z} \quad \text{if} \quad z \neq 0. \tag{1.4.6}$$

Note that the nth root function, as defined by (1.4.5) is giving only one of the nth roots of z. Which one is determined by the branch of the log function that is used. The other nth roots are obtained by multiplying this one by the nth roots of unity (Exercise 1.4.9). For example, given one square root of z, the other is obtained by multipling it by -1.

Example 1.4.9. If $\sqrt{z}$ is defined using the principal branch of the log function, analyze the behavior of $\sqrt{z}$ near the cut line for this branch.

Solution: For the principal branch of the log function the interval I is $(-\pi, \pi]$ and so the cut line is the line defined by $\theta = \pi$ in polar coordinates – that is, it is the negative real axis. If $z = r\mathrm{e}^{i\theta}$ is just above the negative real axis, then $\log z$ is nearly $\log r + i\pi$ and

$$\sqrt{z} = \mathrm{e}^{(1/2)\log z}$$

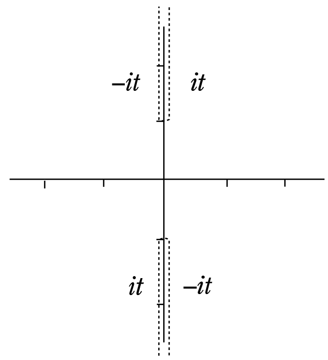

Figure 1.6: The Cut Line Discontinuities of $\sqrt{z^2+1}$.

is nearly $i\sqrt{r}$. On the other hand, if z is just below the negative axis, then $\log z$ is nearly $\log r - i\pi$, $(1/2)\log z$ is close to $(\log r - i\pi)/2$ and $\sqrt{z}$ is close to $-i\sqrt{r}$. In other words, as z crosses the cut line, $\sqrt{z}$ jumps from one square root of z to its negative, which is the other square root of z.

Example 1.4.10. Analyze the function $\sqrt{z^2+1}$, where the square root function is defined, as above, using the principal branch of the log function.

Solution: The cut line for $\sqrt{z}$ is the same as for log – the half line of negative reals. The function $\sqrt{z}$ jumps from values on the positive imaginary axis to their negatives as z crosses this line in the counterclockwise direction. The number z^2+1 crosses the negative real half line in the counterclockwise direction as z crosses either $(i, i\infty)$ or $(-i\infty, -i)$ in the counterclockwise direction. Thus, these two half lines on the imaginary axes are where discontinuities of $\sqrt{z^2+1}$ occur. As these lines are crossed in the counterclockwise direction a typical value of this function jumps from one of the form it, $t > 0$ to its negative $-it$ (see Figure 1.6).

Raising a complex number to a complex power is another function that can be defined using a branch of the log function. If $z \neq 0$ and a is any complex number, then we set

$$z^a = \mathrm{e}^{a \log z}. \tag{1.4.7}$$

Here, we are thinking of a as being fixed and z is the independent variable of the function. If we want the exponent to be the variable, we would write

$$a^z = \mathrm{e}^{z \log a}. \tag{1.4.8}$$

Note that for any function defined in terms of a branch $\log z$ of the log function, we can expect trouble along the cut line for log. Since, log has a jump or discontinuity as we cross the cut line, we can expect the same for functions defined in terms of it.

Also, we emphasize that functions defined this way depend on the choice of a branch of the log function. If one wants such a function to agree with the standard one on the positive real axis, then one must choose a branch of the log function which is the ordinary natural logarithm on the positive real numbers. By Theorem 1.4.8, Part(e), this happens if and only if the interval I, in which $\arg(z)$ is required to lie, contains 0. In particular, the principal branch has this property.

Exercise Set 1.4

1. Put each of the complex numbers $-1,\ i,\ -i, 1+\sqrt{3}\,i$, and $5-5i$ in polar form.

2. put each of the complex numbers $\mathrm{e}^{4\pi i},\ 3\mathrm{e}^{2\pi i/3},\ 5e^{5\pi i/2}$ and $2\mathrm{e}^{-3\pi i/4}$ in standard form.

3. Find all powers of $e^{\pi i/8}$. How many distinct powers of this number are there?

4. Show that $(1-i)^7 = 8(1+i)$ by converting to polar form, taking the seventh power and then converting back to standard form.

5. Using a calculator, calculate $(1.2\,\mathrm{e}^{.5\,i})^n$ for $n = 1, \cdots, 6$ and graph the resulting points. Do the same for $(.8\,\mathrm{e}^{.5\,i})^n$.

6. Prove that if z is a number on the unit circle, then z has finitely many distinct powers z^n if and only if the argument of z is a rational multiple of 2π.

7. What are the 4th roots of unity?

8. What are the cube roots of -9?

9. Show that, given one nth root of z, the others are obtained by multiplying it by the nth roots of unity.

10. Find $\arg_I z$ if (a) $z = -i$ and $I = (-\pi, \pi]$, (b) $z = -i$ and $I = [0, 2\pi)$, (c) $z = 1$ and $I = [3\pi/2, 7\pi/2)$.

11. For the principal branch of the log function, find $\log(1-i)$.

12. For the branch of the log function determined by the interval $[0, 2\pi)$, find $\log(1-i)$.

13. Prove (a) and (b) of Theorem 1.4.8.

14. Prove (c) and (d) and (e) of Theorem 1.4.8.

15. Analyze the function z^i, defined by (1.4.7) using the principal branch of the log function. What kind of a jump, if any, does it have as z crosses the negative real axis?

16. Analyze the function $\sqrt{1-z^2}$, where the square root function is defined by the principal branch of the log function. Where does it have discontinuities (jumps)?

17. Let the square root function be defined by the principal branch of the log function. Compare the functions $\sqrt{z^2-1}$ and $\sqrt{z+1}\sqrt{z-1}$. Where are the discontinuities of each function.

18. The identity $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$ was derived in Example 1.2.7. Use this to derive Lagrange's trigonometric identity:

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2}+\frac{\sin{(n+1/2)\theta}}{2\sin\theta/2}.$$

Hint: take the reals parts of both sides in the first identity.