Math 2280 - Practice Exam 1

Exam 1 on Tuesday, October 3

This exam is closed-book and closed-note. You may use a scientific calculator but **not** one which is capable of graphing or solving differential or linear algebra equations. **In order to receive credit, you must show all work and justify your conclusions**.

1. Consider a differential equation which describes a population growth,

$$\frac{dP}{dt} = aP^2 - BP$$

   (a) Find all the equilibrium solutions.

   (b) Suppose that at the initial population is 1 thousand individuals and there are 2 thousands births per month and there are 1 thousands deaths per month. Write the corresponding differential equation for this scenario.

   (c) Sketch the phase line diagram for the equation you just obtained in (b). What are the equilibrium solutions? Which one is stable/unstable?

   (d) Write down a formula for the solution with the above initial conditions. Sketch your solution and indicate the long term behavior of the system. (Hint: Use partial fraction and you may leave the solution in implicit form)

2. A brine tank holds 15000 gallons of continuously mixed liquid. Let $x(t)$ be the amount of salt (in pounds) in the tank at time t. Brine is flowing and out at 150 gallons per hour and the concentration of the salt flowing in is 1 pound per 10 gallons of water.
   Find the differential equation for $x(t)$ and explain how you get this equation. Find the solution assuming that there is no salt in the water inside the tank initially. What is the limiting amount of salt as t approaches infinity?

3. A vertical cylinder tank of radius 3 feet containing water has a hole in the bottom of radius 2 inches. Let $y(t)$ be the height of the water at time t. Find the function $y(t)$ if the height at time 0 is 4 feet.

4. We know that any expression

$$A\cos(\omega t) + B\sin(\omega t)$$

   can be rewritten as

$$C\cos(\omega t - \alpha)$$

   Use the addition angle formula for cosine to rederive the formula for C in terms of A and B.

5. Consider the differential equation,

$$y''' + 4y' = 0$$

(a) Find a general solution to this equation.

(b) Show that this general solution consists of three linearly independent solutions. You may use Wronskian to prove linear independence.

6. Consider a mass spring system with damping in which the mass is 3 kg.

(a) Suppose that the spring is such that it would exerts a force of 18 Newtons when stretched for 2 meters. The dashpot on the other hand provides a resistance of 12 N for each meter per second of velocity. Write down the differential equation for the displacement $x(t)$ of the mass from its resting position.

(b) Find the general solution to this equation. What kind of damping is exhibited by the spring system.

(c) Solve the initial value problem for the spring system above assuming that the mass is initially at its resting position and its initial velocity is 2 m/s.

(d) Sketch a qualitative graph of your solution $x(t)$ from part (c) as a function of t.

7. Find a general solution for each of the equation listed below

(a) $y'' - 6y' + 9y = 2e^{3x}$

(b) $y'' + y' - 6y = 2\cos x$

(c) $y''' - 2y'' - 4y' = x$

(d) $(x^2 + 1)y' + 3xy = 6x$

(e) $(1 - x^2)y' = 2y$