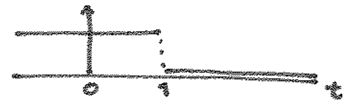
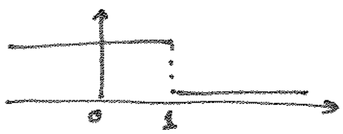
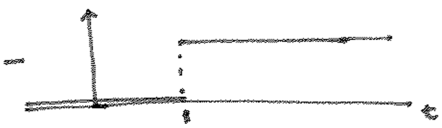
(7) $x'' + x = f(t)$

$x(0) = x_0 \qquad x'(0) = v_0$

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$$

Need to find a step fcn that looks like 

(remember that the value for $t < 0$ is irrelevant when doing LT)

 $= 1 -$  $= 1 - H(t-1)$

So,

$$f(t) = [1 - H(t-1)]\,t = t - H(t-1)\,t$$

LT formula :

$$\mathcal{L}\{H(t-a)g(t-a)\} = G(s)\,e^{-as}$$

Need the argument of g to be the same as the argument for H

$$H(t-1)\cdot t = H(t-1)\cdot(t-1) + H(t-1)$$

$$\Rightarrow f(t) = t - H(t-1)\,t$$
$$= t - H(t-1)(t-1) + H(t-1)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} + \frac{e^{-s}}{s}$$

LT of the eqn :

$$s^2 X - s x_0 - v_0 + X = \mathcal{L}\{f(t)\}$$

$$(s^2+1)X = \left[\frac{1}{s^2} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}\right] + s x_0 + v_0$$

$$X(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} + \frac{e^{-s}}{s(s^2+1)} + x_0\frac{s}{s^2+1} + v_0\frac{1}{s^2+1}$$

Invert

1) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t$

2) $\mathcal{L}^{-1}\left\{\frac{-e^{-s}}{s^2(s^2+1)}\right\} = -\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)\right\}$

$$= -\left[H(t-1)(t-1) - H(t-1)\sin(t-1)\right]$$

$$= H(t-1)\sin(t-1) - H(t-1)(t-1)$$

3) $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\}$

$= H(t-1) - H(t-1)\cos(t-1)$

4) $\mathcal{L}^{-1}\left\{x_0 \frac{s}{s^2+1}\right\} = x_0 \cos t$

5) $\mathcal{L}^{-1}\left\{v_0 \frac{1}{s^2+1}\right\} = v_0 \sin t$

So ...

$x(t) = t - \sin t + H(t-1)\sin(t-1) - H(t-1)(t-1) + H(t-1) - H(t-1)\cos(t-1)$

$+ x_0 \cos t + v_0 \sin t$

$= t - \sin t + x_0 \cos t + v_0 \sin t + H(t-1)\left[\sin(t-1) - t + 2 - \cos(t-1)\right]$