Humboldt Universität zu Berlin
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

November 9, 2009

Exercise Sheet 4

Hand in solutions not later than Monday, November 16.

Exercise 1 *[Gathmann's notes, Ex. 2.6.3]* Which of the following algebraic sets are isomorphic over the complex numbers?

i) $\mathbb{A}^1$
ii) $V(xy) \subset \mathbb{A}^2$
iii) $V(x^2 + y^2) \subset \mathbb{A}^2$
iv) $V(y - x^2, z - x^3) \subset \mathbb{A}^3$.

Exercise 2 *[Gathmann's notes, Ex. 2.6.5]* Are the following statements true or false: if $f : \mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map (i.e. $f(P) = (f_1(P), \ldots, f_m(P))$ with $f_i \in k[x_1, \ldots, x_n]$), and ...

i) $X \subset \mathbb{A}^n$ is an algebraic set, then the image $f(X) \subset \mathbb{A}^m$ is an algebraic set;
ii) $X \subset \mathbb{A}^m$ is an algebraic set, then the inverse image $f^1(X) \subset \mathbb{A}^n$ is an algebraic set;
iii) $X \subset \mathbb{A}^n$ is an algebraic set, then the graph $G = \{(x, f(x)) | x \in X\} \subset \mathbb{A}^{n+m}$ is an algebraic set.

Exercise 3 Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $2g + 1$ with distinct roots. Define $g(z) := z^{2g+2} f(\frac{1}{z})$. Let X be the zero locus of $y^2 = f(x)$ and U its open subset $\{(x, y) \in X | x \neq 0\}$. Let Y be the zero locus of $w^2 = g(z)$ and V its open subset $\{(z, w) \in Y | z \neq 0\}$. Define an isomorphism $\phi : U \to V$ as

$$\phi(x, y) = \left(\frac{1}{x}, \frac{y}{x^{g+1}} \right).$$

Let Z be the prevariety obtained by glueing X and Y along U and V via ϕ.

i) Show that Z is a variety.

ii) Let $\pi : Z \to \mathbb{P}^1$ be the projection on the first coordinate. Determine which points in $\mathbb{P}^1$ have one preimage, and which points have two preimages under π.

Remark In general, given a surjective morphism of curves $\psi : X \to Y$, let d be the maximal number of preimages. Then there exist a finite number of points in Y having a number of preimages less than d. These points are called the *branch points* for ψ.

iii) Consider Z as the compactification of X. How many points are there in $Z \setminus X$?

iv) Find an involution ι of Z (i.e. $\iota : Z \to Z$ such that $\iota \circ \iota = id$).

Exercise 4 *[Gathmann's notes, Ex. 2.6.9]* Let X be a prevariety. Consider pairs (U, f) where U is an open subset of X and $f \in \mathcal{O}_X(U)$ a regular function on U. We call two such pairs (U, f) and (U', f') equivalent if there is an open subset $V \in X$ with $V \subset U \cap U'$ such that $f|_V = f'|_V$.

i) Show that this defines an equivalence relation.
ii) Show that the set of all such pairs modulo this equivalence relation is a field. It is called the *field of rational functions on* X and denoted $K(X)$.
iii) If X is an affine variety, show that $K(X)$ is just the field of rational functions.
iv) If $U \subset X$ is any non-empty open subset, show that $K(U) = K(X)$.