Humboldt Universität zu Berlin October 26, 2009
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

Exercise Sheet 2

Hand in solutions not later than Monday, November 2.

Exercise 1 Let R be a commutative ring, $\mathfrak{a} \subset R$ an ideal. Prove that

$$\sqrt{\mathfrak{a}} \quad = \quad \bigcap_{\substack{\mathfrak{p} \supset \mathfrak{a} \\ \mathfrak{p}\,\text{prime}}} \mathfrak{p}.$$

Exercise 2 Find the radicals of the following ideals:

a) $(xy, xz, y^2, yz) \subset \mathbb{R}[x, y, z]$;
b) $(72) \subset \mathbb{Z}$.

Exercise 3 *[Gathmann's notes, Ex. 1.4.2]* Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements, although X has codimension 2 in $\mathbb{A}^3$.

Exercise 4 *[Gathmann's notes, Ex. 1.4.9]* Let $X \subset \mathbb{A}^2$ be an irreducible algebraic set. Show that either

- $X = Z(0)$, i.e. X is the whole space $\mathbb{A}^2$, or
- $X = Z(f)$ for some irreducible polynomial $f \in k[x, y]$, or
- $X = Z(x - a, y - b)$ for some $a, b \in k$, i.e. X is a single point.

Deduce that $\dim(\mathbb{A}^2) = 2$. (Hint: Show that the common zero locus of two polynomials $f, g \in k[x, y]$ without common factor is finite.)