Math 2280 - Practice Exam 2

Exam 2 on Friday, November 17

This exam is closed-book and closed-note. You are allowed to use a calculator and will be given a sheet containing the Laplace transform table on the front flap of your textbook. In order to receive full credit however you must show all work and justify your answers.

1. Consider the following mixture problems involving two brine tanks where

   - fresh water flows into tank 1 at rate r gallons/minute,
   - solution from tank 1 is pumped into tank 2 at the same rate r, and
   - solution flows out of tank 2 also at the rate r.

   Suppose that tank 1 contains V_1 gallons of solution in the beginning and tank 2 contains V_2 gallons of solution.

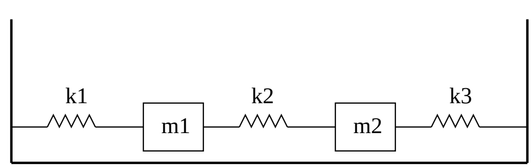
   (a) If $x_1(t)$ and $x_2(t)$ are the amounts of salt in tank 1 and tank 2 respectively, at time t. Write down a system of differential equations describing the dynamics of x_1 and x_2.

   (b) Suppose $r = 10$, $V_1 = 50$, and $V_2 = 25$. Assume also that initially there are 10 lb of salk in tank 1 and tank 2 is filled with pure water. Solve the resulting initial value problems

2. Find the general solution to

$$\mathbf{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \mathbf{x}$$

3. Consider the following configuration of a mass and spring system,



   (a) Let x_1 and x_2 be the the respective displacement of m_1 and m_2 from the rest position. Derive the system of differential equations that describe the motion of m_1 and m_2. Assume that there are no external force applied.

(b) Set $m_1 = 1$, $m_2 = 2$ and $k_1 = 2$, $k_2 = k_3 = 4$. Describe the two fundamental modes of free oscillation to the system.

4. Let A be the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the fundamental matrix solution to the problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$

(b) Calculate the matrix exponential $e^{\mathbf{A}t}$

(c) Find the solution to the initial value problem,

$$\mathbf{x}' = A\mathbf{x} \qquad \mathbf{x}(\mathbf{0}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

5. Consider the Lotka-Volterra model describing two populations competing for the same resource,

$$\begin{aligned} x' &= (3 - 2x - 2y)x \\ y' &= (2 - x - y)y \end{aligned}$$

(a) Find all four equilibrium solutions to the system. Describe the physical meaning of each equilibrium points.

(b) Do linear analysis to classifiy each critical points. Use eigenvalues and eigenvectors to sketch local phase portrait near each equilibrium solutions

(c) Sketch the phase portrait for the full nonlinear system using the analysis you just did in (b). Describe in the long term fate of each population x and y in this system.

6. Find the Laplace transforms or inverse Laplace transforms below,

(a) $\mathcal{L}\{t^2 \cos kt\}(s)$

(b) $\mathcal{L}\left\{\frac{sint}{t}\right\}(s)$

(c) $\mathcal{L}\{e^{-at} * \sin kt\}(s)$

(d) $\mathcal{L}^{-1}\left\{e^{-3s}\frac{s}{s^2+1}\right\}(t)$

(e) $\mathcal{L}^{-1}\left\{\frac{5s-10}{(s-1)^2+4}\right\}(t)$

7. Use Laplace tranform to solve the following initial value problems

$$x'' + x = f(t) \quad \text{where} \quad f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$$