ALGEBRAIC MONODROMY GROUPS OF $G$-VALUED $l$-ADIC REPRESENTATIONS OF $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

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ABSTRACT. A connected reductive algebraic group $G$ is said to be an $l$-adic algebraic monodromy group for $\Gamma_{\mathbb{Q}}$ if there is a continuous homomorphism
$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G(\overline{\mathbb{Q}}_l)$$
with Zariski-dense image. In this paper, we give a classification of connected $l$-adic algebraic monodromy groups for $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, in particular producing the first such examples for $\text{SL}_n$, $\text{Sp}_{2n}$, $\text{Spin}_n$ and $E_6^{sc}$. To do this, we start with a mod-$l$ representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ coming from the Weyl group of $G$ and use a variation of Stefan Patrikis’ generalization of a method of Ravi Ramakrishna to produce its desired $l$-adic lifts.

1. Introduction

For a split connected reductive group $G$ and a prime number $l$, it is natural to study two types of continuous representations of $\Gamma_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$: the mod-$l$ representations
$$\bar{\rho} : \Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{F}}_l)$$
and the $l$-adic representations
$$\rho : \Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$$
where we use the $l$-adic topology for $G(\overline{\mathbb{Q}}_l)$. Mod-$l$ representations of $\Gamma_{\mathbb{Q}}$ are closely related to the inverse Galois problem for finite groups of Lie type, which asks for the existence of surjective homomorphisms $\bar{\rho} : \Gamma_{\mathbb{Q}} \to G(k)$ for $k$ a finite extension of $\mathbb{F}_l$. It is still wide open, even for small groups such as $\text{SL}_2$. This is difficult since we stick with $\mathbb{Q}$ as the base field. If we replace $\Gamma_{\mathbb{Q}}$ by $\Gamma_F$ for some number field $F$, it is not hard to show that every finite group is a Galois group over some number field, but if we insist on $\Gamma_{\mathbb{Q}}$ then the problem becomes very difficult. On the other hand, we can ask for its analogues in the $l$-adic world:

**Question 1**: Are there continuous homomorphisms $\rho : \Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$ with Zariski-dense images?

We also ask a refined question which takes geometric Galois representations (in the sense of [FM93]) into account:

**Question 2**: Are there continuous geometric Galois representations $\rho : \Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$ with Zariski-dense images?

This paper gives an almost complete answer to Question 1. Before explaining our result, let us make some remarks on Question 2. We shall call a reductive group $G$ an $l$-adic algebraic monodromy group, or simply an $l$-adic monodromy group for $\Gamma_{\mathbb{Q}}$ if the homomorphisms in Question 1 exist, and a geometric $l$-adic monodromy group for $\Gamma_{\mathbb{Q}}$ if the homomorphisms in Question 2 exist. Many classical groups are known as geometric
l-adic monodromy groups for \( \Gamma_Q \): for example, the Tate module of a suitable \( n \)-dimensional abelian variety \( A \) over \( \mathbb{Q} \) realizes \( \text{GSp}_{2n} \) as a geometric l-adic monodromy group for \( \Gamma_Q \). But we expect it to be impossible to realize some algebraic groups as geometric l-adic monodromy groups for \( \Gamma_Q \).

**Example 1.1.** (A heuristic argument) There is no homomorphism \( \rho : \Gamma_Q \to \text{SL}_2(\overline{\mathbb{Q}}_l) \) that is unramified almost everywhere, potentially semi-stable at \( l \), and has Zariski-dense image. In fact, assuming the Fontaine-Mazur and the Langlands conjectures (see [7] and [3]), if such \( \rho \) exists, then \( \rho = \rho_{\pi} \) for some cuspidal automorphic representation \( \pi \) on \( \text{GL}_2(\mathbb{A}_Q) \). But \( \rho \) is even, i.e., \( \det(\rho)(c) = 1 \), so \( \pi_{\infty} \) (the archimedean component of \( \pi \)) is a principal series representation, and \( \pi \) is associated to a Maass form. Therefore by the Fontaine-Mazur conjecture, \( \rho_{\pi} \) has finite image, a contradiction. \( \square \)

In contrast, 1.3 shows in particular that \( \text{SL}_2 \) is an l-adic monodromy group for \( \Gamma_Q \). On the other hand, \( \text{SL}_2 \) can be a geometric l-adic monodromy group for \( \Gamma_F \) for some finite extension \( F/\mathbb{Q} \):

**Example 1.2.** Let \( f \) be a non-CM new eigenform of weight 3, level \( N \), with a nontrivial nebentypus character \( \varepsilon \). Such \( f \) exist for suitable \( N \), see [9]. We write \( E \) for the field of coefficients of \( f \). Then for all \( l \) and \( \lambda | l \), there is a continuous representation \( r_{f,\lambda} : \Gamma_Q \to \text{GL}_2(E_{\lambda}) \) which is unramified outside \( \{ v : v | Nl \} \) and \( \text{Tr}(r_{f,\lambda}(\text{Fr}_p)) = a_p \) for \( p \) not dividing \( Nl \), with \( a_p \) the \( p \)-th Hecke eigenvalue of \( f \). We have \( \det(r_{f,\lambda}) = \kappa^2 \varepsilon \) where \( \kappa \) is the l-adic cyclotomic character. By a theorem of Ribet [14], for almost all \( l \), \( \bar{r}_{f,\lambda}(\Gamma_Q) \) contains \( \text{SL}_2(k) \) for a subfield \( k \) of \( k_\lambda \) (the residue field of \( E_\lambda \)). It follows that \( r_{f,\lambda} \) has Zariski dense image. Let \( F \) be a finite extension of \( \mathbb{Q} \) that trivializes \( \varepsilon \), then the image of \( r' := \kappa^{-1} \cdot r_{f,\lambda}|_{\Gamma_F} \) lands in \( \text{SL}_2(E_\lambda) \) and is Zariski-dense.

Most of the exceptional algebraic groups are known as geometric l-adic monodromy groups, see [21], [12], and [2]. For example, in [12], Patrikis constructs geometric Galois representations (in the sense of [7]) for \( \Gamma_Q \) with full algebraic monodromy groups for essentially all exceptional groups of adjoint types. Patrikis has obtained an extension to general reductive groups of Ravi Ramakrishna’s techniques for lifting odd two-dimensional Galois representations to geometric l-adic representations. In [2], a better answer is given for type \( E_6 \): they show that \( E_6 \) (to be precise, the \( L \)-group of an outer form of \( E_6 \)) is the algebraic monodromy group of a Galois representation appearing in the cohomology of a Shimura variety over \( \mathbb{Q} \).

As 1.1 indicates, the question of determining whether a simple algebraic group \( G \) is a geometric l-adic monodromy group for \( \Gamma_Q \) is sensitive to the form \( G \) takes: \( \text{PGL}_2 \) comes from geometry (Tate module of elliptic curves), whereas its simply-connected form \( \text{SL}_2 \) should not. Thus, it is a interesting (and seemingly very difficult) question to determine what simple algebraic groups (adjoint, simply-connected, or others) are geometric l-adic monodromy groups for \( \Gamma_Q \). Moreover, for those that are not, we ask if they are l-adic monodromy groups at least. We have the following:

**Theorem 1.3.** For a simple algebraic group \( G \) and for a prime number \( l \), there are infinitely many non-conjugate continuous homomorphisms

\[
\rho_l : \Gamma_Q \to G(\overline{\mathbb{Q}}_l)
\]
with Zariski-dense images, where we take $l$ to be large enough for all $G$ not equal to $B_n^{sc}, C_n^{sc}, E_7^{sc}$, $l \equiv 1(4)$ for $G = B_n^{sc}, C_n^{sc},$ and $l \equiv 1(3)$ for $G = E_7^{sc}$.

Remark 1.4. Patrikis has shown that $E_7^{ad}, E_8, F_4, G_2$ and the $L$-group of an outer form of $E_6^{ad}$ are geometric $l$-adic monodromy groups for $\Gamma_\mathbb{Q}$. The congruence conditions on $l$ for $G = B_n^{sc}, C_n^{sc}, E_7^{sc}$ can be removed using the methods in an upcoming paper by Fakhruddin, Khare and Patrikis ([8]). I believe Theorem 1.3 contains the first sighting of $SL_n, Sp_{2n}, Spin_n, E_6^{ad}, E_6^{sc}, E_7^{sc}$ as any sort of arithmetic monodromy groups for $\Gamma_\mathbb{Q}$.

1.3 and 1.4 imply the following theorem (see Section 5):

**Theorem 1.5.** For a connected reductive algebraic group $G$, there are continuous homomorphisms

$$\rho_l: \Gamma_\mathbb{Q} \to G(\overline{\mathbb{Q}}_l)$$

with Zariski-dense image for large enough primes $l$ if and only if the dimension of $Z(G)$ is at most one.

In the rest of this introduction, we will sketch the strategy of Theorem 1.3, which makes use of Patrikis’ generalization of Ramakrishna’s techniques but is very different from his arguments in many ways. For the rest of this section, we will assume that $G$ is a simple algebraic group split over $\mathbb{Z}_l$ with a split maximal torus $T$. Let $\Phi = \Phi(G, T)$ be the associated root system. Let $\mathcal{O}$ be the ring of integers of an extension of $\mathbb{Q}_l$ whose reduction modulo its maximal ideal is isomorphic to $k$, a finite extension of $\mathbb{F}_l$. We start with a well-chosen mod $l$ representation and then use a variant of Ramakrishna’s method to deform it to $\mathcal{O}$ with big image. Achieving this is a balancing act between two difficulties: the Inverse Galois Problem for $G(k)$ is difficult, so we want the residual image to be relatively ‘small’. On the other hand, Ramakrishna’s method works when the residual image is ‘big’.

1. Building the residual representations: Weyl groups. Before explaining our construction, let us first recall a different construction used in [12]. Patrikis uses the principal $GL_2$ homomorphism to construct the residual representation

$$\bar{\rho}: \Gamma_\mathbb{Q} \xrightarrow{\tilde{\varphi}} GL_2(k) \xrightarrow{\tilde{\varphi}} G(k)$$

for $\tilde{\varphi}$ a surjective homomorphism constructed from modular forms and $\varphi$ a principal $GL_2$ homomorphism (for the definition, see [15] and [12], Section 7.1). But the principal $GL_2$ is defined only when $\rho^\vee$ (the half-sum of coroots) is in the cocharacter lattice $X_*(T)$, which is not the case for $G = SL_{2n}, Sp_{2n}, E_7^{sc}$, etc. Nonetheless, the principal $SL_2$ is always defined; but it is not known whether $SL_2(k)$ is a Galois group over $\mathbb{Q}$ and the surjectivity of $\tilde{\varphi}$ is crucial in applying Ramakrishna’s method.

For this reason, we use a different construction which is similar to the construction of residual representations in [2]. By abuse of notation, we still use $G, T$ to denote $G(k), T(k)$, respectively. We consider the following exact sequence finite groups, to which we shall refer as the N-T sequence:

$$1 \to T \to N_G(T) \xrightarrow{\overline{r}} W \to 1$$

where $W = N_G(T)/T$ is the Weyl group of $G$. We want to take $N = N_G(T)$ as the image of the residual representation $\bar{\rho}$. It turns out that the adjoint action of $N$ on the Lie algebra
\( \mathfrak{g} \) over \( k \) decomposes into at most three irreducible pieces (2.2), which is a very pleasant property for making our variant of Ramakrishna’s technique work. It has been known for a long time that \( W \) is a Galois group over \( \mathbb{Q} \), but what we need is realizing \( N \) as a Galois group over \( \mathbb{Q} \). A natural approach would be solving the embedding problem posed by the N-T sequence, i.e., to suppose there is a Galois extension \( K'/Q \) realizing \( W \), and then to find an finite Galois extension \( K'/Q \) containing \( K \) such that the natural surjective homomorphism \( \text{Gal}(K'/Q) \to \text{Gal}(K/Q) \) realizes \( \pi : N \to W \).

This embedding problem is solvable when the sequence splits, by an elementary case of a famous theorem of Igor Shafarevich, see [16], Claim 2.2.5. In [1], the splitting/non-spliitng the N-T sequence is determined completely: for instance, it does not split for \( G = \text{SL}_n, \text{Sp}_{2n}, \text{Spin}_n, E_7 \). We find our way out by replacing \( N \) with a suitable subgroup \( N' \) for which the decomposition of the adjoint representation remains the same, then realizing \( N' \) as a Galois group over \( \mathbb{Q} \) with certain properties, see Section 2.1.3-2.1.6. Finally, we define our residual representation \( \tilde{\rho} \) to be the composite

\[
\Gamma_Q \to N' \to G = G(k)
\]

where the first arrow comes from the realization of \( N' \) as a Galois group over \( \mathbb{Q} \) and the second arrow is the inclusion map. We write \( \tilde{\rho}(\mathfrak{g}) \) for the Lie algebra \( \mathfrak{g}/k \) equipped with a \( \Gamma_Q \)-action induced by the homomorphism

\[
\Gamma_Q \xrightarrow{\tilde{\rho}} G \xrightarrow{\text{Ad}} \text{GL}(\mathfrak{g})
\]

2. A variant of Ramakrishna’s method. For a residual representation

\[
\tilde{\rho} : \Gamma_Q \to G(k)
\]

unramified outside a finite set of places \( S \) containing the archimedean place and a global deformation condition for \( \tilde{\rho} \) (which consists of a local deformation condition for each \( v \in S \)), a typical question in Galois deformation theory is to try to find continuous \( l \)-adic lifts

\[
\rho : \Gamma_Q \to G(\mathcal{O})
\]

of \( \tilde{\rho} \) with \( \mathcal{O} \) the ring of integers of a finite extension of \( \mathbb{Q}_l \) whose residue field is \( k \), such that for all \( v, \rho|_{\Gamma_{Q_v}} \) (we fix an embedding \( \mathbb{Q} \to \mathbb{Q}_v \)) satisfies the prescribed local deformation condition at \( v \). Ravi Ramakrishna has an ingenious method [13] for obtaining the desired lifts which has been generalized and axiomatized in [18], [6], [12], etc. By the Poitou-Tate exact sequence, if the dual Selmer group

\[
H^1_{\mathcal{L}^\perp}(\Gamma_{Q,S}, \tilde{\rho}(\mathfrak{g})(1))
\]

associated to the global deformation condition for \( \tilde{\rho} \) vanishes, then such lifts exist. Here \( \mathcal{L} \) (resp. \( \mathcal{L}^\perp \)) is the Selmer system (resp. dual Selmer system) of tangent spaces (resp. annihilators of the tangent spaces under the local duality) of the given local deformation conditions. Ramakrishna discovered that if one imposes finitely many additional local deformation conditions of ‘Ramakrishna type’ in place of the unramified ones at a finite set of well-chosen places of \( \mathbb{Q} \) disjoint from \( S \), then the new dual Selmer group will vanish. However, this technique is very sensitive to the image of \( \tilde{\rho} \), which has to be ‘big’ to make things work: if \( \tilde{\rho}(\mathfrak{g}) \) is irreducible then all is good, but finding such a \( \tilde{\rho} \) can be very difficult. In practice, we would prefer those \( \tilde{\rho} \) for which \( \tilde{\rho}(\mathfrak{g}) \) does not decompose too much. Unfortunately, the form of Ramakrishna’s method in [12] (see 3.4) does not work for our \( \tilde{\rho} \).
Inspired by the use of Ramakrishna’s method in [6], we hurdle this by making two observations. For our \( \tilde{\rho}, \tilde{\rho}(g) \) decomposes as \( \tilde{\rho}(t) \) (the Lie algebra of \( T \) over \( k \) equipped with an irreducible action of \( \tilde{\rho}(\Gamma) \)) and a complement (see 2.2). Our first observation is that if

\[
H^1_L(\Gamma_{Q,S}, \tilde{\rho}(t))
\]

(see 3.6 for the definition) vanishes, then we can kill the full dual Selmer group using Ramakrishna’s method; moreover, we cannot find an auxiliary prime \( w \notin S \) with a Ramakrishna deformation \( L_w^{\text{Ram}} \) (see 3.1) such that

\[
h^1_L(\Gamma_{Q,S}, \tilde{\rho}(t)(1)) \leq h^1_L(\Gamma_{Q,S}, \tilde{\rho}(g)(1)) \]

where \( L_w \) is the intersection of \( L_w^{\text{Ram}} \) and the unramified condition at \( w \). It turns out that (see the proof of 3.11) the right side of the inequality equals \( h^1_L(\Gamma_{Q,S}, \tilde{\rho}(t)(1)) \); thus a double invocation of Wiles’ formula gives

\[
h^1_L(\Gamma_{Q,S}, \tilde{\rho}(t)) < h^1_L(\Gamma_{Q,S}, \tilde{\rho}(t)).
\]

By induction, we can enlarge \( \mathcal{L} \) finitely many times to make \( H^1_L(\Gamma_{Q,S}, \tilde{\rho}(t)) \) vanish, which in turn allows us (see the proof of 3.14) to enlarge \( \mathcal{L} \) even further to make \( H^1_L(\Gamma_{Q,S}, \tilde{\rho}(g)(1)) \) vanish, as remarked in the first observation. Thus we obtain an \( l \)-adic lift \( \rho : \Gamma \to G(\mathcal{O}) \) satisfying the prescribed local deformation conditions. By specifying carefully the local deformation conditions, we make \( \rho \) have Zariski-dense image in \( G(\mathcal{O}_l) \).

In the end, let us remark that the process described in the previous paragraph is very delicate, depending heavily on the algebraic structure of the summand of \( \tilde{\rho}(g) \) we want to annihilate. Fortunately, it works well for \( \tilde{\rho}(t) \).

Remark 1.6. An intricate generalization of Ramakrishna’s techniques is developed in the upcoming paper [8]. Their method applies to the case when \( \tilde{\rho}(g) \) is semi-simple, multiplicity-free, and each summand is absolutely irreducible. In particular, we can annihilate the above dual Selmer group using their method. But it is much more complicated than ours. We will only use it to remove the congruence conditions on \( l \) in 1.3. A simpler method for removing these conditions might exist, but the author does not know at this point.

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Notation: For a field \( F \) (typically \( \mathbb{Q} \) or \( \mathbb{Q}_p \)), we let \( \Gamma_F \) denote \( \text{Gal}(\overline{F}/F) \) for some fixed choice of algebraic closure of \( \overline{F} \) of \( F \). When \( F \) is a number field, for each place \( v \) of \( F \) we fix once and for all embeddings \( \overline{F} \to \overline{F}_v \), giving rise to inclusions \( \Gamma_{F_v} \to \Gamma_F \). If \( S \) is a finite set of places of \( F \) we let \( \Gamma_{F,S} \) denote \( \text{Gal}(F_S/F) \), where \( F_S \) is the maximal extension of \( F \) in \( \overline{F} \) unramified outside of \( S \). If \( v \) is a place of \( F \) outside \( S \), we write \( \text{Fr}_v \) for the corresponding arithmetic frobenius element in \( \Gamma_{F,S} \). When \( F = \mathbb{Q} \), we will sometimes write \( \Gamma_v \) for \( \Gamma_{F_v} \) and \( \Gamma_S \) for \( \Gamma_{Q,S} \). For a representation \( \rho \) of \( \Gamma_F \), we let \( F(\rho) \) denote the fixed field of \( \text{Ker}(\rho) \).
Consider a group $\Gamma$, a ring $A$, an algebraic group $G$ over $\text{Spec}(A)$, and a homomorphism $\rho : \Gamma \to G(A)$. We write $\mathfrak{g}$ for both the Lie algebra of $G$ and the $A[G]$-module induced by the adjoint action. We let $\rho(\mathfrak{g})$ denote the $A[\Gamma]$-module with underlying $A$-module $\mathfrak{g}$ induced by $\rho$. Similarly, for a $A[G]$-submodule $M$ of $\mathfrak{g}$, we write $\rho(M)$ for the $A[\Gamma]$-module with underlying $A$-module $M$ induced by $\rho$.

We call an algebraic group simple if it is connected, nonabelian and has no proper normal algebraic subgroups except for finite subgroups. It is sometimes called an almost simple group in the literature. Consider a simple algebraic group $G$, we write $G^{\text{sc}}$ (resp. $G^{\text{ad}}$) for the simply-connected form (resp. adjoint form) of $G$.

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_l$. We let $\text{CNL}_\mathcal{O}$ denote the category of complete noetherian local $\mathcal{O}$-algebras for which the structure map $\mathcal{O} \to R$ induces an isomorphism on residue fields.

All the Galois cohomology groups we consider will be $k$-vector spaces for $k$ a finite extension of $\mathbb{F}_l$. We abbreviate $\dim H^n(-)$ by $h^n(-)$.

We write $\kappa$ for the $l$-adic cyclotomic character, and $\bar{\kappa}$ for its mod $l$ reduction.

2. CONSTRUCTIONS OF RESIDUAL REPRESENTATIONS

In this section, we construct residual representations

$$\bar{\rho} : \Gamma_Q \to G(\overline{\mathbb{F}_l})$$

for $G$ a simple, simply-connected algebraic group.

2.1. Constructions based on the Weyl groups. I would like to thank Stefan Patrikis for suggesting me to take variants of Weyl groups as the residual images, which turns out to work remarkably well.

2.1.1. Some group-theoretic results. We recall a property of the Weyl group of an irreducible root system $\Phi$:

**Lemma 2.1.** $W$ acts irreducibly on the $\mathbb{C}$-vector space spanned by $\Phi$ and transitively on roots of the same length.

Let $k$ be a finite extension of $\mathbb{F}_l$. Let $G = G(k)$ and $T = T(k)$ be a maximal split torus of $G$. Let $\Phi = \Phi(G, T)$ and $N = N_G(T)$.

**Corollary 2.2.** For any $\alpha, \beta \in \Phi$ of the same length, there exists $w \in N$ such that $\text{Ad}(w)\mathfrak{g}_\alpha = \mathfrak{g}_\beta$. The adjoint action $\text{Ad}(N)$ on $\mathfrak{g}$ decomposes into submodules $\mathfrak{t}$ and

$$\mathfrak{g}_{\Phi} := \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

when $\Phi$ is simply-laced; and is the direct sum of $\mathfrak{t}$ and

$$\mathfrak{g}_l := \sum_{\alpha \in \Phi, \alpha \text{ is long}} \mathfrak{g}_\alpha$$
and
\[ g_s := \sum_{\alpha \in \Phi, \alpha \text{ is short}} g_\alpha \]
otherwise.

Moreover, \( t \) is irreducible, and \( g_\Phi, g_l, g_s \) are irreducible if \( l \) is sufficiently large.

**Proof.** It suffices to show that \( g_\Phi, g_l, g_s \) are irreducible. We will only show that \( g_\Phi \) is irreducible, for the other two cases are similar. Take a nonzero vector \( X \in g_\Phi \), write \( X = \sum_{1 \leq i \leq k} X_i \) where \( 0 \neq X_i \in g_{\alpha_i} \) for some distinct roots \( \alpha_1, \alpha_2, \ldots, \alpha_k \in \Phi \). Since \( l \) is sufficiently large, we can choose \( t \in T \) such that \( \alpha(t), \alpha \in \Phi \) are all distinct. We have
\[ ad(t^j)X = \sum_i \alpha_i(t)^j X_i \]
where \( 0 \leq j \leq k - 1 \). It follows that \( X, ad(t)X, ad(t^2)X, \ldots, ad(t^{k-1})X \) are linearly independent and hence span the same subspace as \( X_1, \ldots, X_k \) do. In particular, \( X_i \in g_{\alpha_i} \) belongs to the \( N \)-submodule of \( g_\Phi \) generated by \( X \). But \( N \) acts transitively on the set of root spaces, it follows that \( X \) generates \( g_\Phi \). Therefore, \( g_\Phi \) is irreducible. \( \Box \)

**Remark 2.3.** The above discussion can be carried out verbatim for a subgroup \( N' \) of \( N \) that maps onto a subgroup \( W' \) of \( W \) acting transitively on roots of the same length.

### 2.1.2. Some basic results in the inverse Galois theory

We list some elementary results with proofs in the inverse Galois theory, some of which are modified in order to suit our purposes. See Serre’s beautiful lecture notes [16] for details.

The first result dates back to David Hilbert:

**Theorem 2.4.** For \( n \geq 2 \) (resp. \( n \geq 3 \)), there are infinitely many polynomials with rational coefficients \( f(T) \) which realize the symmetric group of \( n \) letters \( S_n \) (resp. the alternating group of \( n \) letters \( A_n \)) as Galois groups over \( \mathbb{Q} \). Moreover, for \( S_n \) (resp. for \( A_n \) with \( n \geq 4 \)), \( f \) can be chosen to have at least a pair of non-real roots.

The next result is an elementary case of a theorem due to Igor Shafarevich [11], which will be used frequently in our constructions:

**Theorem 2.5.** Let \( G \) be a finite group. Suppose that there is a finite Galois extension \( K/\mathbb{Q} \) such that \( \text{Gal}(K/\mathbb{Q}) \cong G \). Let \( H \) be a finite abelian group with exponent \( m \). Suppose that there is a **split** exact sequence of finite groups
\[ 1 \rightarrow H \rightarrow S \rightarrow G \rightarrow 1 \]
Then there is a finite Galois extension \( M/\mathbb{Q} \) containing \( K \) such that the natural surjective homomorphism \( \text{Gal}(M/\mathbb{Q}) \rightarrow \text{Gal}(K/\mathbb{Q}) \) realizes the surjective homomorphism \( S \rightarrow G \). In other words, the split embedding problem has a proper solution.

Moreover, for any prime \( l \) that is outside the ramification locus of \( K/\mathbb{Q} \) and prime to \( m \), we can choose \( M \) so that \( l \) is unramified in \( M \).
Proof. The argument is a minor modification of the proof of [16], Claim 2.2.5. Put \( L = K(\mu_m) \). \( H \) can be regarded as a \( m \)-torsion module on which \( \text{Gal}(L/\mathbb{Q}) \) acts. So there is a finite free \((\mathbb{Z}/m\mathbb{Z})[\text{Gal}(L/\mathbb{Q})]\)-module \( F \) of which \( H \) is a quotient. Suppose that \( r \) is the number of copies of \((\mathbb{Z}/m\mathbb{Z})[\text{Gal}(L/\mathbb{Q})]\) in \( F \). Let \( S' \) be the semi-direct product of \( \text{Gal}(L/\mathbb{Q}) \) and \( F \). To solve the embedding problem posed by \( 1 \to H \to S \to G \to 1 \), it suffices to solve the embedding problem posed by

\[
1 \to F \to S' \to \text{Gal}(L/\mathbb{Q}) \to 1
\]

Claim: There is a Galois extension \( M'/\mathbb{Q} \) that solves the above embedding problem. Moreover, for any prime \( l \) that does not divide \( m \) and is outside of the ramification locus of \( K/\mathbb{Q} \), we can choose \( M' \) so that \( l \) is unramified in \( M' \).

To see this, we choose places \( v_1, \ldots, v_r \) of \( \mathbb{Q} \) away from \( l \) such that \( v_i \) splits completely in \( L \). Let \( w_i \) be a place of \( L \) extending \( v_i \), \( 1 \leq i \leq r \). Any place of \( L \) extending \( v_i \) can be written uniquely as \( \sigma w_i \) for some \( \sigma \in \text{Gal}(L/\mathbb{Q}) \). Let \( w_0 \) be a place of \( L \) extending \( l \). For \( 1 \leq i \leq r \), choose \( \theta_i \in \mathcal{O}_L \) such that for \( 0 \leq j \leq r \), \((\sigma w_j)(\theta_i) = 1 \) if \( \sigma = 1 \) and \( i = j \) and 0 if not. The existence of \( \theta_i \) follows from the Weak Approximation Theorem. Let

\[
M' = L(\sqrt[r]{\sigma \theta_i} | \sigma \in \text{Gal}(L/\mathbb{Q}), 1 \leq i \leq r)
\]

\( M' \) is Galois over \( \mathbb{Q} \), being the composite of \( L \) and the splitting field of the polynomial \( \prod \sigma(T^m - \sigma \theta_i) \in \mathbb{Q}[T] \). It is easy to see that \( \text{Gal}(M'/L) \cong F \) as \( \text{Gal}(L/\mathbb{Q}) \)-modules. In fact, fix \( i \), there is an isomorphism

\[
\phi_i : \text{Gal}(L(\sqrt[r]{\sigma \theta_i} | \sigma \in \text{Gal}(L/\mathbb{Q}))/L) \cong (\mathbb{Z}/m\mathbb{Z})[\text{Gal}(L/\mathbb{Q})]
\]

For an element \( g \) on the left side and any \( \sigma \in \text{Gal}(L/\mathbb{Q}) \), \( g(\sqrt[r]{\sigma \theta_i}) = \zeta_\sigma \cdot \sqrt[r]{\sigma \theta_i} \) for some \( \zeta_\sigma \in \mu_m \cong \mathbb{Z}/m\mathbb{Z} \). We then define

\[
\phi_i(g) = \sum_{\sigma} \zeta_\sigma \cdot \sigma \in (\mathbb{Z}/m\mathbb{Z})[\text{Gal}(L/\mathbb{Q})]
\]

It is clear that \( \phi_i \) is an isomorphism by our choice of \( \theta_i \). It follows that \( \text{Gal}(M'/L) \cong F \) by linear disjointness.

Therefore we obtain an exact sequence

\[
1 \to F \to \text{Gal}(M'/\mathbb{Q}) \to \text{Gal}(L/\mathbb{Q}) \to 1
\]

Since \( F \cong \text{Ind}_{\{1\}}^{\text{Gal}(L/\mathbb{Q})} \mathbb{Z}/m\mathbb{Z} \), by Shapiro’s lemma, \( H^2(\text{Gal}(L/\mathbb{Q}), F) = H^2(\{1\}, \mathbb{Z}/m\mathbb{Z}) = 0 \) and hence the sequence splits. Thus, \( \text{Gal}(M'/\mathbb{Q}) \cong S' \).

It remains to show that \( l \) is unramified in \( M' \). We have \( \forall \sigma \in \text{Gal}(L/\mathbb{Q}), \forall i, w_0 \) is unramified in \( L(\sqrt[r]{\sigma \theta_i}) \), because \( w_0(\sigma \theta_i) = 0 \) and \( l \) does not divide \( m \). So \( w_0 \) is unramified in their composite \( M' \). On the other hand, \( l \) is unramified in \( K \) by assumption and is unramified in \( \mathbb{Q}(\mu_m) \) since \( l \) does not divide \( m \), so \( l \) is unramified in \( L \). It follows that \( l \) is unramified in \( M' \). We have proved the claim.

Finally, let \( M \) be the fixed field of the kernel of the natural surjective homomorphism \( S' \to S \), we obtain a solution to the original embedding problem. \( \square \)
2.1.3. SL\(_n\). We use the notation in 2.1.1. Let \(G = \text{SL}_n(k)\), so \(W \cong S_n\). By [1], the N-T sequence only splits when \(n\) is odd. We consider the subgroup \(W' = A_n\) of \(W\). Let \(T\) be the maximal torus consisting of diagonal elements in \(\text{SL}_n(k)\) and \(\Phi = \Phi(G,T)\).

**Lemma 2.6.** Suppose \(n \geq 4\). Then \(A_n\), as a subgroup of \(W\), acts transitively on \(\Phi\).

**Proof.** This follows from the fact that \(A_n\) acts doubly transitively on \([1,2,\ldots,n]\) if and only if \(n \geq 4\). \(\Box\)

Let \(N' = \pi^{-1}(W')\).

**Lemma 2.7.** The following exact sequence of finite groups splits:

\[
1 \to T \to N' \to W' \to 1
\]

**Proof.** We think of \(A_n\) as a subgroup of \(N = N_G(T)\) by realizing it as the group of \(n \times n\) even permutation matrices. Then \(A_n\) normalizes \(T\) and \(N'\) is a semi-direct product of them. \(\Box\)

Let \(l\) be large enough. Since \(T\) is abelian of exponent \(|k| - 1\) which is prime to \(l\), by 2.5, \(N'\) is a Galois group over \(\mathbb{Q}\) that is unramified at \(l\). In other words, there is a surjection

\[
\Gamma_{\mathbb{Q}} \twoheadrightarrow N'
\]

which is unramified at \(l\). Define \(\bar{\rho}\) to be the composite

\[
\Gamma_{\mathbb{Q}} \twoheadrightarrow N' \xrightarrow{i} \text{SL}_n(k)
\]

which is unramified at \(l\).

**Remark 2.8.** It is trickier to realize \(N\) as a Galois group over \(\mathbb{Q}\). This can be reduced to realizing the Tits group \(\mathcal{T}\) of \(\text{SL}_n\) as a Galois group. \(\mathcal{T}\) can be identified with the group of \(n \times n\) signed permutation matrices with determinant one, which is an index two subgroup of the group of \(n \times n\) signed permutation matrices. The latter is isomorphic to the Weyl group of type \(B_n\), hence a Galois group over \(\mathbb{Q}\). As the Tits group sequence splits if and only if \(n\) is odd, so \(\mathcal{T}\) can be realized over \(\mathbb{Q}\) for \(n\) odd. When \(n\) is even, this problem is open except for small \(n\), as far as I know.

Let \(g = \mathfrak{sl}_n(k)\).

**Proposition 2.9.** For \(l\) sufficiently large and \(n \geq 4\), \(\bar{\rho}(g)\) decomposes into irreducible \(\Gamma_{\mathbb{Q}}\)-modules \(\bar{\rho}(\mathfrak{t})\) and \(\bar{\rho}(\mathfrak{g}_\Phi)\).

**Proof.** This follows from 2.2 and 2.3. \(\Box\)

There remains the case when \(G\) is SL\(_2\) or SL\(_3\). For SL\(_3\), see Section 2.2. For SL\(_2\), see Section 4.2.3.
2.1.4. Sp\textsubscript{2n}. We use the notation in 2.1.1. Let \( G = \text{Sp}\textsubscript{2n}(k) \), then \( W \) is isomorphic to a semi-direct product of \( S_n \) and \( D := (\mathbb{Z}/2\mathbb{Z})^n \), see [4], for example. We fix a maximal split torus \( T \) in \( \text{Sp}\textsubscript{2n}(k) \). The N-T sequence does not split by [1].

**Lemma 2.10.** Consider the N-T sequence for \( \text{Sp}\textsubscript{2n} \). \( S_n \subset W \) has a section to \( N \), \( D \subset W \) does not have a section to \( N \) but there is a subgroup \( \tilde{D} \) of \( N \) such that \( \pi(\tilde{D}) = D \) and \( \tilde{D} \cong (\mathbb{Z}/4\mathbb{Z})^n \). Moreover, as subgroups of \( N \), \( S_n \) normalizes \( \tilde{D} \) and \( S_n \cap \tilde{D} = \{1\} \). We let \( W_1 \) be the subgroup of \( N \) generated by \( S_n \) and \( \tilde{D} \).

Therefore, by 2.5, \( W_1 \) can be realized as a Galois group over \( \mathbb{Q} \). \( N \) is generated by \( W_1 \) and \( T \). \( W_1 \) normalizes \( T \) and \( W_1 \cap T = T_2 \cong (\mathbb{Z}/2\mathbb{Z})^n \). Let \( S \) be the (abstract) semidirect product of \( W_1 \) and \( T \). By 2.5, for \( l \) large enough, \( S \) can be realized as a Galois group over \( \mathbb{Q} \) unramified at \( l \). But there is a natural quotient map \( S \to N \), so \( N \) is a Galois group over \( \mathbb{Q} \) unramified at \( l \) as well. In other words, there is a surjection

\[
\Gamma_{\mathbb{Q}} \twoheadrightarrow N
\]

unramified at \( l \). Define \( \bar{\rho} \) to be the composite

\[
\Gamma_{\mathbb{Q}} \twoheadrightarrow N \xrightarrow{i} \text{Sp}\textsubscript{2n}(k)
\]

which is unramified at \( l \).

**Remark 2.11.** Note that \( W_1 \cong T \) and hence the Tits group for \( \text{Sp}\textsubscript{2n} \) is easily realized as a Galois group over \( \mathbb{Q} \).

Let \( g = \text{sp}\textsubscript{2n}(k) \). The root system \( \Phi \) of \( g \) is not simply laced.

**Proposition 2.12.** For \( l \) sufficiently large, \( \bar{\rho}(g) \) decomposes into irreducible \( \Gamma_{\mathbb{Q}} \)-modules \( \bar{\rho}(t) \), \( \bar{\rho}(u) \) and \( \bar{\rho}(w) \).

**Proof.** This follows from 2.2. \( \square \)

2.1.5. Spin\textsubscript{2n} and Spin\textsubscript{2n+1}. For spin groups, the N-T sequence does not split by [1]. We consider a subgroup of \( W \). For \( G = \text{Spin}\textsubscript{2n} \), \( W \) is isomorphic to a semi-direct product of \( S_n \) and \( D := (\mathbb{Z}/2\mathbb{Z})^{n-1} \) and we take \( W' \) be the subgroup generated by \( A_n \) and \( D \). For \( G = \text{Spin}\textsubscript{2n+1} \), \( W \) is isomorphic to a semi-direct product of \( S_n \) and \( D := (\mathbb{Z}/2\mathbb{Z})^n \) and we take \( W' \) be the subgroup generated by \( A_n \) and \( D \). Similar to the symplectic case, we will show that \( N' = \pi^{-1}(W') \) is a Galois group over \( \mathbb{Q} \).

**Lemma 2.13.** Consider the N-T sequence for \( G = \text{Spin}\textsubscript{2n}(k) \) or \( \text{Spin}\textsubscript{2n+1}(k) \). \( A_n \subset W \) has a section to \( N \), and there is an abelian subgroup \( \tilde{D} \) of \( N \) such that \( \pi(\tilde{D}) = D \). Moreover, as subgroups of \( N \), \( A_n \) normalizes \( \tilde{D} \) and let \( W_1 \) be their product.

Therefore, by 2.5, \( W_1 \) can be realized as a Galois group over \( \mathbb{Q} \). \( N' \) is generated by \( W_1 \) and \( T \) and \( W_1 \) normalizes \( T \). Let \( S \) be the (abstract) semidirect product of \( W_1 \) and \( T \). By 2.5, for \( l \) large enough, \( S \) can be realized as a Galois group over \( \mathbb{Q} \) unramified at \( l \). But there is a natural quotient map \( S \to N' \), so \( N' \) is a Galois group over \( \mathbb{Q} \) unramified at \( l \) as well. In other words, there is a surjection

\[
\Gamma_{\mathbb{Q}} \twoheadrightarrow N'
\]
unramified at \( l \). For \( G = \text{Spin}_{2n} \) or \( \text{Spin}_{2n+1} \), define \( \bar{\rho} \) to be the composite

\[
\Gamma_{\mathbb{Q}} \to N' \xrightarrow{i} G(k)
\]

which is unramified at \( l \).

Let \( \mathfrak{g} \) be the Lie algebra of \( G(k) \). The corresponding root system \( \Phi \) is simply laced if \( G = \text{Spin}_{2n} \) and not if \( G = \text{Spin}_{2n+1} \).

**Remark 2.14.** The above construction can also realize the Tits groups for \( \text{Spin}_{2n} \) and \( \text{Spin}_{2n+1} \) as Galois groups over \( \mathbb{Q} \). For, \( S_n \) has a section to the Tits group for the orthogonal group and its pullback in the spin group is a two-fold central extension of \( S_n \) which is a Galois group over \( \mathbb{Q} \) by a result of Jack Sonn [17]. But we will not need this.

**Proposition 2.15.** For \( l \) sufficiently large and \( n \geq 4 \), \( \bar{\rho}(\mathfrak{g}) \) decomposes into irreducible \( \Gamma_{\mathbb{Q}} \)-modules \( \bar{\rho}(\mathfrak{t}) \) and \( \bar{\rho}(\mathfrak{g}_{\mathfrak{s}}) \) when \( G = \text{Spin}_{2n} \); and it decomposes into irreducible \( \Gamma_{\mathbb{Q}} \)-modules \( \bar{\rho}(\mathfrak{t}), \bar{\rho}(\mathfrak{g}_{\mathfrak{t}}) \) and \( \bar{\rho}(\mathfrak{g}_{\mathfrak{s}}) \) when \( G = \text{Spin}_{2n+1} \).

**Proof.** Note that the action of \( W' \) on \( \Phi \) is transitive if and only if \( n \geq 4 \). Then the proposition follows from 2.2 and 2.3.

**Remark 2.16.** There remains the case when \( G \) is one of \( \text{Spin}_4, \text{Spin}_5, \text{Spin}_6, \text{Spin}_7 \). But \( \text{Spin}_4(\mathbb{C}) \cong \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}), \text{Spin}_6(\mathbb{C}) \cong \text{Sp}_4(\mathbb{C}), \text{Spin}_6(\mathbb{C}) \cong \text{SL}_4(\mathbb{C}) \) which are included in other cases. For \( \text{Spin}_7 \), the half sum of coroots \( \rho^\vee = 3\alpha_1^\vee + 5\alpha_2^\vee + 3\alpha_3^\vee \) has integer coefficients (see [4], for example), so the principal \( \text{GL}_2 \) map is well-defined, see Section 2.2.

2.1.6. \( E_7^\text{sc} \). We refer the reader to [4] for detailed information on the root system of type \( E_7 \). Let \( G = E_7^\text{sc}(k) \). The Weyl group \( W \) is isomorphic to the direct product of \( [W, W] \) and \( \mathbb{Z}/2\mathbb{Z} \). By [1], the N-T sequence does not split. We choose a subgroup \( W' \) of \( W \) which lifts to \( N \) as follows. Consider the extended Dynkin diagram of type \( E_7 \); there is a sub-root system \( \Phi' \) of \( \Phi \) which is of type \( A_7 \). The alternating group \( A_8 \) is a subgroup of \( S_8 \cong W(A_7) \leq W = W(E_7) \).

**Lemma 2.17.** \( A_8 \leq W \) lifts to \( N \).

**Lemma 2.18.** The action of \( A_8 \) on \( \Phi \) has an orbit of size 56 and an orbit of size 70.

**Proof.** We first consider the action of \( S_8 \cong W(A_7) \) on \( \Phi \). By 2.1, \( S_8 \) acts transitively on \( \Phi' \), which has 56 roots. A straightforward calculation shows that for some \( \alpha \in \Phi - \Phi' \), \( S_8 \cdot \alpha \) has exactly 70 roots and the stabilizer of \( \alpha \) in \( S_8 \) is isomorphic to \( S_4 \times S_4 \subset S_8 \). Since \( 56 + 70 = 126 \) is the number of roots in \( \Phi \), the lemma is true for \( S_8 \). Now we consider the alternating group \( A_8 \). 2.6 implies that \( A_8 \) still acts transitively on \( \Phi' \). As \((S_4 \times S_4) \cap A_8 \) (the stablizer of \( \alpha \) in \( A_8 \)) has order \( \frac{1}{2}|S_4 \times S_4| = 288 \), \( A_8 \cdot \alpha \) has exactly \( |A_8|/288 = 70 \) roots.

It is clear that \( A_8 \), considered as a subgroup of \( N \), normalizes \( T \) and \( A_8 \cap T = \{1\} \). Let \( N' \) be the subgroup of \( G = E_7^\text{sc}(k) \) generated by \( A_8 \) and \( T \). By 2.4 and 2.5, for \( l \) large enough, \( N' \) can be realized as a Galois group over \( \mathbb{Q} \) unramified at \( l \). In other words, there is a surjection

\[
\Gamma_{\mathbb{Q}} \twoheadrightarrow N'
\]

that is unramified at \( l \). Define \( \bar{\rho} \) to be the composite

\[
\Gamma_{\mathbb{Q}} \to N' \xrightarrow{i} E_7^\text{sc}(k)
\]
which is unramified at \( l \).

Let \( \mathfrak{g}_a \) (resp. \( \mathfrak{g}_b \)) be the direct sum of the root spaces corresponding to the orbit of size 56 (resp. size 70) in 2.18.

**Proposition 2.19.** For \( l \) sufficiently large, \( \bar{\rho}(\mathfrak{g}) \) decomposes into irreducible \( \Gamma_Q \)-modules \( \bar{\rho}(t) \), \( \bar{\rho}(\mathfrak{g}_a) \) and \( \bar{\rho}(\mathfrak{g}_b) \).

**Proof.** The proof is very similar to the proof of 2.2. \( \square \)

2.2. **Constructions based on the principal GL\(_2\).** We record some facts on the principal SL\(_2\). For more details, see [15] and [12]. Let \( G/k \) be a simple algebraic group with a Borel \( B \) containing a split maximal torus \( T \). Let \( \Phi = \Phi(G,T) \) be the root system of \( G \) with the set of simple roots \( \Delta \) corresponding to \( B \). We fix a pinning \( \{ X_\alpha \}_{\alpha \in \Delta} \) and let

\[
X = \sum_{\alpha \in \Delta} X_\alpha
\]

which can be extended to an \( \mathfrak{sl}_2 \)-triple \( (X,H,Y) \) with

\[
H = \sum_{\alpha > 0} H_\alpha
\]

for \( H_\alpha \) the coroot vector corresponding to \( \alpha \).

A **principal SL\(_2\) homomorphism** is a homomorphism

\[
\varphi : \text{SL}_2 \to G
\]

such that

\[
\varphi\left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(tX)
\]

and

\[
\varphi\left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = \exp(tH) = 2\rho^\vee(t)
\]

with \( \rho^\vee \) the half-sum of coroots. \( \rho^\vee \) is always defined when \( G \) is adjoint. Suppose that \( \rho^\vee : \mathbb{G}_m \to G^{\text{ad}} \) lifts to \( G \) and fix a lift which is again denoted \( \rho^\vee \). A **principal GL\(_2\) homomorphism** is a homomorphism

\[
\varphi : \text{GL}_2 \to G
\]

such that

\[
\varphi\left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(tX)
\]

and

\[
\varphi\left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right) = \rho^\vee(t)
\]

By definition, a principal GL\(_2\) factors through PGL\(_2\).

By examining the list in [4], we get

**Lemma 2.20.** For \( G \) a simple algebraic group, \( \rho^\vee : \mathbb{G}_m \to G^{\text{ad}} \) lifts to \( G^{\text{sc}} \) if and only if \( G \) is one of the following types: \( A_{2n}, B_{4n}, B_{4n+3}, D_{4n}, D_{4n+1}, E_6, E_8, F_4, G_2 \).
The operator $ad(H)$ preserves $g^X$ (the centralizer of $X$ in $g$) and

$$g^X = \sum_{m > 0} V_{2m}$$

where $V_{2m}$ is the eigenspace of $H$ corresponding to the eigenvalue $2m$. The following proposition is due to Kostant.

**Proposition 2.21.** The dimension of $g^X$ is equal to the rank of $g$. $V_{2m}$ is nonzero if and only if $m$ is an exponent of $g$. Letting $GL_2$ act on $g$ via $\varphi$, there is an isomorphism of $GL_2$-representations

$$g \cong \bigoplus_{m > 0} \text{Sym}^{2m}(k^2) \otimes \text{det}^{-m} \otimes V_{2m}$$

In [12], the following crucial lemma is verified using Magma for exceptional Lie algebras.

**Lemma 2.22.** Assume $l$ is large enough for $G$. For $g$ of type $A_n$, $B_n$, $C_n$ and exceptional types, there is a root $\alpha \in \Phi$ such that every irreducible submodule of $g$ in 2.21 has a vector with nonzero $l_\alpha$ component and a vector with nonzero $g_{-\alpha}$ component.

**Remark 2.23.** Note that we have excluded the $D_n$ case in the above lemma since the calculation is very messy for $D_n$ and I do not know a conceptual proof.

Suppose that $\rho^\vee : \mathbb{G}_m \to G^{ad}$ lifts to $G$. We take $f$ to be as in 1.2. By Ribet’s theorem, the projective image of $\tilde{r}_{f,\lambda}$ is either $\text{PGL}_2(k)$ or $\text{PSL}_2(k)$ for a subfield $k$ of $k_\lambda$. Define

$$\bar{\rho} : \Gamma_{Q} \to \text{GL}_2(k) \xrightarrow{\varphi} G(k)$$

**Remark 2.24.** This construction works for all exceptional groups but $E_7^{sc}$ for which $\rho^\vee : \mathbb{G}_m \to E_7^{ad}$ does not lift to $E_7^{sc}$. We will only use this construction for $G$ of type $E_6$, $A_2$ and $B_3$.

### 3. Annihilating the Dual Selmer Group: Ramakrishna’s Method and Its Variant

Given $\bar{\rho} : \Gamma_{Q} \to G(k)$ defined in the previous section, we want to obtain its $l$-adic lift $\rho : \Gamma_{Q} \to G(O)$ with $O$ the ring of integers of a finite extension of $Q_l$ whose residue field is $k$ satisfying a given global deformation condition. Just as in [12], we use Ramakrishna’s method to annihilate the associated dual Selmer group. The new feature is a double use of Patrikis’ extension of Ramakrishna’s method when the original form fails to work, see Section 3.2.

#### 3.1. Review on Ramakrishna’s Method

**3.1.1. Ramakrishna deformations.** In this section, we review Patrikis’ extension of Ravi Ramakrishna’s techniques for lifting odd two-dimensional Galois representations to geometric $l$-adic representations. We will only list the key points and results here. For proofs, see [12], Section 4.2. For a review on deformation theory of Galois representations, see [12], Section 3.
We begin by defining a type of local deformation condition called Ramakrishna condition which will be used at the auxiliary primes of ramification in Ramakrishna’s global argument. Let $F$ be a finite extension of $\mathbb{Q}_p$ for $p \neq l$, and let $\bar{\rho} : \Gamma_F \rightarrow G(k)$ be an unramified homomorphism such that $\bar{\rho}(\text{Fr}_F)$ is a regular semi-simple element. Let $T$ be the connected component of the centralizer of $\bar{\rho}(\text{Fr}_F)$; this is a maximal $k$-torus of $G$, but we can lift it to an $O$-torus uniquely up to isomorphism, which we also denote by $T$, and then we can lift the embedding over $k$ to an embedding over $O$ which is unique up to $G(O)$-conjugation. By passing to an etale extension of $O$, we may assume that $T$ is split.

**Definition 3.1.** ([12], Definition 4.9) Let $\bar{\rho}, T$ be as above. For $\alpha \in \Phi(G, T)$, $\bar{\rho}$ is said to be of Ramakrishna type $\alpha$ if

$$\alpha(\bar{\rho}(\text{Fr}_F)) = \bar{\kappa}(\text{Fr}_F)$$

Let $H_\alpha = T \cdot U_\alpha$ be the subgroup generated by $T$ and the root subgroup $U_\alpha$ corresponding to $\alpha$. Ramakrishna deformation is a functor

$$\text{Lift}^{\text{Ram}}_{\bar{\rho}} : \text{CNL}_O \rightarrow \text{Sets}$$

such that for a complete local noetherian $O$-algebra $R$, $\text{Lift}^{\text{Ram}}_{\bar{\rho}}(R)$ consists of all lifts $\rho : \Gamma_F \rightarrow G(R)$ of $\bar{\rho}$ such that $\rho$ is $\hat{G}(R)$-conjugate to a homomorphism $\Gamma_F \xrightarrow{\rho'} H_\alpha(R)$ with the resulting composite

$$\Gamma_F \xrightarrow{\rho'} H_\alpha(R) \xrightarrow{\text{Ad}} \text{GL}(g_\alpha \otimes R) = R^\times$$

equal to $\kappa$.

We shall call such a $\rho$ to be of Ramakrishna type $\alpha$ as well.

We denote by $\text{Def}^{\text{Ram}}_{\bar{\rho}}$ the corresponding deformation functor.

**Lemma 3.2.** ([12], Lemma 4.10) $\text{Lift}^{\text{Ram}}_{\bar{\rho}}$ is well-defined and smooth.

Consider the sub-torus $T_\alpha = \text{Ker}(\alpha)^0$ of $T$, and denote by $t_\alpha$ its Lie algebra. There is a canonical decomposition $t_\alpha \oplus t_\alpha = t$ with $t_\alpha$ the one-dimensional torus generated by the coroot $\alpha^\vee$.

The next lemma is crucial to the global argument.

**Lemma 3.3.** ([12], Lemma 4.11) Assume $\bar{\rho}$ is of Ramakrishna type $\alpha$. Let $W = t_\alpha \oplus g_\alpha$, $W^\perp$ the annihilator of $W$ under the Killing form on $g$. Let $L^{\text{Ram}}_{\bar{\rho}}$ (resp. $L^{\text{Ram,\perp}}_{\bar{\rho}}$) be the tangent space of $\text{Def}^{\text{Ram}}_{\bar{\rho}}$ (resp. the annihilator of $L^{\text{Ram}}_{\bar{\rho}}$ under the local duality pairing). Then

1. $L^{\text{Ram}}_{\bar{\rho}} \cong H^1(\Gamma_F, \bar{\rho}(W))$.
2. $\dim L^{\text{Ram}}_{\bar{\rho}} = h^0(\Gamma_F, \bar{\rho}(g))$.
3. $L^{\text{Ram,\perp}}_{\bar{\rho}} \cong H^1(\Gamma_F, \bar{\rho}(W^\perp)(1))$. 

3.1.2. The global argument. In this section, we assume $G$ is semi-simple. Let $\overline{\rho} : \Gamma_{Q,S} \rightarrow G(k)$ be a continuous homomorphism for which $h^0(\Gamma_{Q}, \overline{\rho}(g)) = h^0(\Gamma_{Q}, \overline{\rho}(g)(1)) = 0$. In particular, the deformation functor is representable.

Proposition 3.4. ([12], Proposition 5.2) Suppose that there is a global deformation condition $L = \{L_v\}_{v \in S}$ consisting of smooth local deformation conditions for each place $v \in S$. Let $K = \mathbb{Q}(\overline{\rho}(g), \mu_l)$. We assume the following:

(1) $\sum_{v \in S} (\dim L_v) \geq \sum_{v \in S} h^0(\Gamma_{Q_v}, \overline{\rho}(g))$

(2) $H^1(\text{Gal}(K/\mathbb{Q}), \overline{\rho}(g))$ and $H^1(\text{Gal}(K/\mathbb{Q}), \overline{\rho}(g)(1))$ vanish.

(3) Assume Item 2 holds. For any pair of non-zero Selmer classes $\phi \in H^1_{L \perp}(\Gamma_{Q,S}, \overline{\rho}(g)(1))$ and $\psi \in H^1_{L}(\Gamma_{Q,S}, \overline{\rho}(g))$, we can restrict them to $\Gamma_K$ where they become homomorphisms, which are non-zero by Item 2. Letting $K_{\phi}/K$ and $K_{\psi}/K$ be their fixed fields, we assume that $K_{\phi}$ and $K_{\psi}$ are linearly disjoint over $K$.

(4) Consider any $\phi$ and $\psi$ as in the hypothesis of Item 3. There exists an element $\sigma \in \Gamma_{Q}$ such that $\overline{\rho}(\sigma)$ is a regular semi-simple element of $G$, the connected component of whose centralizer we denote $T$, and such that there exists a root $\alpha \in \Phi(G,T)$ satisfying

(a) $\alpha(\overline{\rho}(\sigma)) = k(\sigma)$.

(b) $k[\psi(\Gamma_K)]$ has an element with non-zero $t_\alpha$ component; and

(c) $k[\phi(\Gamma_K)]$ has an element with non-zero $g_{-\alpha}$ component.

Then there exists a finite set of primes $Q$ disjoint from $S$, and a lift $\rho : \Gamma_{Q,S\cup Q} \rightarrow G(O)$ of $\overline{\rho}$ such that $\rho$ is of type $L_v$ at all $v \in S$ and of Ramakrishna type at all $v \in Q$.

3.2. A variant of Ramakrishna’s method in the Weyl group case. In this section, we let $G$ be a simple algebraic group of classical type or type $E_7$ and let $\overline{\rho} : \Gamma_{Q} \rightarrow G(k)$ be as in Section 2.1. In particular, the adjoint Galois module $\overline{\rho}(g)$ decomposes into the direct sum of irreducibles: $\overline{\rho}(t)$, $\overline{\rho}(g_\Phi)$ when $\Phi$ is simply-laced, $\overline{\rho}(t)$, $\overline{\rho}(g_l)$, $\overline{\rho}(g_s)$ otherwise. Let us remark that the method in this section would also work for other exceptional groups if one can realize the normalizer of a split maximal torus in $G(k)$ (or an suitable variant of which) as a Galois group over $\mathbb{Q}$.

Let $M$ be a finite dimensional $k$-vector space with a continuous $\Gamma_{Q}$-action. Define its Tate dual $M^\vee = \text{Hom}(M, \mu_\infty)$ to be a space with the following $\Gamma_{Q}$-action:

$$(\sigma f)(m) := \sigma(f(\sigma^{-1}m))$$
Proposition 3.5. For any continuous homomorphism \( \bar{\rho} : \Gamma_{\mathbb{Q}} \to G(k) \), \( \bar{\rho}(g)^{\vee} \cong \bar{\rho}(g)(1) \). For \( \rho \) as in Section 2.1, \( \bar{\rho}(t)^{\vee} \cong \bar{\rho}(t)(1) \).

Proof. The Killing form is a non-degenerate \( G \)-invariant symmetric bilinear form on \( g \), which identifies the contragredient representation \( g^{\ast} \) with \( g \), and hence identifies \( \bar{\rho}(g)^{\vee} \) with \( \bar{\rho}(g)(1) \) as Galois modules. If \( \rho \) is as in Section 2.1, then the Galois action on \( t \) factors through \( W \). It is easy to see that the standard bilinear form on \( t \) is non-degenerate and \( W \)-invariant. Just as above, we deduce that \( \bar{\rho}(t)^{\vee} \cong \bar{\rho}(t)(1) \) as Galois modules. \( \square \)

Definition 3.6. Let \( \mathcal{L} = \{ L_v \}_{v \in S} \) be the Selmer system corresponding to a global deformation condition for \( \bar{\rho} \) that is unramified outside a finite set of places \( S \), and let \( \mathcal{L}^{\perp} = \{ L_v^{\perp} \}_{v \in S} \) be the associated dual Selmer system. Suppose that \( \bar{\rho}(g) \) is semi-simple. Then for any \( \Gamma_{\mathbb{Q}} \)-submodule \( M \) of \( \bar{\rho}(g) \), \( M^{\vee} \) is isomorphic to \( M(1) \subset \bar{\rho}(g)(1) \). Moreover, the isomorphism is canonical if \( \bar{\rho}(g) \) decomposes into non-isomorphic irreducible submodules. Define the \( M \)-Selmer as follows:

\[
H_{\mathcal{L}}^{1}(\Gamma_{\mathbb{Q}}, M) = \text{Ker} \left( H^{1}(\Gamma_{\mathbb{Q}}, M) \to \bigoplus_{v \in S} H^{1}(\Gamma_v, M)/(L_v \cap H^{1}(\Gamma_v, M)) \right)
\]
and define the \( M \)-dual Selmer as follows:

\[
H_{\mathcal{L}^{\perp}}^{1}(\Gamma_{\mathbb{Q}}, M^{\vee}) = \text{Ker} \left( H^{1}(\Gamma_{\mathbb{Q}}, M^{\vee}) \to \bigoplus_{v \in S} H^{1}(\Gamma_v, M^{\vee})/(L_v^{\perp} \cap H^{1}(\Gamma_v, M^{\vee})) \right)
\]

The motivation of the arguments in this section is the difficulty to fulfill Item 4 of 3.4 for our choice of the residual representation. Let us explain it. In practice, we need to find a regular semi-simple element \( \Sigma \) in \( \bar{\rho}(\Gamma_{\mathbb{Q}}) \), the connected component of whose centralizer \( G \) we denote \( T' \), for which there exists a root \( \alpha \in \Phi' := \Phi(G, T') \) such that

1. \( \alpha(\Sigma) \in (\mathbb{Z}/l\mathbb{Z})^{\times} \).
2. Every irreducible summand of \( \bar{\rho}(g) \) has an element with non-zero \( t_{\alpha} \) component.
3. Every irreducible summand of \( \bar{\rho}(g) \) has an element with non-zero \( g_{-\alpha} \) component.

Since \( \bar{\rho} \) is unramified at \( l \) and \( \mathbb{Q}(\mu_l) \) is totally ramified at \( l \), \( \mathbb{Q}(\bar{\rho}) \) and \( \mathbb{Q}(\mu_l) \) are linearly disjoint over \( \mathbb{Q} \). Thus, there exists an element \( \sigma \in \Gamma_{\mathbb{Q}} \) such that \( \bar{\rho}(\sigma) = \Sigma \) and \( \kappa(\sigma) = \alpha(\Sigma) \). It follows that \( \alpha(\bar{\rho}(\sigma)) = \kappa(\sigma) \).

If we take \( \Sigma \) to be a regular semi-simple element in \( T \), then there is no \( \alpha \in \Phi \) fulfilling both (2) and (3). Therefore, we need to look for \( \Sigma \in N = N_G(T) \) for which \( \pi(\Sigma) \neq 1 \) in \( W \).

Let \( t' \) be a regular semi-simple element in \( G \) and \( t \in T \) be an element conjugate to \( t' \). Then \( t \) and \( t' \) determines a unique bijection between \( \Phi = \Phi(G, T) \) and \( \Phi' = \Phi(G, T') \) with \( T' = Z_G(t')^{\Phi} \): for any \( \alpha \in \Phi \), define \( \alpha' \in \Phi' \) such that

\[
\alpha'(h) = \alpha(g^{-1}hg)
\]
for any \( h \in T' \), where \( g \) is an element in \( G \) such that \( g^{-1}t'g = t \). Since \( t \) is regular semi-simple, \( \alpha' \) is independent of \( g \).
Lemma 3.7. For a long root $\alpha \in \Phi$, let $\Sigma$ be a regular semi-simple element in $N$ (which plays the role of $t'$) such that $\pi(\Sigma) = s_\alpha$. We fix an element $t \in T$ conjugate to $\Sigma$ and let $T' = Z_G(\Sigma)^0$. $\Sigma$ and $t$ determine a bijection between $\Phi$ and $\Phi'$ as above. If $\Phi$ is of type $A_n$ or $D_n$, then the root $\alpha' \in \Phi'$ corresponding to $\alpha$ fulfills (1) and (3). For (2), $g_0$ has an element with non-zero $t_{\alpha'}$ component, but $t$ does not. If $\Phi$ is of type $B_n$, $C_n$, or $E_7$, then $\alpha'$ fulfills (1) and $t$ has a vector with non-zero $g_{-\alpha'}$ component. Moreover, $t' \cap t = W \cap t$ with $W = t_{\alpha'} + g_{\alpha'}$ in either case.

Lemma 3.8. Assume that $\Phi$ is of type $B_n, C_n$ or $E_7$, so $\bar{\rho}(g) = \bar{\rho}(t) \oplus \bar{\rho}(g_l) \oplus \bar{\rho}(g_s)$ for $B_n$ and $C_n$, $\bar{\rho}(g) = \bar{\rho}(t) \oplus \bar{\rho}(g_a) \oplus \bar{\rho}(g_b)$ for $E_7$.

(1) (Type $B_n$ and $C_n$) For a pair of non-perpendicular $\beta, \gamma \in \Phi$ with $\beta$ long and $\gamma$ short, let $\Sigma$ be a regular semi-simple element in $N$ such that $\pi(\Sigma) = s_\beta \cdot s_\gamma$. We fix an element $t \in T$ conjugate to $\Sigma$ and let $T' = Z_G(\Sigma)^0$. $\Sigma$ and $t$ determine a bijection between $\Phi$ and $\Phi' = \Phi(G, T')$. Then for all long root $\alpha'$ in the span of $\beta'$ and $\gamma'$ (which is a sub-system of $\Phi'$ of type $C_2$), (3) is satisfied. For (2), $g_l$ (resp. $g_s$) has an element with non-zero $t_{\alpha'}$ component, but $t$ does not. $\alpha'(\Sigma)$ has order 4 in $k^\times$, hence (1) is satisfied only for $l \equiv 1(4)$.

(2) (Type $E_7$) For a pair of non-perpendicular $\beta, \gamma \in \Phi$ with $g_0 \subset g_a$ and $g_s \subset g_b$, let $\Sigma$ be a regular semi-simple element in $N$ such that $\pi(\Sigma) = s_\beta \cdot s_\gamma$. We fix an element $t \in T$ conjugate to $\Sigma$ and let $T' = Z_G(\Sigma)^0$. $\Sigma$ and $t$ determine a bijection between $\Phi$ and $\Phi' = \Phi(G, T')$. Then for all root $\alpha'$ in the span of $\beta'$ and $\gamma'$ (which is a sub-system of $\Phi'$ of type $A_2$), (3) is satisfied. For (2), $g_a$ (resp. $g_b$) has an element with non-zero $t_{\alpha'}$ component, but $t$ does not. $\alpha'(\Sigma)$ has order 3 in $k^\times$, hence (1) is satisfied only for $l \equiv 1(3)$.

Moreover, $W \cap t \subset t'$ with $W = t_{\alpha'} + g_{\alpha'}$.

Remark 3.9. In general, no matter what element of $N$ we pick for $\Sigma$ and what root $\alpha'$ we pick from the associated root system $\Phi'$, $t$ does not have an element with non-zero $t_{\alpha}$ component. This means the irreducible summand of $t$ is ‘too small’ in some sense. But the next lemma indicates how we may find out way out despite of this.

Lemma 3.10. Let $\sigma$ be an element in $\Gamma_Q$ such that $\alpha(\bar{\rho}(\sigma)) = \bar{\sigma}(\sigma)$ and $\Sigma := \bar{\rho}(\sigma)$ is a regular semi-simple element in $N_G(T)$ satisfying 3.7 for $\Phi$ of type $A_n$ or $D_n$, and 3.8 for $\Phi$ of type $B_n, C_n$ or $E_7$. Assume there is a Selmer system $\mathcal{L} = \{L_v\}_{v \in S}$ for which the $t$-Selmer $H^1(\mathcal{L}, \bar{\rho}(t))$ is trivial. Then Item 4 of 3.4 is satisfied.

The next proposition achieves the vanishing assumption of the $t$-Selmer in 3.10 by using of a variant of the cohomological arguments in Ramakrishna’s method.

Proposition 3.11. Suppose that

$$h^1_{\mathcal{L}}(\Gamma_S, \bar{\rho}(t)) \leq h^1_{\mathcal{L}^\perp}(\Gamma_S, \bar{\rho}(t)(1))$$

Then there is a finite set of places $Q$ disjoint from $S$ and a Ramakrishna deformation condition for each $w \in Q$ with tangent space $L_w^{\text{Ram}}$ such that

$$H^1_{\mathcal{L} \cup \{L_w^{\text{Ram}}\}_{w \in Q}}(\Gamma_{S \cup Q}, \bar{\rho}(t)) = 0$$
We may assume that $H^1_{L^\perp}(\Gamma_S, \bar{\rho}(t))$ is nontrivial, for otherwise we are done. The inequality in 3.11 then implies that $H^1_{L^\perp}(\Gamma_S, \bar{\rho}(t)(1))$ is nontrivial. Let $0 \neq \phi \in H^1_{L^\perp}(\Gamma_S, \bar{\rho}(t)(1))$.

**Lemma 3.12.** There exists $\sigma \in \Gamma_\mathbb{Q}$ with the following properties:

1. $\bar{\rho}(\sigma)$ is a regular semisimple element of $G(k)$, the connected component of whose centralizer we denote $T'$.
2. There exists $\alpha \in \Phi(G, T')$, such that $\alpha(\bar{\rho}(\sigma)) = \bar{\kappa}(\sigma)$.
3. $k[\phi(\Gamma_K)]$ has an element with nonzero $g_{\alpha}$-component.

**Proof.** This follows immediately from 3.7. \qed

**Lemma 3.13.** There exist infinitely many places $w \not\in S$ such that $\bar{\rho}|_{\Gamma_w}$ is of Ramakrishna type $\alpha$ and $\phi|_{\Gamma_w} \not\in L_w^{Ram, \perp}$.

**Proof.** The proof is identical to the proof of [12], Lemma 5.3, which uses 3.3, 3.12 and Chebotarev’s density theorem. \qed

**Proof of 3.11:** Let $w$ be chosen as in 3.13. We will show that

$$h^1_{L^\perp \cup L_w^{Ram, \perp}}(\Gamma_{S \cup w}, \bar{\rho}(t)(1)) < h^1_{L^\perp}(\Gamma_S, \bar{\rho}(t)(1))$$

(1)

and

$$h^1_{L^\perp \cup L_w^{Ram}}(\Gamma_{S \cup w}, \bar{\rho}(t)) - h^1_{L^\perp \cup L_w^{Ram, \perp}}(\Gamma_{S \cup w}, \bar{\rho}(t)(1)) = h^1_{L}(\Gamma_S, \bar{\rho}(t)) - h^1_{L^\perp}(\Gamma_S, \bar{\rho}(t)(1))$$

(2)

which will imply that

$$h^1_{L^\perp \cup L_w^{Ram}}(\Gamma_{S \cup w}, \bar{\rho}(t)) < h^1_{L}(\Gamma_S, \bar{\rho}(t))$$

from which 3.11 follows by induction.

We first show (2): A double invocation of Wiles’ formula gives

\[ \text{LHS of (2)} - \text{RHS of (2)} = - \dim(L^{Ram}_{w} \cap H^1(W, \bar{\rho}(t))) - h^0(W, \bar{\rho}(t)) = h^1(W, \bar{\rho}(W \cap t)) - h^0(W, \bar{\rho}(t)) = \dim(W \cap t) - \dim(t \cap t) = 0 \text{ (3.7).} \]

It remains to prove (1). Let $L_w = L_{w}^{unr} \cap L^{Ram}_{w}$, so $L^\perp_w = L_{w}^{unr, \perp} + L^{Ram, \perp}_{w}$. We have the following obvious inclusions

$$H^1_{L^\perp \cup L_w^{Ram}}(\Gamma_{S \cup w}, \bar{\rho}(t)) \subset H^1_{L^\perp \cup L_w^{Ram}}(\Gamma_{S \cup w}, \bar{\rho}(t)) \quad (A)$$

$$H^1_{L^\perp \cup L_w^{Ram, \perp}}(\Gamma_{S \cup w}, \bar{\rho}(t)(1)) \subset H^1_{L^\perp \cup L_w^{Ram, \perp}}(\Gamma_{S \cup w}, \bar{\rho}(t)(1)) \quad (A')$$

$$H^1_{L^\perp \cup L_w^{Ram}}(\Gamma_{S \cup w}, \bar{\rho}(t)) \subset H^1_{L^\perp \cup L_{w}^{unr}}(\Gamma_{S \cup w}, \bar{\rho}(t)) \quad (B)$$

$$H^1_{L^\perp}(\Gamma_S, \bar{\rho}(t)(1)) = H^1_{L^\perp \cup L_{w}^{unr, \perp}}(\Gamma_{S \cup w}, \bar{\rho}(t)(1)) \subset H^1_{L^\perp \cup L_{w}^{unr}}(\Gamma_{S \cup w}, \bar{\rho}(t)(1)) \quad (B')$$

As $\phi|_{\Gamma_w} \not\in L_{w}^{Ram, \perp}$, $(A')$ is a strict inclusion. We claim that $(B')$ is an isomorphism which implies (1). To prove our claim, we consider $(B)$ first. There is an exact sequence

$$0 \to H^1_{L^\perp \cup L_w}(\Gamma_{S \cup w}, \bar{\rho}(t)) \to H^1_{L}(\Gamma_S, \bar{\rho}(t)) \to (L_{w}^{unr} \cap H^1(W, \bar{\rho}(t)))/(L_w \cap H^1(W, \bar{\rho}(t)))$$
The top of its last term has dimension \( \dim(t' \cap t) \), the bottom of its last term has dimension \( \dim(t_0 \cap t) \). By 3.7, these dimensions are equal. So the last term is zero, and hence \((B)\) is an isomorphism. A double invocation of Wiles’ formula gives

\[
\dim(B) = \dim(t_0) = \dim(t) = \dim(W) = \dim(t' \cap t) = \dim(t' \cap t).
\]

Because \((B)\) is an isomorphism and \( \dim(L_w \cap H^1(\Gamma_w, \bar{\rho}(t))) = h^0(\Gamma_w, \bar{\rho}(t)) \), the right hand side of the above identity is zero. Therefore \((B')\) is an isomorphism which completes the proof of the proposition. \(\square\)

**Theorem 3.14.** Let \( \mathcal{L} = \{L_v\}_{v \in S} \) be a family of smooth local deformation conditions for \( \bar{\rho} \) unramified outside a finite set of places \( S \) containing the real place and places where \( \bar{\rho} \) is ramified. Suppose that

\[
\sum_{v \in S} \dim L_v \geq \sum_{v \in S} h^0(\Gamma_v, \bar{\rho}(g)).
\]

and

\[
\sum_{v \in S} \dim(L_v \cap H^1(\Gamma_v, \bar{\rho}(t))) \leq \sum_{v \in S} h^0(\Gamma_v, \bar{\rho}(t)).
\]

Assume \( l \) is large enough, and when \( \Phi \) is doubly-laced, assume \( l \equiv 1(4) \).

Then there is a finite set of places \( Q \) disjoint from \( S \) and a continuous lift

\[
\rho : \Gamma_{S \cup Q} \to \text{G}(\mathcal{O})
\]

of \( \bar{\rho} \) such that \( \rho \) is of type \( L_v \) for \( v \in S \) and of Ramakrishna type for \( v \in Q \).

**Proof.** The second inequality and Wiles’ formula imply that \( h^1(\Gamma_S, \bar{\rho}(t)) \leq h^1(\Gamma_S, \bar{\rho}(t)(1)) \).

By 3.11, we can enlarge \( \mathcal{L} \) by adding finitely many Ramakrishna deformation conditions to get a new Selmer system \( \mathcal{L}' = \{L_v\}_{v \in S'} \) with \( S' \supset S \) such that \( H^1(\mathcal{L}', \Gamma_{S'}, \bar{\rho}(t)) = \{0\} \). By 3.3.2, replacing \( \mathcal{L} \) by \( \mathcal{L}' \) preserves the first inequality.

If \( \Phi \) is of type \( A_n \) or \( D_n \), we take \( \Sigma, \alpha \) as in 3.7. There exists \( \sigma \in \Gamma_Q \) such that \( \bar{\rho}(\sigma) = \Sigma \) and \( \bar{\kappa}(\sigma) = \alpha(\Sigma) \) which imply that \( \alpha(\bar{\rho}(\sigma)) = \bar{\kappa}(\sigma) \). By Chebotarev’s density theorem, there are infinitely many places \( w \notin S' \) such that \( \bar{\rho}|_{\Gamma_w} \) is of Ramakrishna type \( \alpha \). If \( \Phi \) is of type \( B_n, C_n \) or \( E_7 \), we take \( \Sigma, \alpha \) as in 3.8. For the same reason, there are infinitely many places \( w \notin S' \) such that \( \bar{\rho}|_{\Gamma_w} \) is of Ramakrishna type \( \alpha \) if \( l \equiv 1(4) \).

For the Ramakrishna deformation \( L_{w, Ram}^w \) associated to \( \bar{\rho}|_{\Gamma_w} \),

\[
H^1(\mathcal{L}' \cup L_{w, Ram}^w, \Gamma_{S' \cup w}, \bar{\rho}(t)) = 0.
\]

Indeed, \( L_{w, Ram}^w \cap H^1(\Gamma_w, \bar{\rho}(t)) = H^1(\Gamma_w, \bar{\rho}(W \cap t)) \subset H^1(\Gamma_w, \bar{\rho}(t' \cap t)) = H^1(\Gamma_w/I_w, \bar{\rho}(t)). \) So

\[
H^1(\mathcal{L}' \cup L_{w, Ram}^w, \Gamma_{S' \cup w}, \bar{\rho}(t)) \subset H^1(\mathcal{L}', \Gamma_{S'}, \bar{\rho}(t)) = \{0\}.
\]

Let us check the assumptions of 3.4. Item (1) is satisfied by assumption. For Item (2), since \( |\text{Gal}(K/Q)| \) divides \( (l - 1)|\bar{\rho}(\Gamma_Q)| \) which is prime to \( l \) by the definition of \( \bar{\rho} \) (see Section 2.1). Thus the cohomology groups vanish in positive degrees. For Item (3), it suffices to show that for any pair of irreducible summands \( M, N \) of \( \bar{\rho}(g) \), distinct or not, \( M \) and \( N(1) \) are non-isomorphic. This can be proved using the argument of [12], Lemma 6.8. As
Let \( H^1_L(\Gamma^*, \tilde{\rho}(t)) = \{0\} \), Item (4) is satisfied by 3.10. Therefore, 3.4 applies and we get desired \( l \)-adic lifts.

3.3. Ramakrishna’s method in the principal \( GL_2 \) case. We use the notation in Section 2.2. Patrikis has shown that all simple algebraic groups of exceptional types are geometric monodromy groups for \( \Gamma_\mathbb{Q} \) except for \( E_6^{ad}, E_7^{sc} \). In this paper, the principal \( GL_2 \) construction is mainly used for proving that \( E_6^{ad}, E_6^{sc}, SL_3, Spin_7 \) are algebraic monodromy groups for \( \Gamma_\mathbb{Q} \). Unlike what happens in Section 3.2, there is no issue with the original form of Patrikis’ extension of Ramakrishna’s method (3.4) for this construction. The proof of the following theorem is identical to that of [12], Theorem 7.4.

**Theorem 3.15.** Let \( L = \{L_v\}_{v \in S} \) be a family of smooth local deformation conditions for \( \tilde{\rho} \) unramified outside a finite set of places \( S \) containing the real place and places where \( \tilde{\rho} \) is ramified. Suppose that

\[
\sum_{v \in S} \dim L_v \geq \sum_{v \in S} h^0(\Gamma_v, \tilde{\rho}(g)).
\]

Let \( l \) be large enough.

Then there is a finite set of places \( Q \) disjoint from \( S \) and a continuous lift

\[
\rho : \Gamma_{\mathbb{Q}, Q} \to G(\mathcal{O})
\]

of \( \tilde{\rho} \) such that \( \rho \) is of type \( L_v \) for \( v \in S \) and of Ramakrishna type for \( v \in Q \).

4. Simple, simply-connected algebraic groups as monodromy groups for \( \Gamma_\mathbb{Q} \)

In this section, we prove 1.3.

4.1. Local deformation conditions.

4.1.1. The archimedean place. Recall that in Section 2.1.3 through Section 2.1.5, we construct the residual representation by first realizing \( S_n \) or \( A_n \) as a Galois group over \( \mathbb{Q} \) and then repeatedly applying 2.5 to build the Galois extension realizing \( N \) or a subgroup of \( N \) over \( \mathbb{Q} \). Let \( f \) be the polynomial in 2.4. We write \( c \) for the nontrivial element in \( \Gamma_{\mathbb{R}} \), the complex conjugation.

**Lemma 4.1.** Let \( G \) be one of \( SL_n, Sp_{2n}, Spin_{2n+1}, Spin_{2n} \) and \( \bar{\rho} : \Gamma_\mathbb{Q} \to G(k) \) be as in Section 2.1. We assume that \( n \geq 4 \) for \( SL_n, Spin_{2n+1}, Spin_{2n} \), and \( n \geq 2 \) for \( Sp_{2n} \). Let \( k \) be the sum of the number of real roots of \( f \) and the number of pairs of non-real roots of \( f \). Let \( \epsilon_1, \cdots, \epsilon_n \) be a sequence of numbers such that \( k \) of the \( \epsilon_i \)’s are 1, and \( n-k \) of the \( \epsilon_i \)’s are \(-1\). Let \( \pi \) be the covering homomorphism from the spin group to the orthogonal group. Then \( \bar{\rho}(c) \sim \text{diag}(\epsilon_1, \cdots, \epsilon_n) \) for \( SL_n, \bar{\rho}(c) \sim \text{diag}(\epsilon_1, \cdots, \epsilon_n, \epsilon_n, \cdots, \epsilon_1) \) for \( Sp_{2n} \), \( \pi(\bar{\rho}(c)) \sim \text{diag}(\epsilon_1, \cdots, \epsilon_n, \epsilon_n, \cdots, \epsilon_1) \) for \( Spin_{2n}, \pi(\bar{\rho}(c)) \sim \text{diag}(\epsilon_1, \cdots, \epsilon_n, 1, \epsilon_n, \cdots, 1) \) for \( Spin_{2n+1} \).

**Corollary 4.2.** Let \( G \) be of classical type and \( \bar{\rho} : \Gamma_\mathbb{Q} \to G(k) \) be as in Section 2.1. Then \( h^0(\Gamma_{\mathbb{R}}, \bar{\rho}(g)) \) is \( rk(g) + f \) where

\[
f = 2\left(\binom{k}{2} + \binom{n-k}{2}\right).
\]
for \( SL_n \),
\[
f = 4 \left( \binom{k}{2} + \binom{n-k}{2} \right) + 2n
\]
for \( Sp_{2n} \),
\[
f = 4 \left( \binom{k}{2} + \binom{n-k}{2} \right) + 2k
\]
for \( Spin_{2n+1} \),
\[
f = 4 \left( \binom{k}{2} + \binom{n-k}{2} \right)
\]
for \( Spin_{2n} \).

**Lemma 4.3.**
\[
\dim_k \left( \text{Sym}^{2n}(k^2) \otimes \det^{-n} \right)^{\text{diag}(1,-1)}
\]
equals \( n \) when \( n \) is odd, and \( n + 1 \) when \( n \) is even.

**Corollary 4.4.** Let \( \bar{\rho} : \Gamma_{Q} \to G(k) \) be as in Section 2.2. Then \( h^0(\Gamma_R, \bar{\rho}(g)) = 4 \) for \( G = SL_3 \), \( h^0(\Gamma_R, \bar{\rho}(g)) = 9 \) for \( G = Spin_7 \), \( h^0(\Gamma_R, \bar{\rho}(g)) = 38 \) for \( G = E_6^{sc} \).

4.1.2. **The place \( l \).** As we are not looking for geometric \( l \)-adic Galois representations in this paper, we impose no condition at the place \( l \). So the tangent space is equal to \( H^1(\Gamma_l, \bar{\rho}(g)) \).

By the local Euler characteristic formula,
\[
h^1(\Gamma_l, \bar{\rho}(g)) = h^0(\Gamma_l, \bar{\rho}(g)) + h^2(\Gamma_l, \bar{\rho}(g)) + \dim_k g
\]

**Lemma 4.5.** Let \( \bar{\rho} \) be as in Section 2.1 or Section 2.2. Then \( h^2(\Gamma_l, \bar{\rho}(g)) = 0 \) for large enough primes \( l \).

**Proof.** By local duality, it suffices to show that \( h^0(\Gamma_l, \bar{\rho}(g)(1)) = 0 \). For \( \bar{\rho} \) in Section 2.1, \( \bar{\rho}(I_{Q_l}) \) is trivial by construction but \( \bar{k}(I_{Q_l}) \) is nontrivial, so \( \bar{\rho}(g)(1)_{\text{I_{Q}}} \) is trivial. In particular, \( h^0(\Gamma_l, \bar{\rho}(g)(1)) = 0 \). For \( \bar{\rho} \) in Section 2.2, a similar argument to that in [20], Proposition 4.4 shows \( h^0(\Gamma_l, \text{Sym}^{2n}(k^2) \otimes \det^{-n} \otimes \bar{k}) = 0 \) for \( n \geq 1 \) and large enough primes \( l \) which implies \( h^0(\Gamma_l, \bar{\rho}(g)(1)) = 0 \). \( \square \)

**Corollary 4.6.** \( h^1(\Gamma_l, \bar{\rho}(g)) = h^0(\Gamma_l, \bar{\rho}(g)) + \dim_k g \).

4.1.3. **A zero-dimensional deformation.** In order to enlarge the image of the \( l \)-adic lift of the residual representation, we need to use some simple local deformation conditions at some unramified places.

Suppose that \( p \neq l \) and \( \bar{\rho} : \Gamma_p \to G(k) \) is an unramified representation and let \( g \in G(\mathcal{O}) \) be a lift of \( \bar{\rho}(\text{Fr}_p) \).

**Definition 4.7.** Define
\[
\text{Lift}_g^\varphi : \text{CNL}_{\mathcal{O}} \to \text{Sets}
\]
such that for a complete local noetherian \( \mathcal{O} \)-algebra \( R \), \( \text{Lift}_g^\varphi(R) \) consists of all lifts
\[
\rho : \Gamma_F \to G(R)
\]
of \( \bar{\rho} \) such that \( \rho \) is unramified and \( \rho(\text{Fr}_p) \) is \( \hat{G}(R) \)-conjugate to \( g \). So the tangent space is zero-dimensional and the deformation condition is smooth.
4.1.4. Steinberg deformations. In [12], a local deformation condition of ‘Steinberg type’ is taken at a place in order to obtain a regular unipotent element in the image of the \( l \)-adic lift. We will only need this to deform \( \bar{\rho} \) constructed from the principal \( \text{GL}_2 \). We refer the reader to [12], Section 4.3 for the definition and properties of the Steinberg deformation.

4.1.5. Minimal prime to \( l \) deformations. This deformation condition is well-known, see [12], Section 4.4 for the definition. We will use this deformation condition at places \( v \neq l \) for which \( \hat{\rho}(I_{Q_v}) \) is nontrivial and \( \hat{\rho}(\Gamma_{Q_v}) \) has order prime to \( l \).

4.2. Algebraic monodromy groups of the \( l \)-adic lifts of \( \bar{\rho} \). In this section, we specify the global deformation condition and compute the Wiles formula, then use the results in Section 3 to prove 1.3. Let us recall Wiles’ formula, for a proof, see [12], Proposition 9.2.

**Proposition 4.8.** Let \( M \) be a finite-dimensional \( k \)-vector space with a continuous \( \Gamma_Q \) action unramified outside a finite set of places \( S \). Let \( \mathcal{L} = \{L_v\}_{v \in S} \) (resp. \( \mathcal{L}^\perp = \{L_v^\perp\}_{v \in S} \)) be a Selmer system (resp. dual Selmer system) for \( M \). Then

\[
h^1_L(\Gamma_S, M) - h^1_{L^\perp}(\Gamma_S, M^\vee) = h^0(\Gamma_S, M) - h^0(\Gamma_S, M^\vee) + \sum_{v \in S} (\dim_k L_v - h^0(\Gamma_v, M)).
\]

We will compute the right hand side of the identity for the global deformation condition to be specified in below. For \( \bar{\rho}(\mathfrak{g}) \) from either Section 2.1 or Section 2.2, note that \( h^0(\Gamma_S, \bar{\rho}(\mathfrak{g})) = h^0(\Gamma_S, \bar{\rho}(\mathfrak{g})(1)) = 0 \).

4.2.1. Weyl group case. For \( G \) a simple, simply connected group of classical type or type \( E_7 \), let \( \bar{\rho} : \Gamma_Q \to G(k) \) be as in Section 2.1. Here we exclude the \( A_1, A_2, B_3 \) cases. We impose no condition at \( v = l \), minimal prime to \( l \) condition at \( v \in S - \{\infty, l\} \) which is legitimate since \( \bar{\rho}(\Gamma_Q) \) has order prime to \( l \) by our construction. Moreover, take a semi-simple element \( \bar{\rho} \) in \( T = T(k) \subset \text{Im}(\bar{\rho}) \) such that \( \alpha(\bar{\rho}) = 1, \beta(\bar{\rho}) \neq 1 \) for all \( \beta \in \Phi - \{\pm \alpha\} \) (which is possible since \( l \) is large enough for \( G \)). Let \( \mathfrak{g} = s \cdot u \in G(\mathcal{O}) \) with \( s \in T(\mathcal{O}) \) lifting \( \bar{\rho} \) and \( u \) a nontrivial element in the root subgroup \( U_\alpha(\mathcal{O}) \) whose reduction modulo the maximal ideal of \( \mathcal{O} \) is trivial. Let \( \sigma \in \Gamma_Q \) such that \( \bar{\rho}(\sigma) = \bar{\rho} \). By Chebotarev’s density theorem, there is a prime \( p \notin S \) such that \( \text{Fr}_p = \sigma \). We take \( \text{Lift}_{\bar{\rho}|_{\mathfrak{g}_p}}^{\mathfrak{g}_p} \) at \( p \). Let \( \mathcal{L} \) be the Selmer system associated to the above local deformations.

**Lemma 4.9.** Let \( f \) be as in Section 4.1.1 which is chosen to have at least a pair of non-real roots. Then \( \sum_{v \in S} \dim_k L_v \geq \sum_{v \in S} h^0(\Gamma_v, \bar{\rho}(\mathfrak{g})). \)

**Proof.** This follows directly from the previous computations. \( \square \)

**Lemma 4.10.** \( \sum_{v \in S} \dim_k (L_v \cap H^1(\Gamma_v, \bar{\rho}(\mathfrak{t}))) \leq \sum_{v \in S} h^0(\Gamma_v, \bar{\rho}(\mathfrak{t})). \)

Let us make the following observation which is from [12], Lemma 7.7. It will be used frequently in the proof of 4.11, 4.15, 4.16: suppose that \( \bar{\rho} : \Gamma_Q \to G(k) \) is a continuous representation with a continuous lift \( \rho : \Gamma_Q \to G(\mathcal{O}) \) in \( G(\mathcal{O}) \). Then \( \text{Lie}(G_\rho), \text{Lie}(G_\rho) \cap \mathfrak{g}_\rho, \) and \( (\text{Lie}(G_\rho) \cap \mathfrak{g}_Q) \otimes \mathcal{O} k \) are \( \Gamma_Q \)-modules. Moreover, the last one is a submodule of \( \bar{\rho}(\mathfrak{g}) \) and thus is a direct sum of some irreducible summands of \( \bar{\rho}(\mathfrak{g}) \). If \( \text{Lie}(G_\rho) = \mathfrak{g}(\mathbb{Q}_l) \) or \( (\text{Lie}(G_\rho) \cap \mathfrak{g}_Q) \otimes \mathcal{O} k = \mathfrak{g} \), then \( G_\rho = G(\mathbb{Q}_l) \) (since \( G \) is connected).
Proposition 4.11. For $G$ a simple, simply-connected group of classical type or type $E_7$, excluding type $A_1$, $A_2$ and $B_3$, and for almost all primes $l$ when $\Phi$ is of type $A_n$ or $D_n$, for almost all primes $l \equiv 1(4)$ when $\Phi$ is of type $B_n$ or $C_n$, for almost all primes $l \equiv 1(3)$ when $\Phi$ is of type $E_7$, there are infinitely many non-conjugate $l$-adic lifts

$$\rho : \Gamma_\mathbb{Q} \to G(\mathcal{O})$$

of $\bar{\rho} : \Gamma_\mathbb{Q} \to G(k)$ defined as in Section 2.1 with Zariski-dense images in $G(\overline{\mathbb{Q}}_l)$.

Proof. By 4.9 and 4.10, we can apply 3.14 to obtain a lift $\rho : \Gamma_\mathbb{Q} \to G(\mathcal{O})$ satisfying the prescribed local conditions. Suppose first that $G$ is of classical type. The condition at $p$ implies that $G_\rho$ has infinitely many elements which implies that $\text{Lie}(G_\rho)$ is nontrivial. By 2.9, 2.12 and 2.15, $(\text{Lie}(G_\rho) \cap g_\mathcal{O}) \otimes \mathcal{O} k$ is then either $t$ or $g$ (since it is a Lie subalgebra of $g$). But the condition at $p$ implies that there is a non-trivial unipotent element in $G_\rho$, so $(\text{Lie}(G_\rho) \cap g_\mathcal{O}) \otimes \mathcal{O} k$ cannot be $t$. Thus $(\text{Lie}(G_\rho) \cap g_\mathcal{O}) \otimes \mathcal{O} k = g$. Now suppose that $G = E_7^\mathbb{C}$.

By 2.19, $(\text{Lie}(G_\rho) \cap g_\mathcal{O}) \otimes \mathcal{O} k$ is either $t$ or $g$, hence $(\text{Lie}(G_\rho) \cap g_\mathcal{O}) \otimes \mathcal{O} k = g$.

Finally, by choosing infinitely many non-conjugate $g \in G(\mathcal{O})$ for $\text{Lift}_{\bar{\rho}|\Gamma_p}$, we obtain infinitely many non-conjugate $l$-adic lifts of $\bar{\rho}$. \qed

4.2.2. Principal $GL_2$ case. For $G$ a simply connected group of one of the following types: $A_2, B_3, E_6$, let $\bar{\rho} : \Gamma_\mathbb{Q} \to G(k)$ be as in Section 2.2.

We begin with the following proposition due to Tom Weston ([20], Proposition 5.3):

Proposition 4.12. Let $\pi = \pi_f$ be a cuspidal automorphic representation corresponding to a holomorphic eigenform $f$ of weight at least 2. Assume that for some prime $p$, $\pi_p$ is isomorphic to a twist of the Steinberg representation of $GL_2(\mathbb{Q}_p)$. Then for almost all $\lambda$, the local Galois representation $\bar{\rho}_{f,\lambda}|_{\Gamma_p}$ (in the notation of 1.2) has the form

$$\bar{\rho}_{f,\lambda}|_{\Gamma_p} \sim \begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}$$

where the extension * in $H^1(\Gamma_p, k_\lambda(\bar{\kappa}))$ is non-zero.

Suppose that $G$ admits a principal $GL_2$. Let $f$ be a non-CM weight 3 cuspidal eigenform that is a newform of level $\Gamma_1(p) \cap \Gamma_0(q)$ for some primes $p$ and $q$; the nebentypus of $f$ is a character $\varepsilon : (\mathbb{Z}/pq\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$. Such a form $f$ exists, for example, [9], 15.3.7.a and 15.3.11.a. At $p$ we take the Steinberg deformation condition. At $q$ we use the minimal prime to $l$ deformation condition, as $\bar{\rho}(\Gamma_q)$ has order prime to $l$. At $l$ we impose no condition. Moreover, choose an element $\sigma \in \Gamma_\mathbb{Q}$ such that $\bar{\rho}(\sigma)$ is regular semi-simple in $T(k)$ together with a lift $g \in T(\mathcal{O})$ such that $\alpha(g)$, $\alpha \in \Delta$ are distinct. By Chebotarev’s density theorem, there is a prime $r \notin \{\infty, l, p, q\}$ such that $\text{Fr}_r = \sigma$. We take $\text{Lift}_{\bar{\rho}|\Gamma_r}^g$ at $r$. Let $\mathcal{L}$ be the Selmer system associated to the above local deformations.

Lemma 4.13. Under the above global deformation condition, the right hand side of Wiles’ formula is 2 for $A_2$, 9 for $B_3$, and 34 for $E_6$. 
Proposition 4.14. For $G = \text{SL}_3$, $\text{Spin}_7$ or $E_6^{sc}$ and for almost all primes $l$, there are infinitely many non-conjugate $l$-adic lifts
\[ \rho : \Gamma \rightarrow G(\mathcal{O}) \]
of $\bar{\rho} : \Gamma \rightarrow G(k)$ defined as in Section 2.2 with Zariski-dense images in $G(\overline{\mathbb{Q}}_l)$.

Proof. The proof is very similar to the proof of [12], Theorem 7.4, so we skip a few details here.

We first show that 3.4 applies to $\bar{\rho}$. Item 1 is satisfied by 4.13; Item 2 and 3 are satisfied by the proof of [12], Theorem 7.4; For Item 4, take $\sigma \in \Gamma$ such that $\bar{\rho}(\sigma) = 2\rho^\vee(a)$ is regular with $1 \neq a \in (\mathbb{Z}/l\mathbb{Z})^\times$ and $\bar{k}(\sigma) = a^2$ (which is possible, again, see the proof of [12], Theorem 7.4). It follows that (a) is satisfied for any simple root $\alpha$. (b) and (c) are also satisfied by 2.22.

Therefore, we can deform $\bar{\rho}$ to a continuous representation $\rho : \Gamma \rightarrow G(\mathcal{O})$ satisfying the prescribed local conditions on $S$ and the Ramakrishna condition on a set of auxiliary primes disjoint from $S$. We write $G_\rho$ for the Zariski closure of the image of $\rho$ in $G(\overline{\mathbb{Q}}_l)$. By [12], Lemma 7.7, $G_\rho$ is reductive. By 4.12, the Steinberg condition at $p$ ensures that $G_\rho$ contains a regular unipotent element (see the proof of [12], Theorem 8.4). By a theorem of Dynkin, $G_\rho$ is then of type $A_1$ or $A_2$ for $G = \text{SL}_3$; type $A_1, G_2$ or $B_3$ for $G = \text{Spin}_7$; and type $A_1, F_4$, or $E_6$ for $G = E_6^{sc}$. But $\alpha(\rho(\text{Fr}_p))$, $\alpha \in \Delta$ are distinct, so $G_\rho = G(\mathbb{Q}_p)$ in all three cases (see the proof of [12], Lemma 7.8).

Finally, by choosing infinitely many non-conjugate $g \in T(\mathcal{O})$ for $\text{Lift}^g_{\rho|_{\Gamma'}}$, we obtain infinitely many non-conjugate $l$-adic lifts of $\bar{\rho}$.

Remark 4.15. For any simple, adjoint algebraic group $G$, the principal GL$_2$ is well-defined and the proof of the above proposition carries over to $G$, modulo some local calculations. Therefore, all simple, adjoint algebraic groups are $l$-adic monodromy groups for $\Gamma_Q$.

4.2.3. SL$_2$. The alternating group $A_n$ admits a unique nontrivial extension $\hat{A}_n$ by $\mathbb{Z}/2\mathbb{Z}$ for $n \neq 6, 7$. By a result of N. Vila and J-F. Mestre [16], $\hat{A}_n$ can be realized as a Galois group over $\mathbb{Q}$. In particular, we get a surjection $\bar{r} : \Gamma \rightarrow \hat{A}_5$. On the other hand, $\hat{A}_5$ can be described as follows: the symmetries of an icosahedron induce a 3-dimensional irreducible faithful representation of $A_5$, i.e., there is an injective homomorphism $A_5 \rightarrow SO(3)$. The pullback of $A_5$ along the two-fold covering map $SU(2) \rightarrow SO(3)$ is a nontrivial central extension of $A_5$ by $\mathbb{Z}/2\mathbb{Z}$, hence is isomorphic to $\hat{A}_5$. In particular, we get an embedding $\hat{A}_5 \rightarrow SL_2(\mathbb{C})$. As the matrix entries of the image lie in a finite extension of $\mathbb{Q}$, we can choose a finite extension $k$ of $\mathbb{F}_l$ for which there is a well-defined embedding $\hat{A}_5 \rightarrow SL_2(k)$. Precomposing it with $\bar{r}$, we obtain a representation $\Gamma \rightarrow SL_2(k)$ which we denote by $\bar{\rho}$. It is easy to see that the adjoint module $\bar{\rho}(\text{sl}_2(k))$ is irreducible. Let $S$ be a finite set of places containing the archimedean place and ramified places. We impose no condition at $l$, and take the minimal prime to $l$ deformation condition at all other places in $S$, which is legitimate since the residual image has order prime to $l$ for $l > 120$. Let $\Sigma \in \hat{A}_5$ be an element of order 4, whose image in $SL_2(k)$ is conjugate to $\text{diag}(\sqrt{-1}, -\sqrt{-1})$. As $\hat{A}_5$ has trivial abelian quotient, $\mathbb{Q}(\mu_4)$ and $\mathbb{Q}(\bar{\rho})$ are linearly disjoint over $\mathbb{Q}$. So there is an element $\sigma \in \Gamma$ such that $\bar{\rho}(\sigma) = \Sigma$ and $\bar{k}(\sigma) = -1$. By Chebotarev’s density theorem, there is a prime $p \notin S$ for which $\text{Fr}_p = \sigma$. Therefore, $\bar{\rho}|_{\Gamma_p}$ is of Steinberg type and we take the Steinberg deformation condition at $p$. 
Proposition 4.16. For \( \bar{\rho} : \Gamma_{\mathbb{Q}} \to \text{SL}_2(k) \) defined as above and for almost all primes \( l \), there is an \( l \)-adic lift

\[
\rho : \Gamma_{\mathbb{Q}} \to \text{SL}_2(\mathcal{O})
\]

of \( \bar{\rho} \) with Zariski-dense image in \( \text{SL}_2(\mathbb{Q}_l) \). Moreover, there are infinitely many non-conjugate representations as such.

Proof. We first show that 3.4 applies to \( \bar{\rho} \). Item 1 is satisfied: the left hand side of the inequality equals the right hand side; Item 2 is satisfied since \( |\text{Gal}(K/\mathbb{Q})| \) has order prime to \( l \) by the definition of \( \bar{\rho} \); Item 3 is satisfied since \( \bar{\rho}(\mathfrak{g}) \) and \( \bar{\rho}(\mathfrak{g})(1) \) are non-isomorphic; For Item 4, we take \( \sigma \) to be as above, the connected component of whose centralizer is denoted \( T \), and \( \alpha \) be a root of \( \Phi(G,T) \) which has rank one, so (a) is satisfied. As \( \bar{\rho}(\mathfrak{g}) \) is irreducible, (b) and (c) are satisfied. Therefore, we can deform \( \bar{\rho} \) to a continuous representation \( \rho : \Gamma_{\mathbb{Q}} \to \text{SL}_2(\mathcal{O}) \) satisfying the prescribed local conditions on \( S \) and the Ramakrishna condition on a set of auxiliary primes disjoint from \( S \). We write \( G_{\rho} \) for the Zariski closure of the image of \( \rho \) in \( \text{SL}_2(\mathbb{Q}_l) \). As \( \bar{\rho}(\mathfrak{g}) \) is irreducible, \( \text{Lie}(G_{\rho}) \) is either trivial or \( \mathfrak{g}_{\mathbb{Q}_l} \). If the former is true, then \( G_{\rho} \) is finite. But \( \rho|_{\Gamma_{\bar{\rho}}} \) is Steinberg, so in particular the image of \( \rho \) is infinite, a contradiction.

Finally, by choosing infinitely many linearly disjoint extension of \( \mathbb{Q} \) realizing \( \tilde{A}_5 \), we obtain infinitely many non-conjugate lifts. \( \square \)

Remark 4.17. For \( G \) a simple but non-simply connected group, suppose there is a homomorphism \( \rho_l : \Gamma_{\mathbb{Q}} \to G_{sc}(\mathbb{Q}_l) \) with Zariski-dense image. We compose \( \rho_l \) with the projection \( G_{sc}(\mathbb{Q}_l) \to G(\mathbb{Q}_l) \), the resulting map has Zariski-dense image in \( G(\mathbb{Q}_l) \). Therefore, 4.11, 4.14 and 4.16 imply 1.3.

5. Connected reductive groups as algebraic monodromy groups for \( \Gamma_{\mathbb{Q}} \)

In this section, we explain how the hoped-for classification of \( l \)-adic monodromy groups (1.5) can be deduced from the case when \( G \) is a simple algebraic group.

Following [10], a connected algebraic group \( G \) is said to be the almost-direct product of its algebraic subgroups \( G_1, \ldots, G_n \) if the map

\[(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n : G_1 \times \cdots \times G_n \to G\]

is a surjective homomorphism with finite kernel; in particular, this means that the \( G_i \) commute with each other and each \( G_i \) is normal in \( G \).

Proposition 5.1. [10] An algebraic group is semi-simple if and only if it is an almost direct product of simple algebraic groups. (Here a simple algebraic group is called almost simple in [10])

Proposition 5.2. (Goursat’s Lemma) Let \( G_1, G_2 \) be groups, let \( H \) be a subgroup of \( G_1 \times G_2 \) such that the two projections \( p_1 : H \to G_1, p_2 : H \to G_2 \) are surjective. Let \( N_1 \) (resp. \( N_2 \)) be the kernel of \( p_2 \) (resp. \( p_1 \)). Then the image of \( H \) in \( G_1/N_1 \times G_2/N_2 \) is the graph of an isomorphism \( G_1/N_1 \cong G_2/N_2 \).

5.1, 5.2, 4.15 together imply
Corollary 5.3. Let $G$ be a connected semi-simple group with trivial center, then there are continuous homomorphisms
$$\Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$$
with Zariski-dense images for almost all $l$.

Proposition 5.4. (Tate)[5] Let $F$ be a number field, let $G' \twoheadrightarrow G$ be a surjection of algebraic groups over $\overline{\mathbb{Q}}_l$ with kernel a central torus, then any continuous homomorphism $\Gamma_F \to G(\overline{\mathbb{Q}}_l)$ lifts to a continuous homomorphism $\Gamma_F \to G'(\overline{\mathbb{Q}}_l)$.

5.3 together with 5.4 imply

Corollary 5.5. Let $G$ be a connected reductive group such that $Z(G)$ is a one-dimensional torus, then there are continuous homomorphisms
$$\Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$$
with Zariski-dense images for almost all $l$.

As $\mathbb{Q}$ only has one $\mathbb{Z}_l$-extension by class field theory, we have

Lemma 5.6. Let $G$ be a connected reductive group such that $\dim Z(G) > 1$, then for any prime $l$, there is no continuous homomorphism
$$\Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$$
with Zariski-dense image.

Therefore, to prove 1.5, it remains to show the analogue of 5.3 and 5.5 in the case when $G$ has non-trivial finite center. By 1.3, 1.4, 5.1, 5.2, we have

Theorem 5.7. Let $G$ be a connected semi-simple group with non-trivial finite center, then there are continuous homomorphisms
$$\Gamma_{\mathbb{Q}} \to G(\overline{\mathbb{Q}}_l)$$
with Zariski-dense images for almost all primes $l$.

Therefore, the proof of 1.5 is complete!

References


ALGEBRAIC MONODROMY GROUPS OF $G$-VALUED $l$-ADIC REPRESENTATIONS OF $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$


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