Humboldt Universität zu Berlin
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

January 18, 2010

Exercise Sheet 11

Hand in solutions not later than Monday, January 25.

Exercise 1. Let $S_k = Spec\ \mathbb{C}[x,y]/(x^2 + y^{k+1})$, for $k > 0$. For which values of k is S_k irreducible? For which values of k is S_k separated? For which values of k is S_k reduced?

Exercise 2. Let $S = Spec\ \mathbb{K}[x,y]/(x + y^3)$, $l_1 = Spec\ \mathbb{K}[x,y]/(x + y)$, $l_2 = Spec\ \mathbb{K}[x,y]/(x)$ and $l_3 = Spec\ \mathbb{K}[x,y]/(y)$. Determine the scheme-theoretic intersections $S \cap l_i$, for $i = 1, 2, 3$.

Exercise 3. Let $pr_i : \mathbb{A}^n \to \mathbb{A}^1$ be the projection on the i-th factor.

i) Determine the fiber product $\mathbb{A}^2 \times_{\mathbb{A}^0} \mathbb{A}^2$.

ii) Determine the fiber product $S_1 := \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^3$ via pr_1 and pr_2. Give the image of $S_1 \to \mathbb{A}^2$ and $S_1 \to \mathbb{A}^3$.

ii) Determine the fiber product $S_2 := \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^2$ via pr_1 and pr_2. Give the two images of $S_2 \to \mathbb{A}^2$.

iii) Determine the fiber product $S_3 := \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^2$ via pr_1 and pr_1. Give the two images of $S_3 \to \mathbb{A}^2$.

Exercise 4. The aim of this exercise is to determine the points of $Spec\ \mathbb{Z}[\sqrt{3}]$. Recall that $\mathbb{Z}[\sqrt{3}]$ is an Euclidean ring, in particular $\mathbb{Z}[\sqrt{3}]$ is a principal ideal domain.

i) Let $p \equiv \pm 5 \bmod 12$ be a prime number. Show that $p\mathbb{Z}[\sqrt{3}]$ is a prime ideal of $\mathbb{Z}[\sqrt{3}]$.

ii) Let $p \equiv \pm 1 \bmod 12$ be a prime ideal. Show that $p\mathbb{Z}[\sqrt{3}]$ is the product of two prime ideals of $\mathbb{Z}[\sqrt{3}]$.

iii) Show that $2\mathbb{Z}[\sqrt{3}]$ is the product of two prime ideals of $\mathbb{Z}[\sqrt{3}]$.

iv) Show that $3\mathbb{Z}[\sqrt{3}]$ is of the form $\mathfrak{p}^2$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}[\sqrt{3}]$.

v) Let $\sigma : \mathbb{Z}[\sqrt{3}] \to \mathbb{Z}[\sqrt{3}]$ be the conjugation $\sigma(a + b\sqrt{3}) = a - b\sqrt{3}$, for $a, b \in \mathbb{Z}$. Show that if $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}\sqrt{3}$, then either $\mathfrak{p} = p\mathbb{Z}[\sqrt{3}]$, or $\mathfrak{p}\sigma(\mathfrak{p}) = p\mathbb{Z}[\sqrt{3}]$, for some prime number p.

vi) Determine the points of $Spec\ \mathbb{Z}[\sqrt{3}]$.