QUIZZES AND EXAMS FOR MATH 1321
ACCELERATED ENGINEERING CALCULUS 2

NICOLA TARASCA

Week 1 Quiz

Find the value of c such that

$$\sum_{n=2}^{\infty}(1+c)^{-n} = \frac{1}{2}. \tag{1}$$

Solution. Since the geometric series (1) converges, we have that

$$\left|\frac{1}{1+c}\right| < 1. \tag{2}$$

We obtain

$$\begin{aligned}
\sum_{n=2}^{\infty}(1+c)^{-n} &= \sum_{n=2}^{\infty}\left(\frac{1}{1+c}\right)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{1+c}\right)^n - \sum_{n=0}^{1}\left(\frac{1}{1+c}\right)^n \\
&= \frac{1}{1-\frac{1}{1+c}} - 1 - \frac{1}{1+c} \\
&= \frac{1}{c(c+1)} = \frac{1}{2}.
\end{aligned}$$

Alternatively

$$\begin{aligned}
\sum_{n=2}^{\infty}(1+c)^{-n} &= \frac{1}{(1+c)^2}\sum_{n=0}^{\infty}\left(\frac{1}{1+c}\right)^n \\
&= \frac{1}{(1+c)^2}\cdot\frac{1}{1-\frac{1}{1+c}} \\
&= \frac{1}{c(c+1)} = \frac{1}{2}.
\end{aligned}$$

In both cases, we deduce

$$c^2 + c - 2 = 0.$$

Date: Fall 2013.

This is an equation in c of degree 2: the two solutions are $c = 1$ and $c = -2$. Note that the value $c = -2$ does not satisfy (2), indeed the series

$$\sum_{n=2}^{\infty}(1+c)^{-n} = \sum_{n=2}^{\infty}(-1)^{-n}$$

diverges. The only solution for (1) is $c = 1$. $\square$

Week 2 Quiz

Test the following series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

What can you deduce about $\lim_{n\to\infty} \frac{n!}{n^n}$?

Solution. We use the ratio test. Let $a_n := \frac{n!}{n^n}$. Then we have

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\
&= \frac{n!(n+1)}{(n+1)(n+1)^n} \cdot \frac{n^n}{n!} \\
&= \frac{n^n}{(n+1)^n} \\
&= \left(\frac{n}{n+1}\right)^n \\
&= \left(\frac{1}{1+\frac{1}{n}}\right)^n .
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n \\
&= \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} \\
&= \frac{1}{e}.
\end{aligned}$$

Since $\frac{1}{e} < 1$, we conclude that the series converges. Since the series converges, then $\lim_{n\to\infty} a_n = 0$. $\square$

Week 3 Quiz

Use power series to compute the following limit

$$\lim_{x\to 0}\frac{1-\cos(x)}{1+x-e^x}.$$

Solution. Since we are considering values of x arbitrarily close to 0, we can replace $\cos(x)$ and e^x with the respective Maclaurin series. We have

$$\begin{aligned}\cos(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\ &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\end{aligned}$$

and

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\\ &= 1+x+\frac{x^2}{2!}+\cdots.\end{aligned}$$

Hence we have

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{1+x-e^x} &= \lim_{x\to 0}\frac{1-\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}{1+x-\sum_{n=0}^{\infty}\frac{x^n}{n!}}\\ &= \lim_{x\to 0}\frac{1-(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots)}{1+x-(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)}\\ &= \lim_{x\to 0}\frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots}{-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots}\\ &= \lim_{x\to 0}\frac{\frac{x^2}{2!}(1-\frac{x^2}{12}+\cdots)}{-\frac{x^2}{2!}(1+\frac{x}{3}+\cdots)}\\ &= \lim_{x\to 0}-\frac{1-\frac{x^2}{12}+\cdots}{1+\frac{x}{3}+\cdots}\\ &= -1.\end{aligned}$$

$\square$

Week 4 Super Quiz

Problem 1. Find the Maclaurin series of $\sin^{-1}(x)$ and its radius of convergence.

Hint: Use the following
$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Problem 2. Do the points $A = (1, 3, 2)$, $B = (3, -1, 6)$, $C = (5, 2, 0)$, $D = (3, 6, -4)$ in $\mathbb{R}^3$ lie in the same plane?

Solution Problem 1. Using the binomial series, one has
$$(1-x^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (-x^2)^m$$
for $|x| < 1$. Using the hint, we deduce
$$\begin{aligned}
\sin^{-1}(x) &= \int \frac{1}{\sqrt{1-x^2}} dx \\
&= \int \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (-x^2)^m dx \\
&= \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} \int (-x^2)^m dx \\
&= \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (-1)^m \frac{x^{2m+1}}{2m+1}.
\end{aligned}$$
Since the Maclaurin series of $\sin^{-1}(x)$ is obtained integrating the binomial series, the radius is 1.
Extra. Note that
$$\begin{aligned}
\binom{-\frac{1}{2}}{m} &= \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2m-1}{2})}{m!} \\
&= (-1)^m \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m \cdot m!}.
\end{aligned}$$
Hence the Maclaurin series for $\sin^{-1}(x)$ can be simplified as
$$\sin^{-1}(x) = \sum_{m=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m \cdot m!} \frac{x^{2m+1}}{2m+1}.$$
□

Solution Problem 2. The points A, B, C, D lie in the same plane if and only if the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ lie in the same plane. We have

$$\begin{aligned} \overrightarrow{AB} &= \ < 2, -4, 4 > \\ \overrightarrow{AC} &= \ < 4, -1, -2 > \\ \overrightarrow{AD} &= \ < 2, 3, -6 > . \end{aligned}$$

The above vectors lie in the same plane if and only if the following determinant

$$\begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix}$$

is zero. One computes

$$\begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2(6+6) + 4(-24+4) + 4(12+2) = 0.$$

Hence the four points lie in the same plane. □

Week 5 Quiz

1) Find the equation of the plane which contains the point $(0, 1, 2)$ and the line L with parametric equation $x = 1 + t$, $y = 1 - t$, $z = 2t$.

2) Find the vector equation of the line passing through the point $(0, 1, 2)$ and meeting the line L orthogonally.

Midterm 1

Problem 1. Find the radius of convergence and interval of convergence of the following series

$$\sum_{n=0}^{\infty} \frac{3^{2n+1} x^n}{(n+2)^2}.$$

Problem 2. Use series to evaluate the following limit

$$\lim_{x \to 0} \frac{1 - (1 - 2x^2)^{2013}}{\ln(1 + 2x) - 2x}.$$

Problem 3. Let $P_1 = (3, 2, 0)$, $P_2 = (0, -1, -3)$ be two points in $\mathbb{R}^3$, and let L be the line with parametric equation $x = 1 + 2t$, $y = -2t$, $z = t - 2$, with $t \in \mathbb{R}$.

i) Show that the line L and the points P_1, P_2 are coplanar, and find the equation of the plane containing L, P_1, P_2.

ii) Find a vector equation of the line passing through P_1 and meeting the line L orthogonally.

Problem 4. Let $z = z(x, y)$ be a function of two variables such that
$$x^5 + x^4 y + y^5 + xy^2 z^2 + z^5 = 5.$$
Use implicit differentiation to find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Week 8 Quiz

Find the local maximum and minimum and saddle points of the following function:
$$f(x,y) = \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\right)\cosh(y).$$

Solution. Let us first find the set of critical points. The gradient of the function $f(x, y)$ is
$$\nabla f = \left\langle (x^3 - x)\cosh(y), \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\right)\sinh(y)\right\rangle.$$
Since $\cosh(y)$ is positive for all y, we have that $f_x = 0$ if and only if $x^3 - x = 0$. This is a polynomial of degree 3 and the three roots are $x = 0$, $x = -1$, $x = 1$. Plugging these values in f_y, we see that for $x \in \{0, -1, 1\}$ we have $f_y = 0$ if and only if $\sinh(y) = 0$, that is, $y = 0$. We conclude that the critical points are $(0, 0)$, $(-1, 0)$ and $(1, 0)$.

The determinant of the matrix of second order partial derivatives of f is
$$\begin{aligned} H_f(x,y) &:= \begin{vmatrix} (3x^2-1)\cosh(y) & (x^3-x)\sinh(y) \\ (x^3-x)\sinh(y) & \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\right)\cosh(y) \end{vmatrix} \\ &= (3x^2-1)\left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\right)\cosh^2(y) - (x^3-x)^2\sinh^2(y). \end{aligned}$$
Let us compute $H_f(x, y)$ at the three critical points. Since $H_f(0, 0) = -\frac{1}{2} < 0$, we have that the point $(0, 0)$ is a saddle point. Since $H_f(1, 0) = H_f(-1, 0) = \frac{1}{2} > 0$ and $f_{xx}(1, 0) = f_{xx} = (-1, 0) = 2 > 0$, the points $(-1, 0)$, $(1, 0)$ are local minimum points of the function f. $\square$

Week 9 Super Quiz

Problem 1. Find the points on the ellipsoid
$$\frac{x^2}{4} + 4y^2 + \frac{z^2}{4} = 4$$
that are nearest to and farthest from the origin.

Solution Problem 1. The problem is asking for the points on the given ellipsoid with extreme values with respect to the distance function $G(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Such points have also extreme values of the function $\widetilde{G}(x, y, z) = x^2 + y^2 + z^2$, hence we can replace G with $\widetilde{G}$. Using the method of the Lagrange multipliers, we solve the system

$$\begin{cases} \frac{x}{2} = \lambda 2x \\ 8y = \lambda 2y \\ \frac{z}{2} = \lambda 2z \\ \frac{x^2}{4} + 4y^2 + \frac{z^2}{4} = 4 \end{cases} . \tag{3}$$

Note that

$$\begin{array}{rcllcl} \dfrac{x}{2} & = & \lambda 2x & \Leftrightarrow & x = 0 & \text{or} \quad \lambda = \dfrac{1}{4} \\ 8y & = & \lambda 2y & \Leftrightarrow & y = 0 & \text{or} \quad \lambda = 4 \\ \dfrac{z}{2} & = & \lambda 2z & \Leftrightarrow & z = 0 & \text{or} \quad \lambda = \dfrac{1}{4}. \end{array}$$

There are only two possible values for λ, namely $\lambda = \frac{1}{4}$ or $\lambda = 4$. If $\lambda = \frac{1}{4}$, then necessarily $y = 0$. Plugging $y = 0$ in the last equation in (3), we obtain $x^2 + z^2 = 16$, that is, the circle on the plane $y = 0$ of radius 4 centered at the origin. If $\lambda = 4$, then necessarily $x = z = 0$, and plugging these constraints in the last equation in (3), we obtain $y^2 = 1$, that is $y = \pm 1$, hence the two points $(0, \pm 1, 0)$. These last two points have distance 1 from the origin, while the points on the circle $x^2 + z^2 = 16$, $y = 0$ have distance $\sqrt{x^2 + 0^2 + z^2} = 4$. It follows that the two point $(0, \pm 1, 0)$ have minimal distance, while the points on the circle $x^2 + z^2 = 16$, $y = 0$ have maximal distance. $\square$

Problem 2. Find the volume of the solid that lies under the surface $z = x^4 + \sin(y) + 2$ and above the rectangle $[-1, 0] \times [-\pi, \pi]$.

Solution Problem 2. The volume of the solid is given by the double integral

$$\iint_{[-1,0]\times[-\pi,\pi]} x^4 + \sin(y) + 2 \, dA.$$

By the Fubini theorem, we have

$$\begin{aligned}\iint_{[-1,0]\times[-\pi,\pi]} x^4+\sin(y)+2\,dA &= \int_{-1}^{0}\int_{-\pi}^{\pi} x^4+\sin(y)+2\,dydx\\ &= \int_{-1}^{0}\left[x^4y-\cos(y)+2y\right]_{y=-\pi}^{y=\pi}dx\\ &= \int_{-1}^{0} 2\pi x^4+4\pi\,dx\\ &= \left[2\pi\frac{x^5}{5}+4\pi x\right]_{x=-1}^{x=0}\\ &= \frac{2\pi}{5}+4\pi\\ &= \frac{22\pi}{5}.\end{aligned}$$

□

Week 10 Quiz

Find the volume of the solid below the paraboloid $z=10-2x^2-2y^2$ and above the $z=0$ plane.

Solution. Using polar coordinates, we have that the volume of the solid is given by

$$\iint_D (10-2r^2)rdrd\theta$$

where D is the disk with border $10-2x^2-2y^2=0$, that is, $D=\{(r,\theta)\mid 0\le r\le \sqrt{5},\ 0\le\theta\le 2\pi\}$. We have

$$\begin{aligned}\iint_D (10-2r^2)rdrd\theta &= \int_0^{2\pi}\int_0^{\sqrt{5}}(10-2r^2)rdrd\theta\\ &= \int_0^{2\pi}d\theta\int_0^{\sqrt{5}}(10-2r^2)rdr\\ &= 2\pi\int_0^{\sqrt{5}}(10r-2r^3)dr\\ &= 2\pi\left[5r^2-2\frac{r^4}{4}\right]_{r=0}^{r=\sqrt{5}}\\ &= 2\pi\left(25-\frac{25}{2}\right)\\ &= 25\pi.\end{aligned}$$

□

Midterm 2

Problem 1. Find the points on the surface $1 - x^2 + (y-1)^4 - z^2 = 0$ which are closest to the point $(0, 1, 0)$.

Solution Problem 1. We have to look for the points which minimize the function $F(x, y, z) = \sqrt{x^2 + (y-1)^2 + z^2}$ under the constraint $1 - x^2 + (y-1)^4 - z^2 = 0$. The local minimum points for the function $F(x, y, z)$ are also local minimum points for the function $(F(x, y, z))^2$, hence we can replace $F(x, y, z)$ with $(F(x, y, z))^2 = x^2 + (y-1)^2 + z^2$. Using the Lagrange multipliers method, we look for the solution of the following system:

$$\begin{cases} 2x = \lambda(-2x) \\ 2(y-1) = \lambda 4(y-1)^3 \\ 2z = \lambda(-2z) \\ 1 - x^2 + (y-1)^4 - z^2 = 0 \end{cases} . \tag{4}$$

Note that

$$\begin{aligned} 2x = \lambda(-2x) \quad &\Leftrightarrow \quad x = 0 \quad \text{or} \quad \lambda = -1 \\ 2(y-1) = \lambda 4(y-1)^3 \quad &\Leftrightarrow \quad y = 1 \quad \text{or} \quad (y-1)^2 = \frac{2}{4\lambda} \\ 2z = \lambda(-2z) \quad &\Leftrightarrow \quad z = 0 \quad \text{or} \quad \lambda = -1. \end{aligned}$$

Suppose that $x = z = 0$. Then from the last equation in (4) we have that $(y-1)^4 = -1$, a contradiction (the left hand side is positive!). Hence necessarily $\lambda = -1$. We use the second and forth equation in (4) to find the remaining information. Since $\lambda = -1$, we cannot have $(y-1)^2 = \frac{2}{4\lambda}$ (the left hand side is positive!). Hence necessarily $y = 1$. From the forth equation in (4), we have that if $y = 1$, then $x^2 + z^2 = 1$. Hence the outcome of the Lagrange multipliers method is the circle given by the two equations $y = 1$, $x^2 + z^2 = 1$. It remains to show whether the points on this set have minimal or maximal distance from $(0, 1, 0)$. We pick an arbitrary point on the surface $1 - x^2 + (y-1)^4 - z^2 = 0$ and we compute the distance from $(0, 1, 0)$: the point $(\sqrt{2}, 2, 0)$ has distance $\sqrt{3}$ from the point $(0, 1, 0)$. The distance from a point on the circle $y = 1$, $x^2 + z^2 = 1$ is 1 $(< \sqrt{3})$. Hence the points on the circle $y = 1$, $x^2 + z^2 = 1$ are the points on the surface $1 - x^2 + (y-1)^4 - z^2 = 0$ closest to the point $(0, 1, 0)$. □

Problem 2. Find a such that the part of the surface $z = 10x^2 + 10y^2$ below the plane $z = a$ has surface area $\frac{182}{75}\pi$.

Solution Problem 2. The area of the surface $z = 10x^2 + 10y^2$ below the plane $z = a$ is given by the formula

$$\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

where D is

$$\begin{aligned} D &:= \{(x, y) \mid 10x^2 + 10y^2 \leq a\} \\ &= \left\{(r, \theta) \mid 0 \leq \theta \leq 2\pi,\, 0 \leq r \leq \sqrt{\frac{a}{10}}\right\}. \end{aligned}$$

We have

$$\begin{aligned} \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA &= \iint_D \sqrt{1 + (20x)^2 + (20y)^2} dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{\frac{a}{10}}} \sqrt{1 + 400r^2}\, r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{a}{10}}} \sqrt{1 + 400r^2}\, r dr \\ &= 2\pi \left[\frac{2}{3}\frac{1}{800}(1 + 400r^2)^{\frac{3}{2}}\right]_0^{\sqrt{\frac{a}{10}}} \\ &= \frac{\pi}{600}\left((1 + 40a)^{\frac{3}{2}} - 1\right). \end{aligned}$$

From the desired equality

$$\frac{\pi}{600}\left((1 + 40a)^{\frac{3}{2}} - 1\right) = \frac{182}{75}\pi$$

we recover the value of a:

$$a = \frac{\left(1457^{\frac{2}{3}} - 1\right)}{40}.$$

$\square$

Problem 3. Find a such that the solid above the surface $z = 10x^2 + 10y^2$ and below the plane $z = a$ has volume $\frac{4}{5}\pi$.

Solution Problem 3. The volume of the solid is given by

$$\iint_D \int_{10r^2}^{a} dz dA$$

where D is as in the previous problem

$$D = \left\{ (r, \theta) \,|\, 0 \leq \theta \leq 2\pi,\, 0 \leq r \leq \sqrt{\frac{a}{10}} \right\}.$$

We have

$$\begin{aligned}
\iint_D \int_{10r^2}^{a} dz dA &= \int_0^{2\pi} \int_0^{\sqrt{\frac{a}{10}}} \int_{10r^2}^{a} dz r dr d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{a}{10}}} \int_{10r^2}^{a} dz r dr \\
&= 2\pi \int_0^{\sqrt{\frac{a}{10}}} (a - 10r^2) r dr \\
&= 2\pi \left[a\frac{r^2}{2} - 10\frac{r^4}{4} \right]_0^{\sqrt{\frac{a}{10}}} \\
&= 2\pi \left(\frac{a^2}{20} - \frac{a^2}{40} \right) \\
&= 2\pi \frac{a^2}{40}.
\end{aligned}$$

Imposing the volume to be equal to $\frac{4}{5}\pi$, we recover the value of a: $a = 4$. □

Week 14 Quiz

Verify that Stokes' Theorem is true for the vector field $\vec{F} = < y, z, x >$ and the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented inward.

Final Exam

Problem 1. Find the local maximum and minimum values and saddle points of the following function

$$f(x, y) = \sin(x) + 3y^2.$$

Problem 2. Use Lagrange multipliers to find the points on the surface $(x - 8)^6 + 8 - y^2 - z^2 = 0$ which are closest to the point $(8, 0, 0)$.

Problem 3. Verify that Stokes' theorem is true for the vector field $\vec{F} = < z, x, y >$ and the surface defined by $x^2 + y^2 + z^2 = 1$ and $0 \leq z \leq \frac{\sqrt{2}}{2}$.

Problem 4. Verify that the divergence theorem is true for the vector field $\vec{F} = < z, y, x >$ and the region $z^2 + y^2 \leq x$ and $1 \leq x \leq 2$.