Math 2250-2 Practice Final Exam

Final Exam on Thursday, August 2 at 10 a.m.

The final exam is 120 minutes in duration. It is closed-book and closed-note. You may use a graphing calculator. However, in order to receive credit, you must show all work and justify your conclusions. A table of Laplace transforms will be provided.

1. Consider the following mixture problem involving two brine tanks. Assume that solution from tank 1 is pumped to tank 2 at the rate r gallons/minute and similarly, from tank 2 to tank 1 at the same rate r. Suppose the volume of each respective tank is V_1 gallons and V_2 gallons.

   (a) Derive a system of differential equation describing $x_1(t)$ and $x_2(t)$, the amount of salt (in lb) in tank 1 and tank 2 respectively. Explain your equations in words.

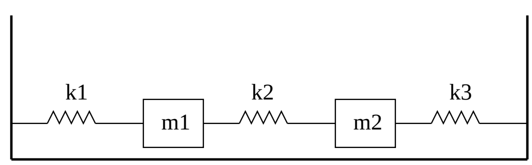
   (b) Let $r = 25$ gal/min, $V_1 = 50$ gallons and $V_2 = 25$ gallons. Solve the resulting system assuming that initially there are 3 lb of salt in tank 1, and tank 2 is filled with pure water.

   (c) Find the limiting amount of salt in each tank as $t \to \infty$.

2. Find the general solution to

$$\mathbf{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \mathbf{x}$$

3. Consider the following configuration of a mass and spring system,



   (a) Let x_1 and x_2 be the the respective displacement of m_1 and m_2 from the rest position. Derive the system of differential equations that describe the motion of m_1 and m_2. Assume that there are no external force applied. (*10 points*)

   (b) Set $m_1 = 2$, $m_2 = 1$ and $k_1 = 100$, $k_2 = 50$, $k_3 = 0$ (i.e. no right wall).
   Describe the two fundamental modes of free oscillation to the system.

4. Consider the matrix $\mathbf{A}$ below and its reduced row echelon form.

$$A = \begin{bmatrix} 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 7 & -10 & -19 & 13 \end{bmatrix} \qquad \text{RREF}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the solution space of $\mathbf{Ax} = \mathbf{0}$.
What is the dimension of the solution space?

(b) Explain what it means for a collection of vectors $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ to be linearly independent or linearly dependent.

(c) Are the first 3 columns of $\mathbf{A}$ linearly independent? Explain why.

(d) Explain what it means for a collection of vectors $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ to span a vector space $\mathbf{V}$.

(e) Do the first three columns of $\mathbf{A}$ span all of $\mathbf{R}^3$? Explain why.

(f) Do the first three columns of $\mathbf{A}$ form a basis for $\mathbf{R}^3$? Explain why.

5. Consider the following system of equations

$$\begin{aligned} x_1 + 3x_2 - 4x_3 - 8x_4 &= 6 \\ x_1 + 2x_3 + x_4 &= 3 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 &= 13 \end{aligned}$$

(a) Write down the system in a matrix-vector form.

(b) Find the solution to this system. You may use the reduced row echelon form given in problem 4.

(c) What is the determinant of the following matrix?

$$\begin{bmatrix} 1 & 3 & -4 \\ 1 & 0 & 2 \\ 2 & 7 & -10 \end{bmatrix}$$

6. Consider the following model of population growth

$$\frac{dP}{dt} = -P^2 + 4P - 3$$

(a) Find all equilibrium points of the system.

(b) Use the phase line diagram and the sign of $\frac{dP}{dt}$ to determine the stability of all equilibrium points.

(c) How large does the initial population have to be in order to avoid extinction?

(d) What is the limiting population size for this model?

7. Consider the following undamped forced oscillator problem.

$$\frac{d^2x}{dt^2} + 9x = 2cos(\omega t)$$

   (a) For what value of forcing frequency ω, do you expect resonance?

   (b) Find the general solution to this equation in the case when there is no resonance.

   (c) Find the general solution when there is resonance.

8. Suppose that a car starts from rest. Its engine provides a constant acceleration of 10 ft/s^2 while air resistance provides a linear drag causing 0.1 ft/s^2 of deceleration for each foot per second of the car's velocity.

   (a) Derive the first order equation for the velocity of the car at time t,

$$\frac{dv}{dt} = 10 - 0.1v$$

   (b) Use equilibrium analysis (phase line diagram) to find out the limiting speed of the car.

   (c) Find out the solution to this initial value problem if $v(0) = v_0$.
   We have learned four ways to solve this equation - linear eqn with constant coefficient, separation, integrating factor, and Laplace transform. Solve it each way.

9. Use Laplace transform to solve the following initial value problems.

   (a) $x'' + 4x = \cos t$, $x(0) = x'(0) = 0$

   (b) $x'' + 4x = \cos 2t$, $x(0) = x'(0) = 0$

   (c) $x'' + 2x' + x = 4te^t$, $x(0) = x'(0) = 0$

10. Use Laplace transform to solve the following non-homogeneous initial value problem

$$\begin{aligned} x' &= x + 2y \\ y' &= x + e^t \end{aligned}$$

where $x(0) = y(0) = 0$.