Notes on Electrical Circuits and Cell Excitability

Math 2280 - Fall 2006

November 2, 2006

1 Electrical Circuits

We consider a basic electrical circuit (RLC circuit) which consists of three components, a resistor R, a capacitor C, and an inductor I. Suppose a battery that supplies a voltage of $E(t)$ is attached to the circuit. How do we mathematically describe this process?

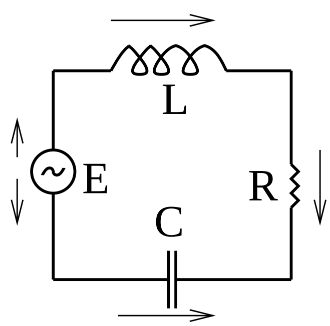


Figure 1: RLC circuit

We will keep track of two quantities namely currents and voltage,

- current $I = \frac{dQ}{dt}$ = amount of charge Q flowing per unit time t.
- voltage V = potential difference that is driving the flow of charges

Let I_R, I_C, and I_L denote currents flowing through R, L, C respectively and V_R, V_C, V_L denote the voltage drop across each of the branch.

Two basic laws, known as Kirchoff rules, must be followed:

1. The amount of charge must be conserved.
   At any node on the circuit, the sum of currents coming in must equal the sum of currents coming out,
   $$I_R = I_L = -I_C$$
2. Energy must be conserved.
   Sum of voltage drops around a closed loop must equal to the voltage supplied by the battery
   $$V_L + V_R - V_C = E(t)$$

We now need to find out how V and I are related to one another for each component of the circuit.

- Typically, resistors follow a linear current-voltage relationship,

$$V_R = R\, I_R$$

  As resistance R gets higher, the larger the potential drop V_R that must be supplied to move charges at rate I_R. This is known as Ohm's law. It is however possible to construct a non-Ohmic resistor so that current and voltage has some nonlinear dependence $V_R = f(I_R)$. This is how one would design a semi-conductor in fact.

- Capacitors consists of two plates that stores opposite charges. The capacitance measures how much voltage is generated across the plates for a given amount of charge stored $C = \frac{Q_C}{V_C}$. Since $I_C = \frac{dQ_C}{dt}$,

$$I_C = C\frac{dV_C}{dt}$$

- The voltage across an inductor is related to the change in current by,

$$V_L = L\frac{dI_L}{dt}$$

Now let's put everything together! Using Kirchoff's rule 2,

$$L\frac{dI_L}{dt} + RI_R - \frac{Q_C}{C} = E(t)$$

Differentiate with respect to t to get

$$L\frac{d^2 I_L}{dt^2} + R\frac{dI_R}{dt} - \frac{I_C}{C} = E'(t)$$

Use Kirchoff's rule 1 to write everything in terms of I_L,

$$L\frac{d^2 I_L}{dt^2} + R\frac{dI_L}{dt} + \frac{I_L}{C} = E'(t)$$

We will drop the subscript L from now on and just write

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

If there's no battery $E(t) = 0$ and the equation is homogeneous. Imagine a situation when in the beginning the circuit is briefly connected to the battery but it's quickly withdrawn just to get some initial current flowing in the system. We can use this initial condition and the homogeneous equation to study what will happen next.

The second order equation we just derived can also be written as a system of two first order equations. In particular, we can keep track of the dynamics of I and V by taking $V = L\frac{dI}{dt}$. The resulting system is,

$$\begin{aligned}\frac{dI}{dt} &= \frac{1}{L}V \\ \frac{dV}{dt} &= E'(t) - \frac{1}{C}I - \frac{R}{L}V\end{aligned}$$

Finally, also note the similarity of this equation to the mechanical mass-spring system

$$mx'' + cx' + kx = f(t)$$

2 Using electrical circuit to study the behavior of excitable cells

Some cells uses make use of ion concentration gradients to function. Typically, a cell at rest has a low concentration of $Na^+$and a high concentration of K^+ions. Meanwhile outside the cell, the reverse is true, high Na^+concentrations and low K^+. The balance of these ionic concentrations is such that the inside of the cell is more negative compared to extracellular space (outside compartment). Biologists says that such cell maintains a negative resting potential. Maintance of this concentration and charge gradients is very important that cells would burn energy continuously to do so.

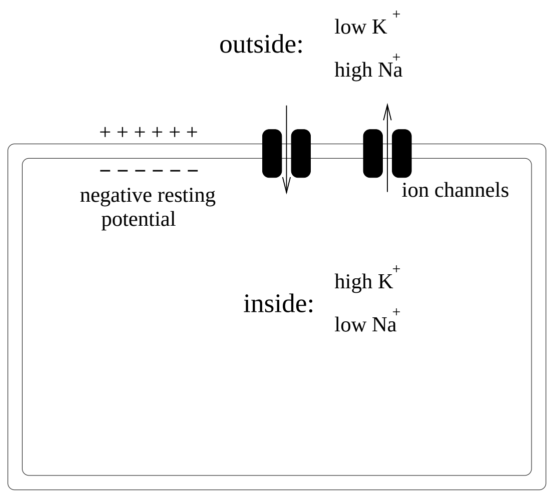


Figure 2: Ionic composition of a cell at rest

Excitable cells such as nerve cells (neurons) and cardiac cells, uses these concentration and charge gradients to amplify and transmit information. This is mediated by ion channels, a specific proteins embedded on the membrane of the cell that allows certain ion to pass through in and out of the cell. These channels open and close in response to the potential difference (voltage) across the cell membrane. This is the basis of why we can represent a cell by an electrical circuit Let

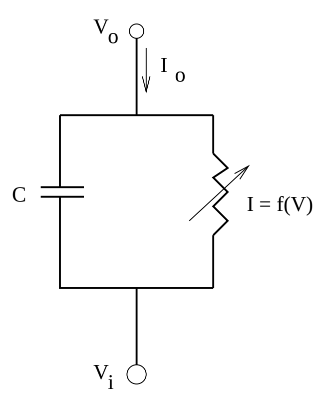
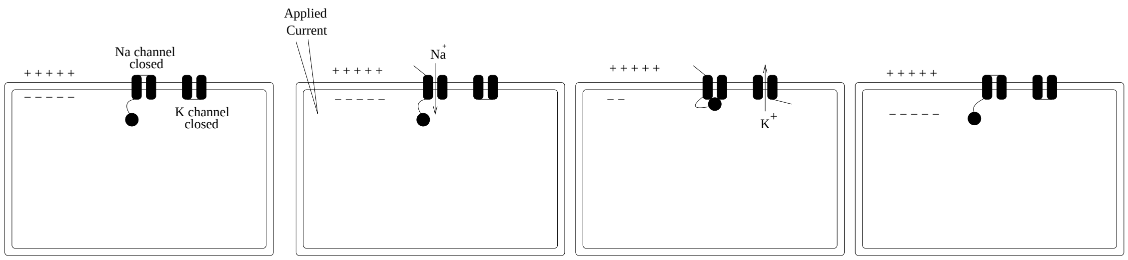


Figure 3: Cell circuit diagram

V be the potential/voltage difference across the cell membrane $V = V_i - V_o$. The membrane of

the cell separates the charges, positive outside the cell and negative inside, and thus acts like a capacitance. The ion channels allows charges (currents) to pass through and can be thought of as a resistor although it may not be an Ohmic/linear resistor.
Typically, the following set of events occurs whenever the cell receive a signal:

1. Na^+channel opens causing $Na^+$to flow inside a cell causing the inside of the cell to become more positive.
2. When the potential difference (voltage) across the membrane has been positive for sometimes, Na^+channel inactivates. In addition, K^+channel opens letting K^+ions to leave the cells
3. Cell returns to its negative resting potential.



This set of events is called an action potential. If one would record the potential difference across the membrane during this process, the trace would look like the one illustrated in the next figure,

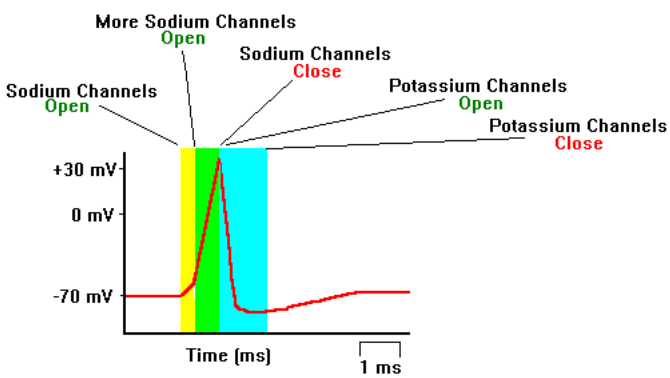


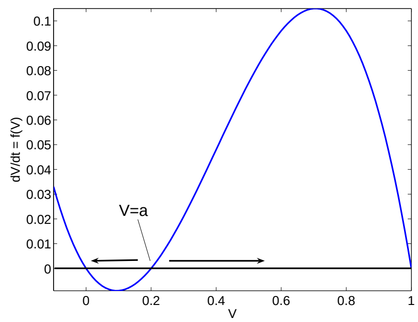
Figure 4: action potential

Through painstaking experiments, Hodgkin and Huxley were able to write down accurate mathematical models to describe this process (1952) and subsequently won a Noble prize for their efforts. Their model consists of four differential equations but we will only study a reduction of their model by Fitzhugh (1961) and Nagumo (1964).

For the circuit diagram above, we have

$$C\frac{dv}{dt} = I_R + I_{\text{app}} = f(v) + I_{\text{app}}$$

Set $C = 1$ and for now suppose $I_{\text{app}} = 0$. We can rescale the potential difference so that the cell is at rest when $v = 0$ and is excited (Na^+channels are completely opened) when $v = 1$. In addition, take $f(v) = -v(v-a)(v-1)$ so a represents the threshold for excitation. Small deviations from resting



potential is not amplified and the potential returns back to rest. However, if the potential is raised above the threshold a the cell moves towards the other equilibrium potential $V = 1$. However, now the cell is stuck at this point and the cell never returns to its rest state and cannot be restimulated.

To account for Na^+channel closing/inactivating, we keep track of a separate variable $w(t)$ that measures how much the channel is blocked at a given time. This variables follows,

$$\frac{dw}{dt} = \epsilon(v - \gamma w)$$

- ϵ is a small parameter which means that $w(t)$ is a slow variable compared to $v(t)$
- When $v = 0$, w approaches 0 so Na^+channel is not blocked when cell is at rest.
- When $v = 1$, w increases to its equilibrium value $1/\gamma$.

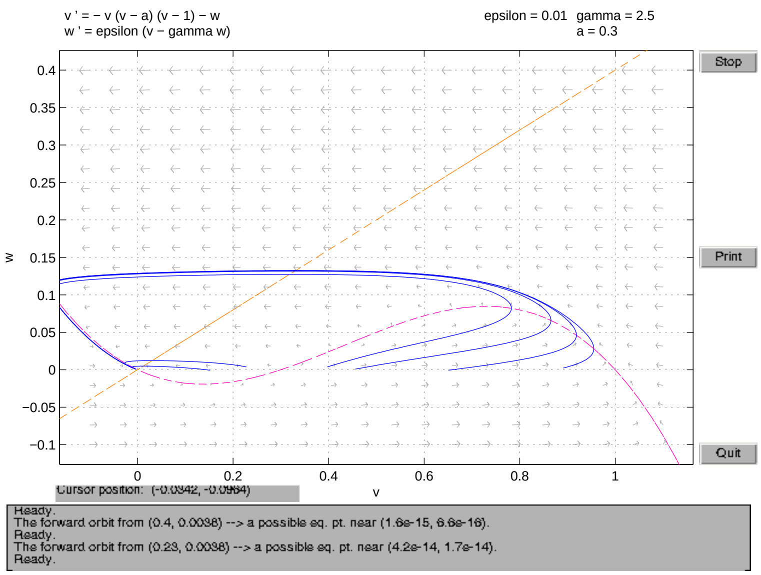
Finally, the equation for v has to be modified. When w is large, channels are blocked and cell should return to its resting potential. We assume that the rate at which v returns to rest is just proportional to w.

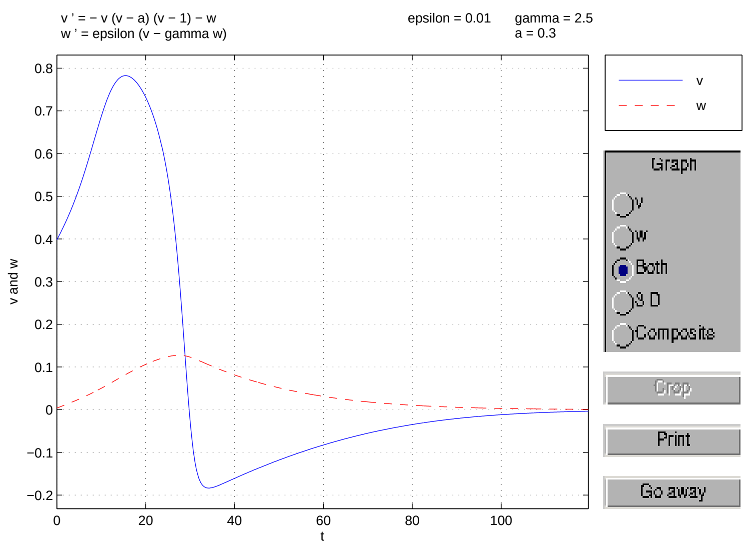
$$\frac{dv}{dt} = f(v) - w$$

With all of these, we end up with the Fitzhugh-Nagumo equation,

$$\begin{aligned} \frac{dv}{dt} &= -v(v-a)(v-1) - w \\ \frac{dw}{dt} &= \epsilon(v - \gamma w) \end{aligned}$$

We can analyze this equation using phase plane analysis. With an initial condition above the threshold, potential of the cell increases. As time goes however, the channels start to inactivates (w increases). When the trajectory crosses the v-nullcline, the potential begins to decreases (trajectory) moving to the left and when it crosses the w-nullcine, the blocking mechanism starts to turn off (trajectory heads down). The trajectory undershoorts $v = 0$ before both v and w returns to their rest state.


v ' = − v (v − a) (v − 1) − w
w ' = epsilon (v − gamma w)
epsilon = 0.01 gamma = 2.5
a = 0.3
Stop
Print
Quit
w
v
Cursor position: (-0.0342, -0.0984)
Ready.
The forward orbit from (0.4, 0.0038) --> a possible eq. pt. near (1.6e-15, 6.6e-16).
Ready.
The forward orbit from (0.23, 0.0038) --> a possible eq. pt. near (4.2e-14, 1.7e-14).
Ready.


v ' = − v (v − a) (v − 1) − w
w ' = epsilon (v − gamma w)
epsilon = 0.01
gamma = 2.5
a = 0.3
v and w
t
v
w
Graph
v
w
Both
3 D
Composite
Crop
Print
Go away