Humboldt Universität zu Berlin November 2, 2009
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

Exercise Sheet 3

Hand in solutions not later than Monday, November 9.

Exercise 1 Let $X = V(y^2 - x^3)$ and $X' = V(y - x^3)$.

i) Compute $\mathcal{O}_{X,(0,0)}$ and its maximal ideal $\mathfrak{m}_{(0,0)}$. Prove that $\mathfrak{m}_{(0,0)}$ cannot be generated by one element.

ii) Compute $\mathcal{O}_{X',(0,0)}$ and its maximal ideal $\mathfrak{m}_{(0,0)}$. Prove that $\mathfrak{m}_{(0,0)}$ is generated by one element.

Exercise 2 Consider the following subsets of $\mathbb{A}^3$: $V = \{(x, y, z)|z = xy\}$, $W = \{(1, t, t)|t \in k\}$ and $Z = \{(x, y, z)|z \neq 0\}$. Determine whether the following restriction maps are surjective or not. Moreover, determine whether they are injective or not and if not determine the kernel:

i) $\mathcal{O}_V(V) \to \mathcal{O}_W(W)$

ii) $\mathcal{O}_V(V) \to \mathcal{O}_V(Z \cap V)$.

Exercise 3 Let $X = V(x^2 + y^2 - 1)$ and $p = (0, 1) \in X$.

i) Let $q = (t, 0)$. Find an equation for the line l_q passing trough p and q.

ii) Show that for $q = (\pm i, 0)$, we have $l_q \cap X = \{p\}$. Find coordinates for $\bar{q}$, where $\bar{q}$ is such that $l_q \cap X = \{p, \bar{q}\}$, for $q \neq (\pm i, 0)$.

iii) Let $f : \mathbb{A}^1 \setminus \{i, -i\} \to X \setminus \{p\}$ be the map defined by $f(q) = \bar{q}$. Show that f is a morphism and find its inverse.

iv) Show that f does not induce a morphism $f^* : \mathbb{K}[x, y]/(x^2 + y^2 - 1) \to \mathbb{K}[t]$.

v) Show that f induces a field isomorphism $f^* : \mathbb{K}(x, y)/(x^2 + y^2 - 1) \to \mathbb{K}(t)$.

Exercise 4 Let $\mathcal{F}, G$ be two sheaves of rings on a topological space X. Prove that $\mathcal{F} \oplus \mathcal{G}$ defined by $(\mathcal{F} \oplus G)(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ for each $U \subset X$ open, is a sheaf of rings.