5 Applications of the Integral

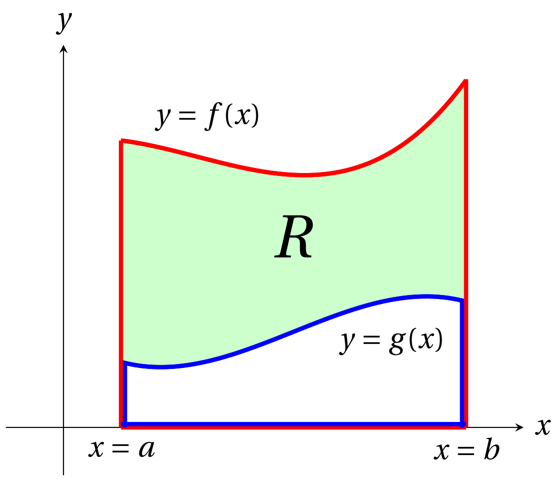
Now that we learnt how to compute definite integrals, we are ready to harness the vast potential of definite integrals. Not surprising, we will use the definite integral to compute areas of regions of complicated shape; but it goes far beyond that: it can be used to find volume of solids, lengths of plane curves, surface areas of solids, work and fluid force, and many more!

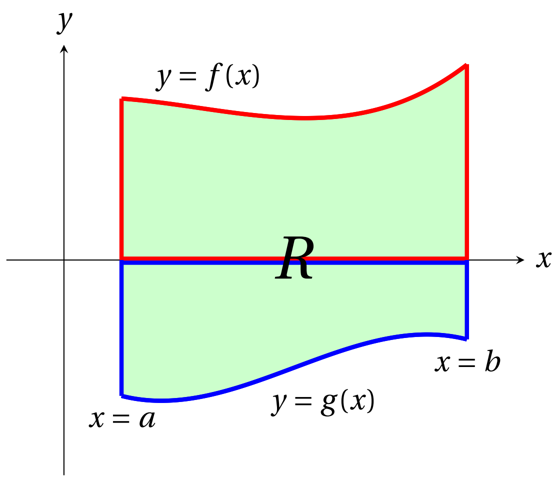
5.1 The Area of a Plane Region

A *plane region* is simply a bounded region in the xy-plane (Cartesian plane). The goal of this section is the following:

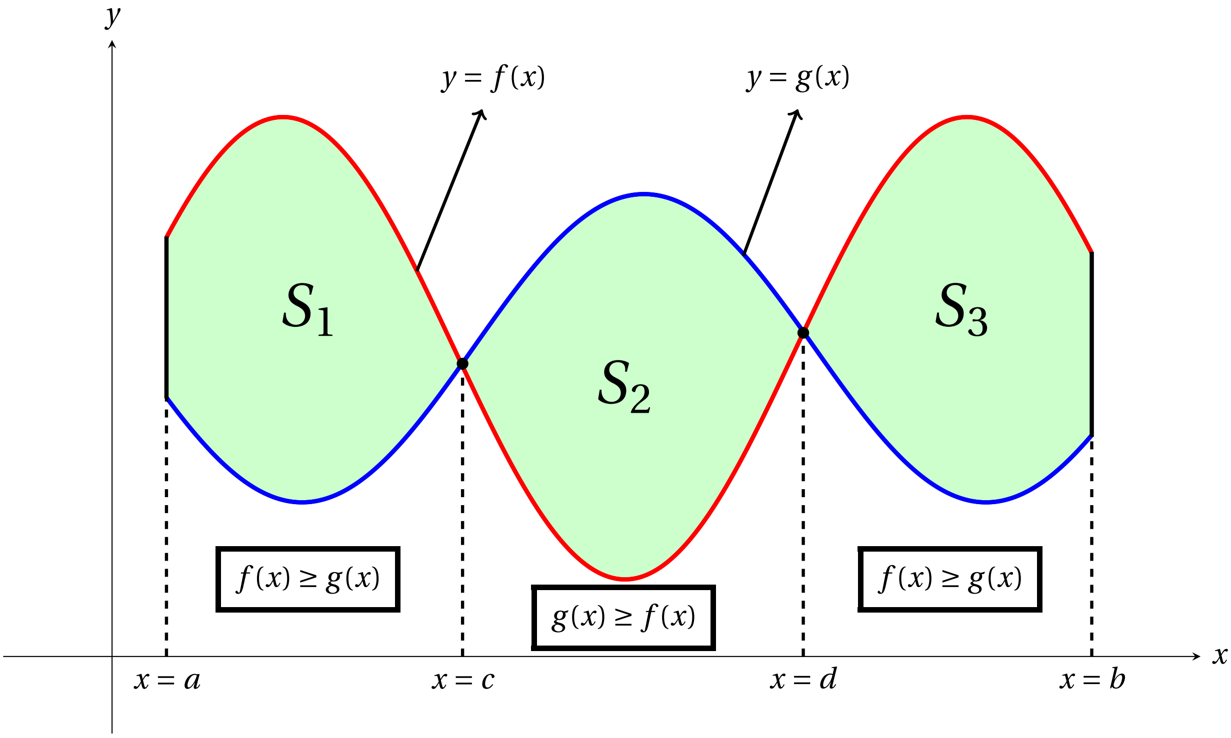
Compute the area of a plane region bounded by curves of functions and lines.

Suppose $f(x)$ and $g(x)$ are functions such that $f(x) \geq g(x)$ for all x in the interval $[a, b]$, and we want to find the area A of the region R (shaded below, in green) between the curves $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$, $x = b$. Let us examine two specific cases:





Because we are computing area (not signed area) which is a positive quantity, it is crucial to note that the previous formula is only valid if $f(x) \geq g(x)$ on the interval $[a,b]$. However, given two arbitrary functions $f(x)$ and $g(x)$, what usually happens is that $f(x)$ will greater than $g(x)$ for some values of x while for others, $g(x)$ will be greater than $f(x)$.



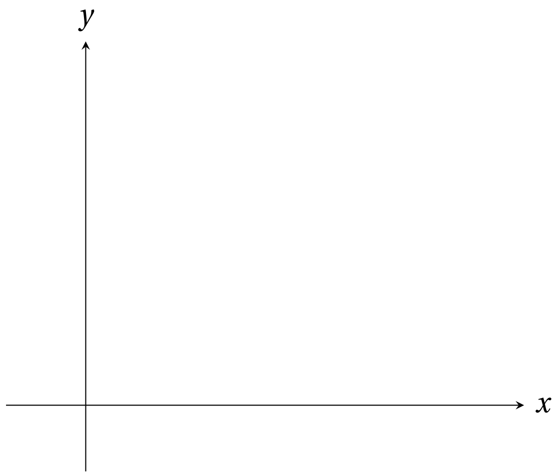
Similar reasoning as before implies that the area of the region $R = S_1 \cup S_2 \cup S_3$ between the curves $y = f(x)$, $y = g(x)$, and the vertical lines $x = a$ and $x = b$ is given by

$$\begin{aligned} A(R) &= A(S_1) + A(S_2) + A(S_3) \\ &= \int_a^c f(x) - g(x)\,dx + \int_c^d g(x) - f(x)\,dx + \int_d^b f(x) - g(x)\,dx \\ &= \end{aligned}$$

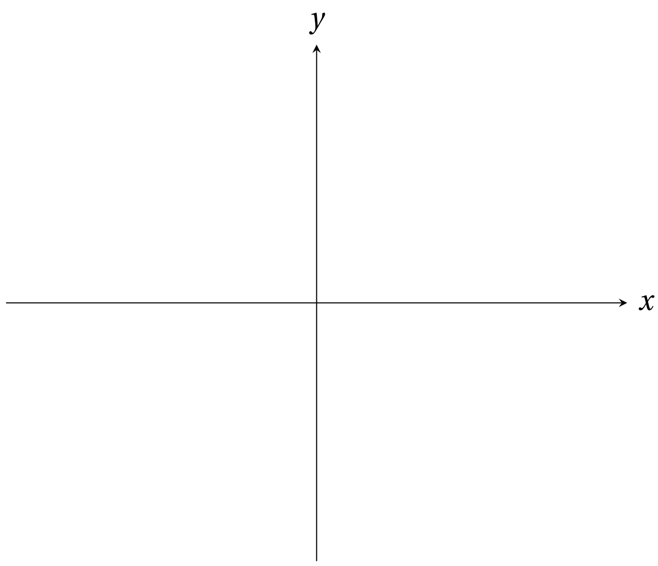
To actually compute $\int_a^b \left| f(x) - g(x) \right| dx$, we need to divide the interval into subintervals where either $f(x) \geq g(x)$ or $g(x) \geq f(x)$. Even if one function is always greater than the other, we still need to know which one that is. One way or another, **we should always start by sketching the region to determine the appropriate subintervals**. So yeah, the formula we have above is pretty much useless unless we sketch..........

Example 5.1. Find the areas of the indicated regions.

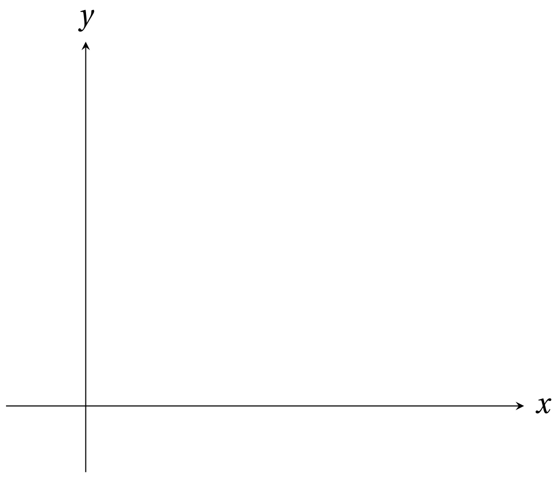
(a) The region between the line $y = x$ and $y = \sin x$, for $0 \le x \le \frac{\pi}{2}$.



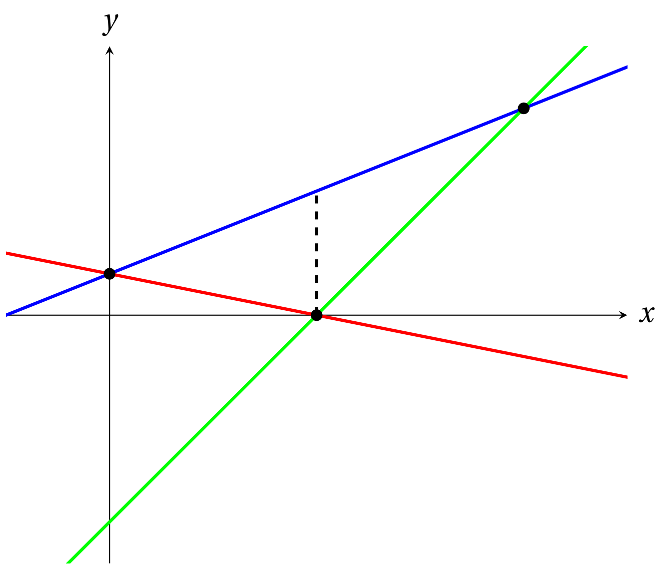
(b) The region enclosed between $y = x^2$ and $y = 3x$.

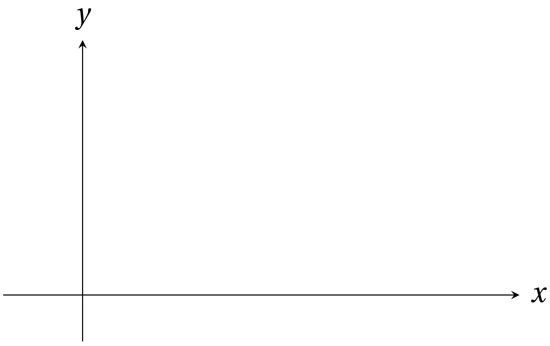


(c) The region between the curves $y = x^3$ and $y = \sqrt{x}$, between $x = 0$ and $x = 2$.



(d) The triangle bounded by the lines $y = 1 - x$, $y = 1 + 2x$, and $y = 5x - 5$.



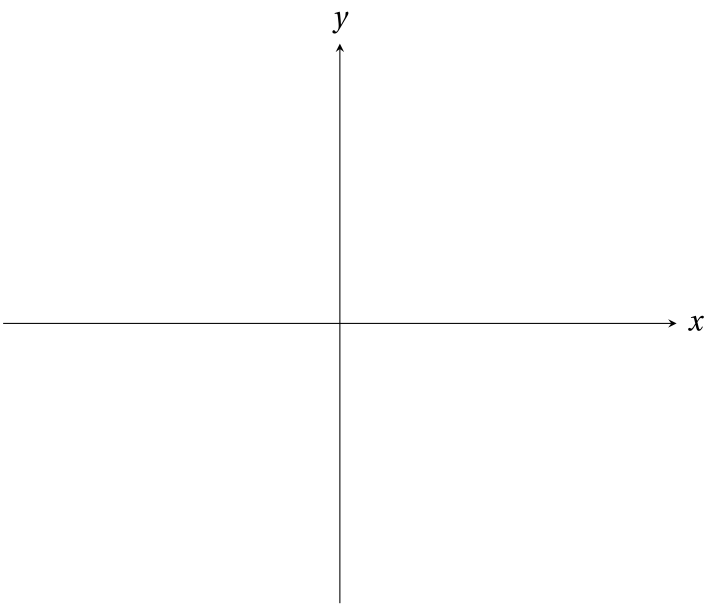


Sometimes it is not convenient to write a plane region as a bounded region between two functions of x. Instead, it is more convenient to consider the region as bounded between two functions of y. The idea is similar as before, the only difference is that now *the integration variable is y instead of x.*

Proposition 5.2. *The area of the region between the curves $x = f(y)$ and $x = g(y)$ and between the horizontal lines $y = c$ and $y = d$ is given by*

$$\int_c^d \Big| f(y) - g(y) \Big| \, dy.$$

Example 5.3. Find the area of the region bounded by the curves $x = y^2 - 4y$ and $x = 2y - y^2$.



5.2 Volumes of Solids: Slabs, Disks, Washers

Let's move on to the next goal: Finding volumes of solids, in particular volumes of solids of revolution. **Imagine the solid as a loaf of bread, a very uneven one because Chee Han is a lousy baker.**

Slice the solid into "slices of bread",
add all the volumes and take the limit
as the slices of bread shrinks in size.

We start with simple solids called **right cylinders**. These are generated by moving a plane region (called the base) through a distance h in a direction perpendicular to that region.

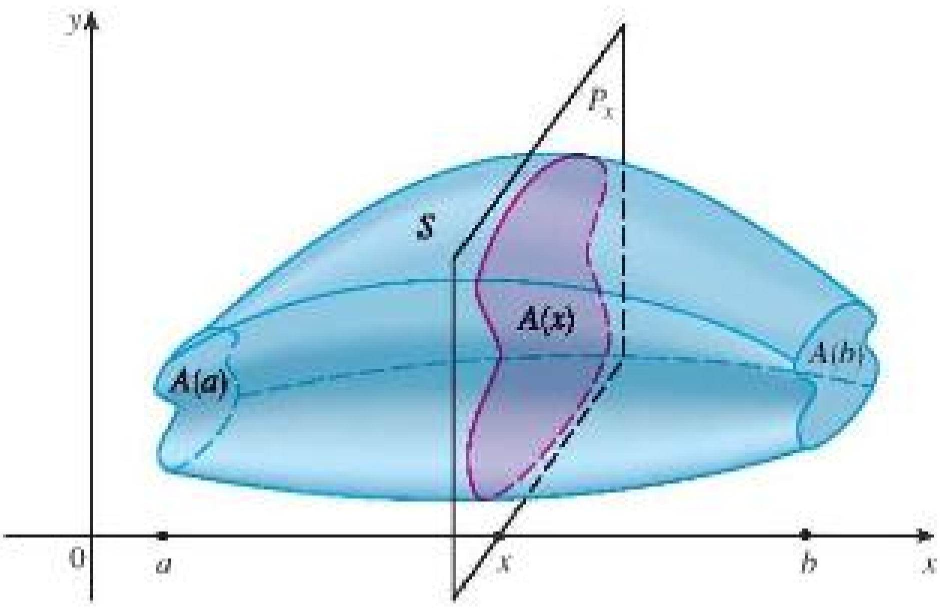
♣ A cube is a right cylinder with a square plane region.

♣ A cylinder is a right cylinder with a circular plane region.

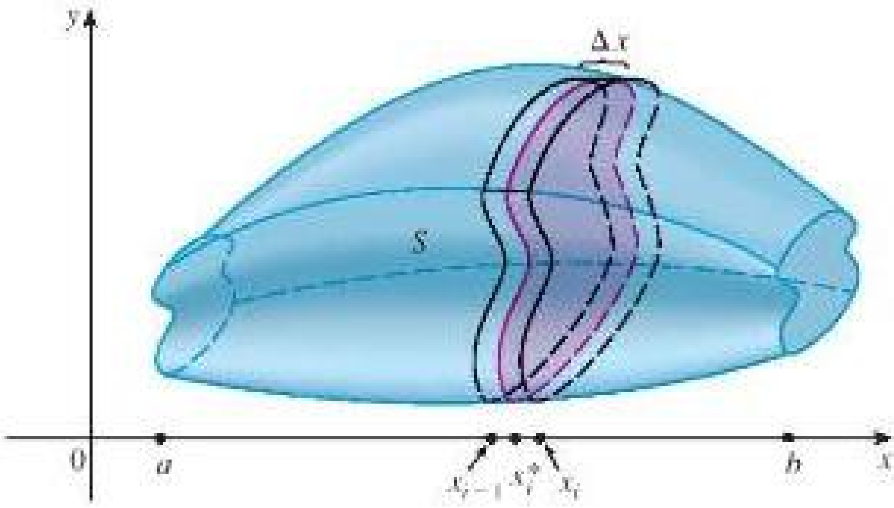
♣ A triangular prism is a right cylinder with a triangular plane region.

Now consider an arbitrary solid which we imagine being located between $x = a$ and $x = b$.

1. For every x in $[a, b]$, we take a slice of our solid perpendicular to the x-axis. This plane region is called the **cross-section**, and this cross-section has some cross-sectional area $A(x)$. Clearly, the shape of each such cross-section varies with x and so does the area $A(x)$.



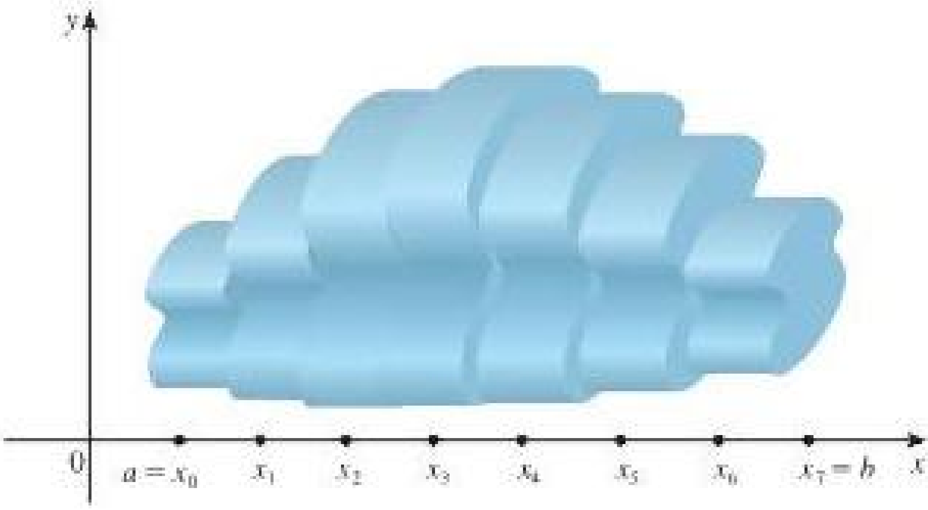
2. We divide the interval $[a,b]$ into n equally-spaced subintervals, each of length Δx; this in effect slices our solid into thin **slabs**. On each subinterval, choose a sample point x_i^* and approximate the cross-sectional area on that subinterval by $A(x_i^*)$.



Thus the exact volume ΔV_i of the thin slab is approximately $A(x_i^*)\Delta x$, the volume of the thin right cylinder with base the cross-section of our solid at x_i^* and height (or thickness) Δx.

3. We may approximate the volume of our solid by summing the volumes of all the thin right cylinders:

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$



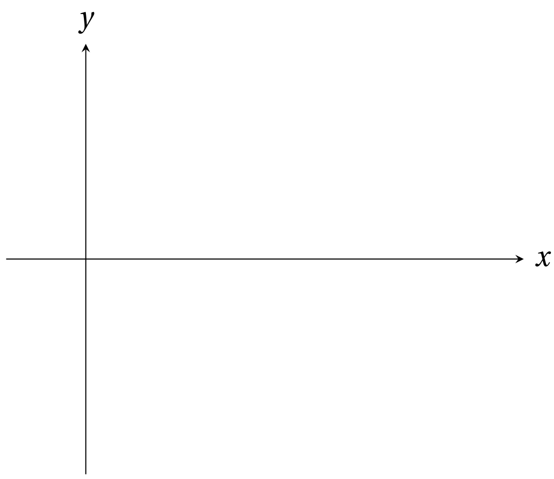
Because the latter expression is a Riemann sum, we take the limit as $n \longrightarrow \infty$ and obtain a definite integral, which we define to be the exact volume of our solid:

$$V = \lim_{n\to\infty}\left(\sum_{i=1}^{n} A(x_i^*)\Delta x\right) =$$

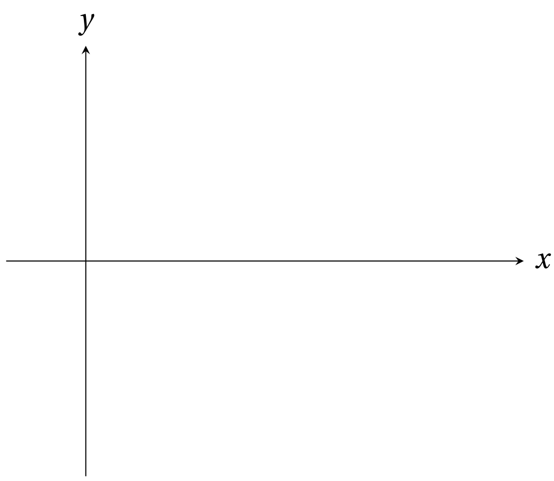
This way of finding volumes is called the **Method of Slabs**.

Example 5.4. Use the method of slabs to find a formula for

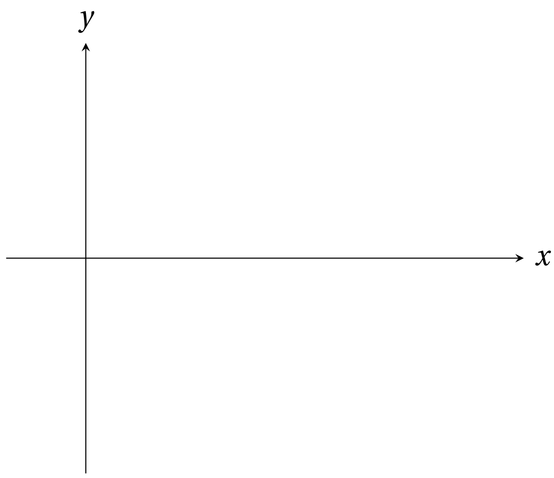
(a) the volume of a cylinder with radius r and height h;



(b) the volume of a cuboid with square base of length ℓ and height h.



Example 5.5. Use the method of slabs to find a formula for the volume of pyramid with square base of length b and height h.

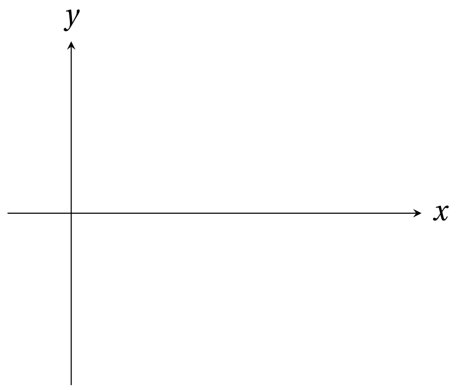


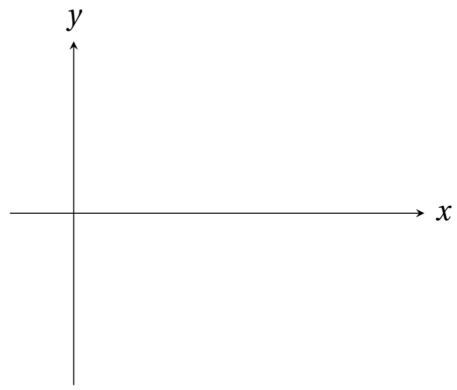
Solids of revolution

When a plane region, lying entirely on one side of a fixed line (often an axis) in the xy-plane, is rotated about that line, it generates a *solid of revolution*. In this setting, the cross-sections taken perpendicular to the *axis of rotation* will either be circles or annuli.

1. **Circular cross sections:** A circle with a small thickness is called a **disk**. With radius $r(x)$,

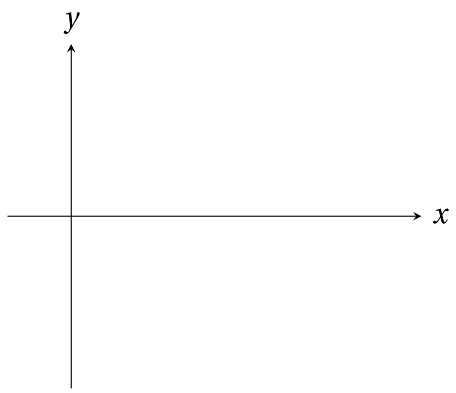
   This way of finding volumes is called the **Method of Disks**.

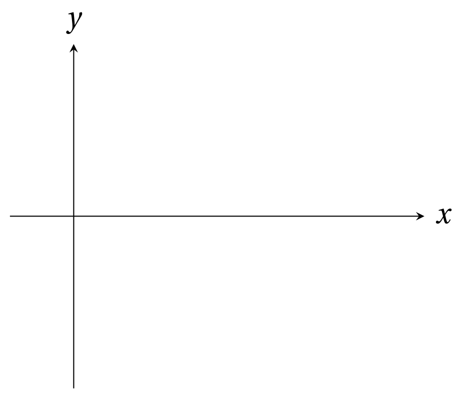
   

2. **Annular cross-sections:** An annulus with a small thickness is called a **washer**. Since we have an inner radius $r(x)$ and an outer radius $R(x)$,

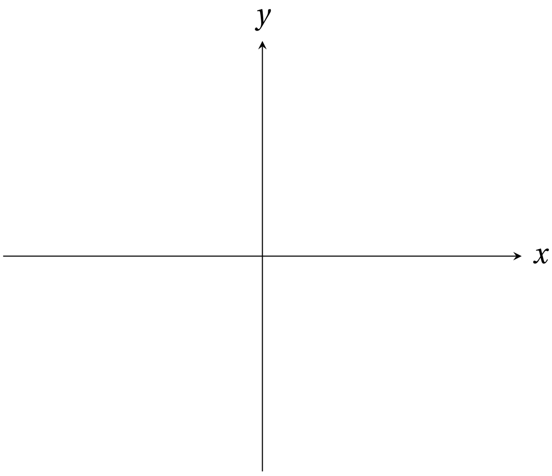
   This way of finding volumes is called the **Method of Washers**.

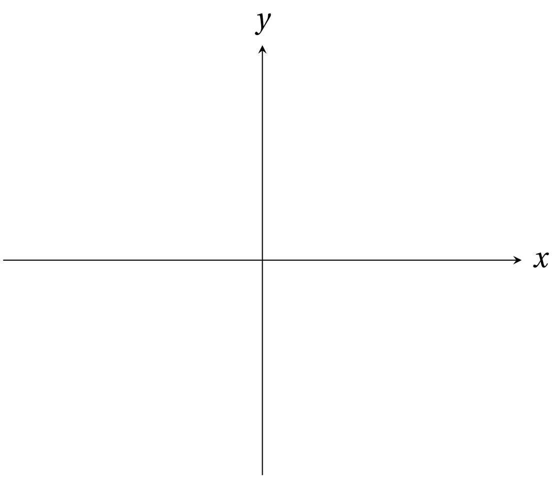
   

These integral formulas are only valid when the region is rotated about a horizontal line. The radius must be a function of y and the integration variable becomes y if the region is rotated about a vertical line instead.

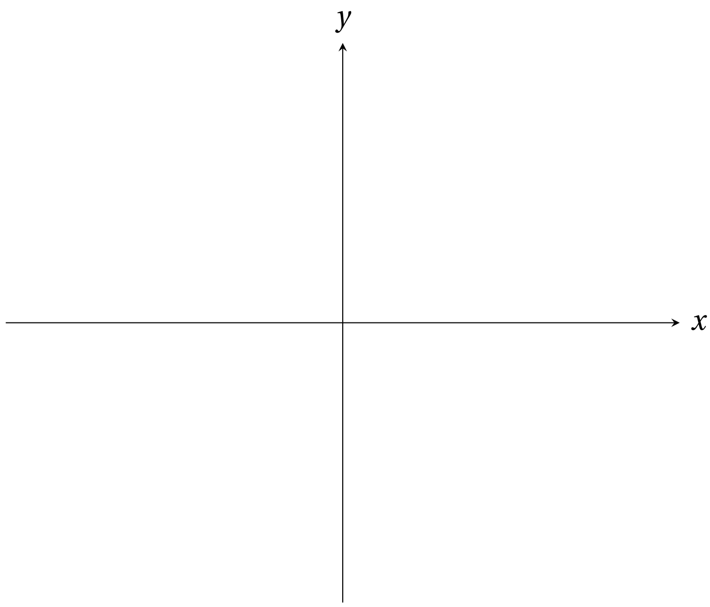
Example 5.6. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \dfrac{1}{x}$, the x-axis, $x = 1$, and $x = 5$ about the x-axis.



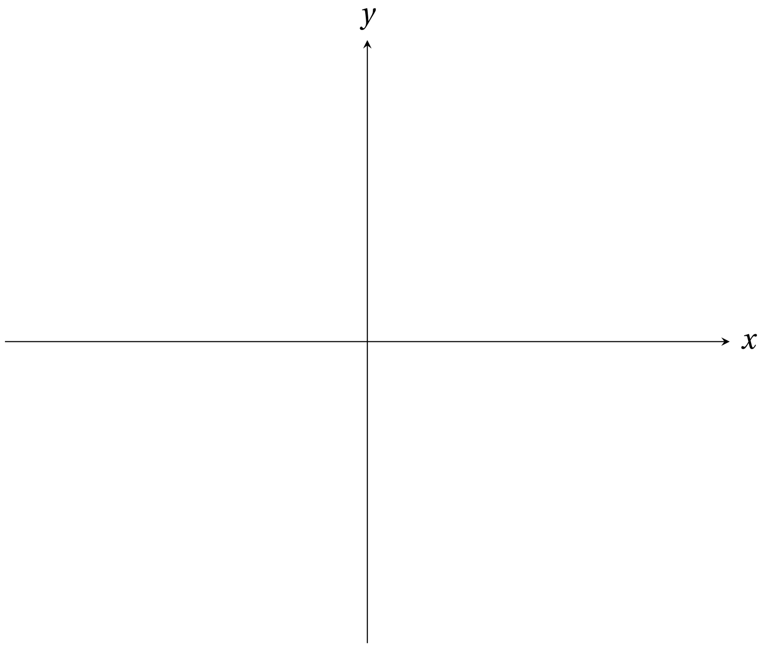
Example 5.7. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 3x - 2$, the y-axis, and $y = 5$ about the y-axis.



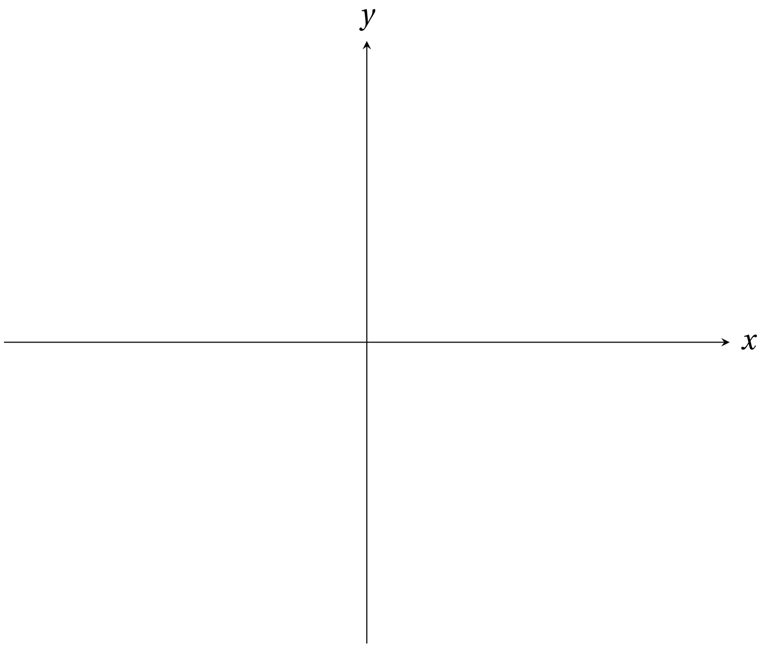
Example 5.8. Find the volume of the solid obtained by rotating the region bounded by the curves $y = -x^2 + 4x + 12$ and $y = 7$ about the line $y = 7$.



Example 5.9. The region bounded by the curves $y = \sqrt{x}$ and $y = \dfrac{x}{2}$ is rotated about the x-axis. Find the volume of the resulting solid.

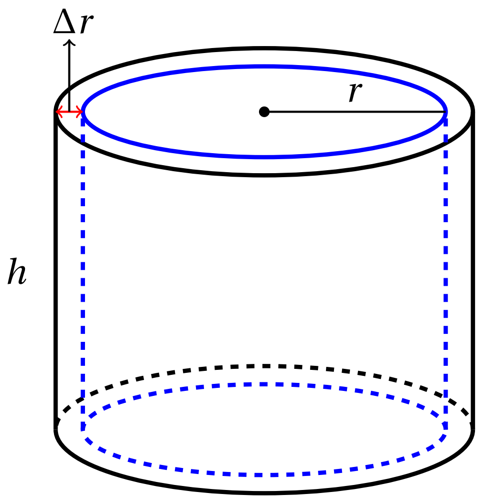


Example 5.10. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \dfrac{1}{x^2}$, $y = 0$, $x = 1$, and $x = 3$ about the line $y = -1$.



5.3 Volumes of Solids of Revolution: Shells

In this section, we introduce the **Method of Cylindrical Shells** as another method for computing the volume of a solid of revolution. Specifically, we will approximate the volume using thin cylindrical shells instead of thin slabs. Consider a thin cylindrical shell of thickness Δr, with an inner radius r and height h.



The volume of the inner cylinder is $V = \pi r^2 h$. Since Δr is very small, we can approximate the volume ΔV of our thin cylindrical shell by the differential dV. That is,

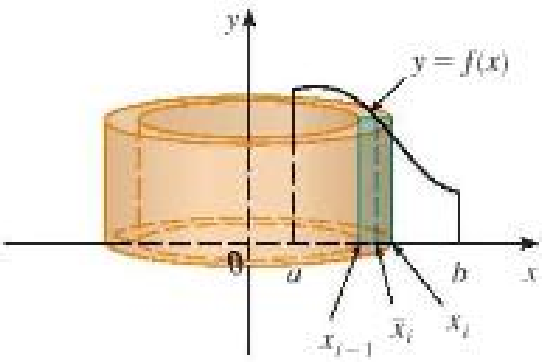
$$\Delta V \approx dV =$$

$$=$$

Here is a *heuristic argument*. Imagine cutting the shell by slitting it down the slide and rolling it out, we roughly get a thin rectangular box with

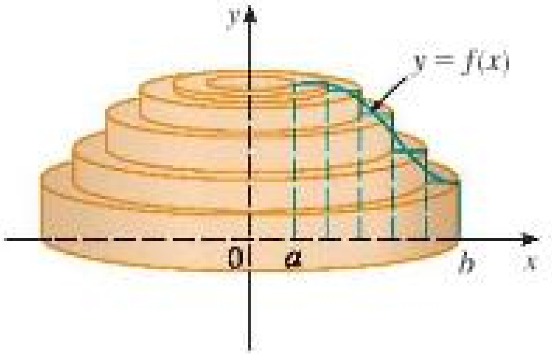
Now consider a solid generated by rotating the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ about the y-axis.

1. We approximate our region using rectangles. Divide the interval $[a, b]$ into n equally-spaced subintervals, each of length Δx; this in effect slices our solid into thin nested shells. On each subinterval, choose a sample point x_i^* and approximate the height on that subinterval by $f(x_i^*)$.



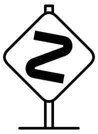
2. When this rectangle of height $f(x_i^*)$ and thickness Δx is rotated about the y-axis, it generates a thin cylindrical shell of volume $2\pi x_i^* f(x_i) \Delta x$ and this should approximately equal to the exact volume ΔV_i of the thin curved shell.

3. We may approximate the volume of our solid by summing the volumes of all the thin cylindrical shells:
$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} 2\pi x_i^* f\left(x_i^*\right) \Delta x.$$



Because the latter expression is a Riemann sum, we may take the limit as $n \longrightarrow \infty$ and obtain a definite integral, which we define to be the exact volume of our solid:
$$V = \lim_{n\to\infty}\left(\sum_{i=1}^{n} 2\pi x_i^* f\left(x_i^*\right)\Delta x\right)$$
$$=$$

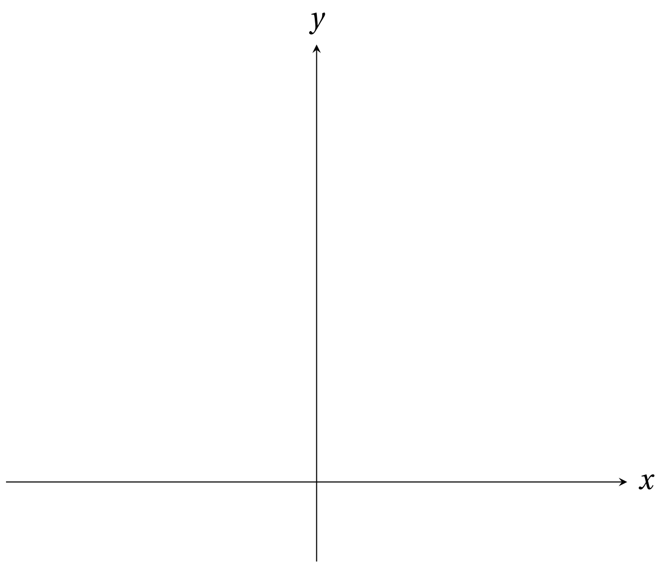
Remark 5.11. A good way of remembering this formula is to note that each thin cylindrical shell has volume
$$\boxed{\Delta V \approx 2\pi x f(x)\Delta x = \text{ circumference} \times \text{height} \times \text{thickness}}$$



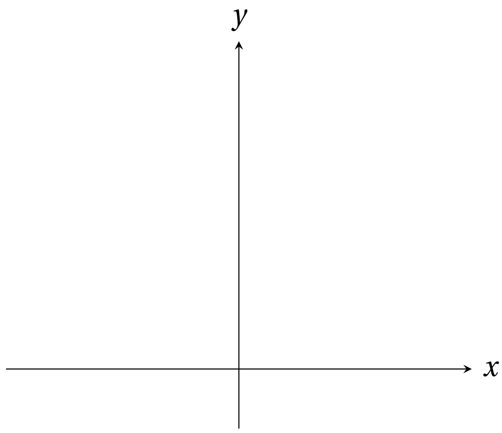
The integral formula above is only valid when the region is under the curve $y = f(x)$ between $x = a$ and $x = b$ and is being rotated about the y-axis. The formula will change slightly when we rotate around a different axis.

Example 5.12. Find the volume of the indicated solid using the method of cylindrical shells.

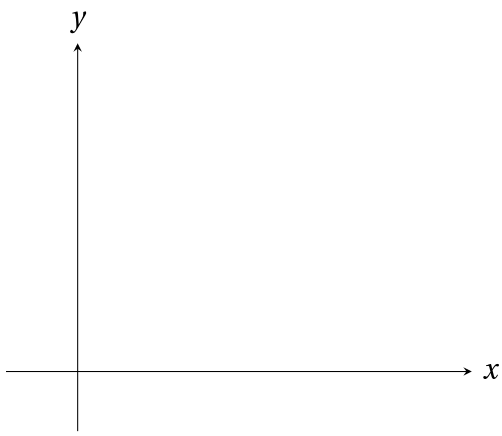
(a) The solid obtained by rotating the region bounded by $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, and $x = 3$ about the y-axis.



(b) The solid obtained by rotating the region bounded by $y = \sin\left(x^2\right)$, $y = 0$, $x = 0$, and $x = \sqrt{\pi}$ about the y-axis.



(c) The solid obtained by rotating the region bounded by $y = 0$, $y = x^3$, and $x = 1$ about the line $x = 2$.



We now have three methods at our disposal for computing volume of a solid of revolution:

♠ Disks - - need to find the ________________________.

♠ Washers - - need to find the ________________________ and the ________________________.

♠ Shells - - need to find the ________________________ and the ________________________.

But at the end of the day, we need to decide which method is the most appropriate (or rather, most convenient) one for a given solid of revolution. For this reason, it is crucial that one understands the step-by-step process of slicing, approximating and then taking limit to get a definite integral.

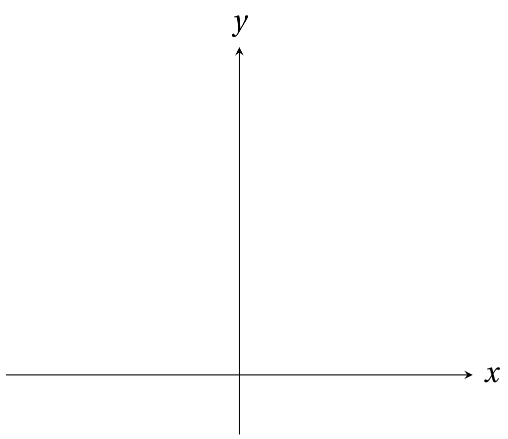
How to decide which method to use?

Below we list three simple steps for deciding which method to use:

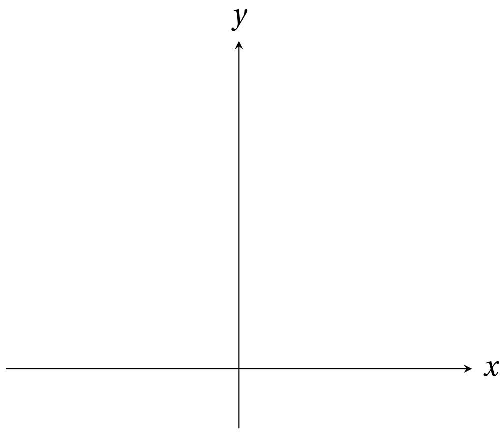
1. Sketch the region.
2. Take either a horizontal or vertical slice of that region.
3. Rotate that slice about the given axis of rotation and determine if the resulting shape is a disk, washer or shell.

Example 5.13. For the following solids of revolution, determine which method (disks, washers, or shells) to use and find the volume of the indicated solid if possible.

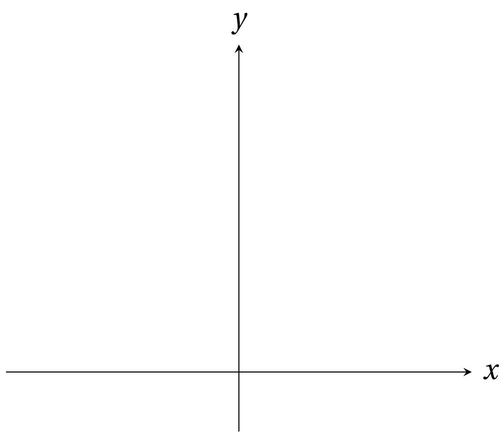
(a) The solid obtained by rotating the region bounded by $x = 0$, $y = 1$, and $y = \sin x$ about the y-axis.



(b) The solid obtained by rotating the region bounded by $y = 1$, $y = x$, and $y = \dfrac{x}{3}$ about the y-axis.



(c) The solid obtained by rotating the region bounded by $y = 1$, $y = x$, and $y = \dfrac{x}{3}$ about the x-axis.

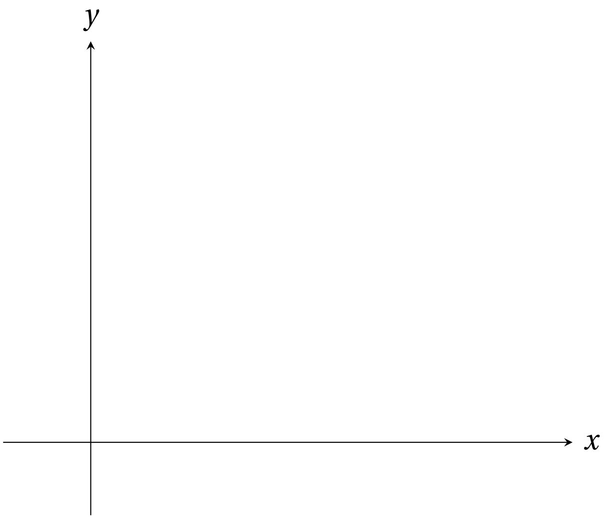


5.4 Length of a Plane Curve

We learnt how to compute areas of plane regions in Section 5.1 and volumes of solids in Section 5.2 and 5.3. In this section, we are going to see how to compute the length of a (smooth parametric) plane curve in the xy-plane; a byproduct of this investigation is the area of a surface of revolution.

Parametric equations

Consider a curve C in the xy-plane and suppose for simplicity that this curve C has a starting point P_{start} and an end point P_{end}. Let us imagine ourselves as a particle situated at the starting point P_{start} and that we move along the curve C until we reach the end point P_{end} in some amount of time.



Let $a \le t \le b$ be the time parameter and suppose $f(t), g(t)$ are continuous functions describing the position of particle at time t, *i.e.*

$$x = f(t), \quad y = g(t), \quad a \le t \le b.$$

- At time $t = a$, $(x, y) = (f(a), g(a)) = P_{\text{start}}$.
- At time $t = b$, $(x, y) = (f(b), g(b)) = P_{\text{end}}$.

We say that

$$x = f(t), \quad y = g(t), \quad a \le t \le b,$$

are **parametric equations** describing the **plane curve** C.

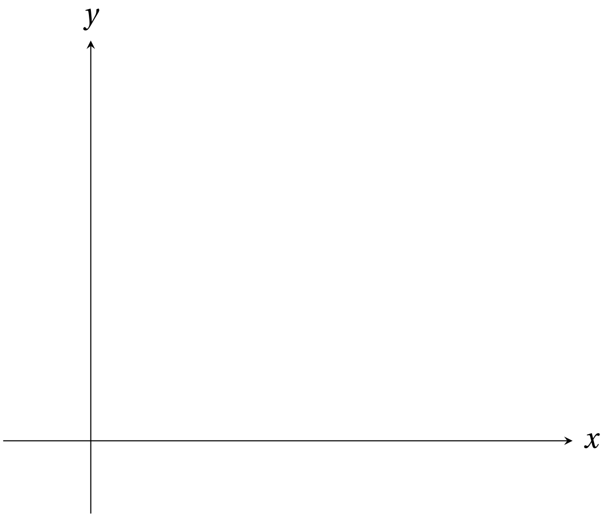
The choice of the parameter t is merely a convention, in general we can replace t with any other letter. Also, there are infinitely many pairs of parametric equations for a given plane curve C. We point out that **graphs of functions are special cases of plane curves**.

Example 5.14. Find a pair of parameter equations describing the circle with centre $(5, -3)$ and radius 2.

Arc length

Consider a smooth curve C given parametrically by $x = f(t)$, $y = g(t)$, $a \le t \le b$, where the word smooth means that both $f'(t)$ and $g'(t)$ exist and are continuous on $[a, b]$, plus some other technical assumptions. Let L denote the length of C.

1. Divide the interval $[a, b]$ into n equally-spaced subintervals, each of length Δt. This cuts the curve into n arc segments with corresponding end points $P_0, P_1, \dots, P_n$. On each subinterval, we approximate that arc segment of the curve by the straight line segment connecting the two end points. Let $P_0P_1, P_1P_2, \dots, P_{n-1}P_n$ denote all the straight line segments.



2. We zoom in on the ith arc segment of the curve. From Pythagorean theorem, its actual length ΔL_i is approximately the length of the line segment $P_{i-1}P_i$:

$$\begin{aligned}
\Delta L_i^2 \approx \left|P_{i-1}P_i\right|^2 &= \big(\text{changes in } x\big)^2 + \big(\text{changes in } y\big)^2 \\
&= \left[f(t_i) - f(t_{i-1})\right]^2 + \left[g(t_i) - g(t_{i-1})\right]^2 \\
&= \left[f'\left(t_i^*\right)\Delta t\right]^2 + \left[g'\left(\tilde{t}_i\right)\Delta t\right]^2 \\
&= \left\{\left[f'\left(t_i^*\right)\right]^2 + \left[g'\left(\tilde{t}_i\right)\right]^2\right\}(\Delta t)^2
\end{aligned}$$

   where t_i^* and $\tilde{t}_i$ are two points in the subinterval (t_{i-1}, t_i) arising from the **Mean Value Theorem for Derivatives**.

3. We may approximate the length of our curve C by summing the lengths of all the line segments:

$$L = \sum_{i=1}^{n} \Delta L_i \approx \sum_{i=1}^{n} \sqrt{\left[f'\left(t_i^*\right)\right]^2 + \left[g'\left(\tilde{t}_i\right)\right]^2}\,\Delta t.$$

   The exact length of our curve C is obtained by taking the limit $n \longrightarrow \infty$. Because t_i^* and $\tilde{t}_i$ make no difference in the limit as $n \longrightarrow \infty$, the sum above is actually a Riemann sum and the limit becomes a definite integral:

$$L = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \sqrt{\left[f'\left(t_i^*\right)\right]^2 + \left[g'\left(\tilde{t}_i\right)\right]^2}\,\Delta t\right) =$$

Arc Length of a Plane Curve

1. If $f'(t)$ and $g'(t)$ are continuous on $[a,b]$, then the **arc length** L of the curve determined by the parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$ is given by

$$\boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

2. Suppose $f'(x)$ is continuous on $[a,b]$. If the curve C is the **graph of $y = f(x)$, $a \le x \le b$**, then we treat x as the parameter and the parametric equations describing C is

   and so

   Thus the arc length L of the curve $y = f(x)$, $a \le x \le b$ is

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

3. Suppose $g'(y)$ is continuous on $[c,d]$. If the curve C is the **graph of $x = g(y)$, $c \le y \le d$**, then we treat y as the parameter and the parametric equations describing C is

   and so

   Thus the arc length L of the curve $x = g(y)$, $c \le y \le d$ is

$$\boxed{L = \int_c^d \sqrt{\left(\frac{dg}{dy}\right)^2 + 1}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy}$$

Example 5.15. Find the length of the graph $y = 1 + 6x^{3/2}$ between $x = 0$ and $x = 1$.

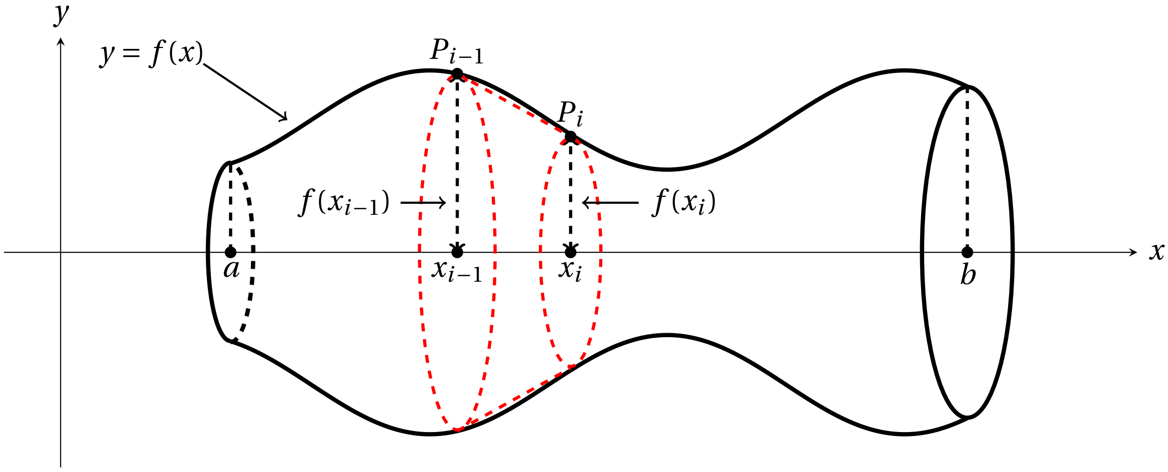
Example 5.16. Find the arc length of the curve defined by

$$x = 3t^2 + 1, \quad y = 4 - 2t^3, \quad 0 \le t \le 4.$$

Surface of revolution

When a smooth plane curve is rotated about a given line, it generates a **surface of revolution**. Consider the specific case of a smooth curve above the x-axis that is the graph of a function $y = f(x)$, $a \le x \le b$, which we rotate about the x-axis to form a surface.

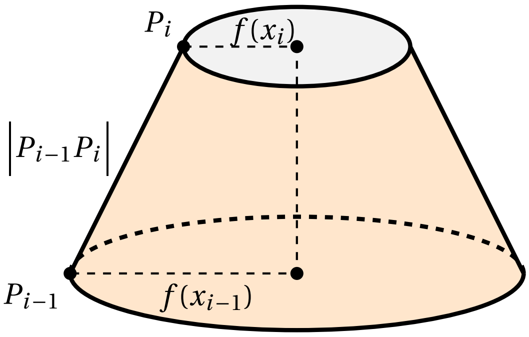
1. We approximate our curve $y = f(x)$ by straight line segments as before. Divide the interval $[a, b]$ into n equally-spaced subintervals, each of length Δx; this in effect divides our surface into narrow bands. Rotating each of these segments around the x-axis generates a **frustum of a cone** (a cone with the top chopped off).



2. We zoom in onto a particular frustum. The slant height of the frustum is

$$\left|P_{i-1}P_i\right|^2 = [\Delta x]^2 + \left[f(x_i) - f(x_{i-1})\right]^2 \approx \left\{1 + \left[f'\left(x_i^*\right)\right]^2\right\}\Delta x^2,$$

where x_i^* is some point in the subinterval (x_{i-1}, x_i) from the **Mean Value Theorem for Derivatives**.



Assuming Δx is small, we may approximate both $f(x_{i-1})$ and $f(x_i)$ as $f(x_i^*)$ since f is continuous. Thus the actual area ΔA_i of the narrow band is approximately the surface area of this frustum:

$$\begin{aligned} \Delta A_i &\approx 2\pi \left[\frac{f(x_{i-1}) + f(x_i)}{2}\right] \left|P_{i-1}P_i\right| \\ &\approx 2\pi f(x_i^*)\sqrt{1 + f'(x_i^*)}\Delta x. \end{aligned}$$

3. We may approximate the area of our surface by summing the areas of all the frustums:

$$A = \sum_{i=1}^{n} \Delta A_i \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + f'(x_i^*)}\Delta x.$$

Because the latter expression is a Riemann sum, we may take the limit as $n \longrightarrow \infty$ and obtain a definite integral, which we define to be the exact surface area of our surface of revolution:

$$A = \lim_{n\to\infty}\left(\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + f'(x_i^*)}\Delta x\right) =$$

Area of a Surface of Revolution

1. If f' is continuous on $[a,b]$, then the **surface area of the surface of revolution** obtained by rotating the curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx}$$

2. If f' and g' are continuous on $[a,b]$, then the surface area of the surface of revolution obtained by rotating the curve determined by the parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, is

$$\boxed{S = \int_a^b 2\pi g(t)\sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt}$$

Example 5.17. Find the surface area of the surface of revolution obtained by rotating the graph of $y = \sqrt{x}$ between $x = 0$ and $x = 1$ about the x-axis.

Example 5.18. Find the surface area of the surface of revolution obtained by rotating the curve defined by

$$x = r\cos t, \quad y = r\sin t, \quad 0 \le t \le \pi,$$

about the x-axis. What type of surface is this?

5.5 Work

In this final section, we examine a specific physical application of the integral: computing work. When an object moves a distance d as a result of being acted on by a constant force F acting in the direction of motion, we define the **work done by the force** on the object as

$$W = Fd.$$

It has *metric units* of newton metres (N·m) called joules (J) and *English units* of foot-pounds (ft·lb). **Keep in mind** that kilograms (kg) are (metric) units of mass, while pounds (lbs) are (English) units of force.

Example 5.19. If we lift an object, then we must apply a lifting force equal and opposite to that of gravity in order to overcome gravity's downward pull. The gravitational acceleration is 9.8 m/s^2 in metric units or 32 ft/s^2 in English units. Recall Newton's second law $F = ma$.

(a) How much work is done in lifting a 5 kilogram weight up a distance of 1 metre?

(b) How much work does it take to lift a 10 pound weight up a distance of 3 feet?

Variable force

In many applications, a force is applied to move an object over a distance, but that force is not constant. For instance, if an object is moved from $x = a$ and $x = b$, then it is likely that the force F required to move the object varies with x, *i.e.* the force is now a function of position $F(x)$.

1. Divide the interval $[a, b]$ into n equally-spaced subintervals, each of length Δx. On each subinterval, choose a sample point x_i^* and approximate the force on that subinterval as the constant force $F\left(x_i^*\right)$.

2. The exact work ΔW_i required to move the object from the left-endpoint x_{i-1} to the right-endpoint x_i is approximately $F\left(x_i^*\right)\Delta x$.

3. We may approximate the work done in moving the object from $x = a$ to $x = b$ by summing the work done across all the subintervals:

$$W = \sum_{i=1}^{n} \Delta W_i \approx \sum_{i=1}^{n} F\left(x_i^*\right)\Delta x.$$

   Because the latter expression is a Riemann sum, we may take the limit as $n \longrightarrow \infty$ and obtain a definite integral, which we define to be the work done in moving the object from $x = a$ to $x = b$:

$$W = \lim_{n\to\infty}\left(\sum_{i=1}^{n} F\left(x_i^*\right)\Delta x\right) =$$

Application to springs

A spring is a natural and common example of variable force. According to **Hooke's Law**, the force $F(x)$ required to maintain a spring stretched or compressed at a position x units from its equilibrium length is given by

$$\boxed{F(x) = kx}$$

where $k > 0$ is the spring constant depending on the spring's form and composition.

Example 5.20. How much work does it take to compress a spring with spring constant $k = 16$ lb/ft from its natural length of 1 foot to a length of 0.75 feet?

Applications to lifting objects

From the simple examples that we started with, we know that *lifting an object is doing work to overcome gravity*. If the object is non-uniform in composition during the interval, an integral is often required to compute the total work done. The exact form of such integral depends on the problem, but the underlying concept is familiar.

Divide the object being lifted into small pieces and calculate the work required to lift each individual piece, then sum up all the work. This will give a Riemann sum approximation to the desired integral.

Example 5.21. Suppose a 20 foot rope with a linear density of 0.8 lb/ft is used to haul a 5 pound bucket to the top of a building. How much work does it take to lift both the rope and the bucket to the top of the building? **Hint:** The weight of the rope changes as we lift the bucket.