Humboldt Universität zu Berlin — January 4, 2010
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

Exercise Sheet 10

Hand in solutions not later than Monday, January 11.

Exercise 1. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ be the rational map defined as $[x : y : z] \mapsto [f_0 : f_1 : f_2 : f_3]$, where

$$\begin{aligned} f_0 &= (z+y)(z-y)(y-2z) \\ f_1 &= (x+y)(x+3z)(x+2y-3z) \\ f_2 &= (z-y)(x+3z)(3x+2y-z) \\ f_3 &= (z+y)(x+2y-3z)(z-x-2y). \end{aligned}$$

i) Check that f_0, f_1, f_2, f_3 form a basis for the K-vector space of homogeneous degree 3 polynomials in x, y, z that vanishes at

$$\begin{aligned} p_1 &= [1:1:1] & p_2 &= [-1:1:1] & p_3 &= [-1:1:0] \\ p_4 &= [-3:-1:1] & p_5 &= [-1:2:1] & p_6 &= [-3:2:1]. \end{aligned}$$

ii) Prove that the image of φ is contained in the cubic hypersurfaces $X = \{[x : y : z : w] \in \mathbb{P}^3 \mid zw^2 - z^2w + 8xyw + 2xyz + 6xy^2 - 16x^2y = 0\}$.

iii) Prove that $\varphi([1 : 0 : 0])$ and $\varphi([0 : 1 : 0])$ are Eckardt points of X.

Exercise 2. Let $X = \{[x : y : z : w] \in \mathbb{P}^3 \mid zw^2 - z^2w + 8xyw + 2xyz + 6xy^2 - 16x^2y = 0\}$ be a cubic hypersurface in $\mathbb{P}^3$ and let $\pi : X \setminus \{p\} \to \mathbb{P}^2$ be the projection from the point $p := [0 : 0 : 1 : 0] \in X$. Let U be the open subset of X where $w \neq 0$.

i) Show that $\pi|_U : U \to \mathbb{A}^2$ is surjective, and that for each point $P \in U$ we have that $\pi|_U^{-1}(P)$ consists of either one or two points.

ii) Let $C = \{p \in U | \#\pi|_U^{-1}(p) = 1\}$ be the ramification curve. Show that C is a curve of degree 4.

iii) Consider the line $L = \overline{\varphi(V(x + y + 4z) \setminus \{p_1, p_3\})}$ on X, with notation as in Exercise 1. Show that $\ell = \pi(L)$ is a line in $\mathbb{P}^2$ that intersects the ramification curve C at two distinct points r_1, r_2. Show that ℓ is tangent to C at both r_1 and r_2.

Exercise 3. Describe the points of *Spec* $\mathbb{Z}[x]$.

Exercise 4. Let $\mathbb{K}$ be an algebraically closed field. Let f be an irreducible polynomial in $\mathbb{K}[x, y, z]$ Describe the points of *Spec* $\mathbb{K}[x, y, z]/(f)$.