Humboldt Universität zu Berlin November 30, 2009
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

Exercise Sheet 7

Hand in solutions not later than Monday, December 7.

Exercise 1. Given a d-plane in $\mathbb{P}^n$ defined by $n-d$ linear independent equations $L_1 = \cdots = L_{n-d} = 0$, define the projection $\pi_E : \mathbb{P}^n \setminus E \to \mathbb{P}^{n-d-1}$ as $\pi_E(x) = [L_1(x) : \cdots : L_{n-d}(x)]$. For $n = 4$, let E be defined as $V(x_0, x_1, x_2)$ and S as $V(x_0^2 + x_1^2 + x_2^2 + x_3^2, x_0^2 + x_1x_2 + x_2^2 + x_4^2)$. Let $\pi_E|_S : S \to \mathbb{P}^2$ be the projection from E restricted to S. Show that $\pi_E|_S$ has finite fibers.

Exercise 2. Let $\Sigma \subset \mathbb{G}(k, n) \times \mathbb{P}^n$ be the *incidence correspondence*

$$\Sigma := \{(\Lambda, x) | x \in \Lambda\}.$$

Let π_1, π_2 be the natural projections from Σ respectively to $\mathbb{G}(k, n)$ and $\mathbb{P}^n$. Let $X \subset \mathbb{P}^n$ be an irreducible variety. For any $k \leq n - \dim X$, consider $\mathcal{C}_k(X) \subset \mathbb{G}(k, n)$ the subvariety of $k-$planes meeting X.

i) Show that $\mathcal{C}_k(X) = \pi_1(\pi_2^{-1}(X))$.
ii) Compute the dimension of $\mathcal{C}_k(X)$.

Exercise 3. Let $X \subset \mathbb{P}^n$ be a variety of dimension $k < n - 1$. Let the secant line map

$$s : (X \times X) \setminus \Delta \to \mathbb{G}(1, n)$$

be the map that sends a pair (p, q) to the line in $\mathbb{P}^n$ passing through p and q. The variety $\mathcal{S}(X)$ of secant lines to X is defined as the image of s.

i) Compute the dimension of $\mathcal{S}(X)$.
ii) Show that $\mathcal{S}(X)$ is a proper subvariety of $\mathcal{C}_1(X)$. Deduce that the general projections $\pi_E : X \to \mathbb{P}^{k+1}$ of X from an $(n-k-2)$-plane E is birational onto its image. In particular every variery is birational to an hypersurface.

Exercise 4. Let X be a complete variety.

i) Let $f : X \to Y$ a morphism of varieties. Show that $f(X)$ is closed in Y and complete.
ii) Show that if X is affine, then $\dim X = 0$. (Hint: Consider the embedding of X in the closure $\overline{X}$ in a suitable projective space.)