Humboldt Universität zu Berlin — December 14, 2009
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

Exercise Sheet 9

Hand in solutions not later than Monday, January 4.

Exercise 1. *[Gathmanns notes, Ex. 4.6.11]* Let $C \subset \mathbb{P}^2$ be a smooth curve, given as the zero locus of a homogeneous polynomial $f \in k[x_0, x_1, x_2]$. Consider the morphism

$$\varphi : C \to \mathbb{P}^2, \quad P \mapsto \left[\frac{\partial f}{\partial x_0}(P) : \frac{\partial f}{\partial x_1}(P) : \frac{\partial f}{\partial x_2}(P)\right].$$

The image $\varphi(C) \subset \mathbb{P}^2$ is called the *dual curve* to C.

i) Find a geometric description of φ. What does it mean geometrically if $\varphi(P) = \varphi(Q)$ for two distinct points $P, Q \in C$?
ii) If C is a conic, prove that its dual $\varphi(C)$ is also a conic.
iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines.

Exercise 2. *[Gathmanns notes, Ex. 4.6.9]* Let $X \subset \mathbb{A}^n$ be an affine variety, and let $P \in X$ be a point. Show that the coordinate ring $A(C_{X,P})$ of the tangent cone to X at P is equal to $\bigoplus_k I(P)^k/I(P)^{k+1}$, where $I(P)$ is the ideal of P in $A(X)$.

Exercise 3. Let $X = V(xyz) \subset \mathbb{A}^3$. Determine the singular locus Σ of X. Determine the singular locus of Σ.

Exercise 4. Let $\mathbb{P}^N$ be the space of hypersurfaces of degree d in $\mathbb{P}^n$, where $N = \binom{n+d}{d} - 1$. Show that the subset of $\mathbb{P}^N$ corresponding to smooth hypersurfaces is non-empty and open (i.e. a general hypersurface in $\mathbb{P}^n$ is smooth).