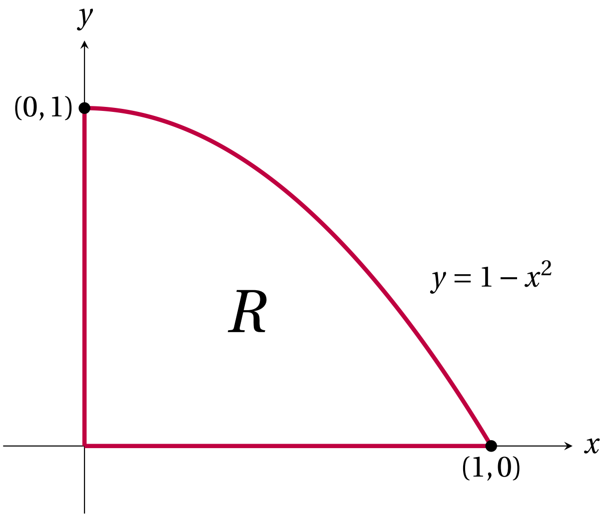
4 The Definite Integral

4.1 Introduction to Area and 4.2 The Definite Integral



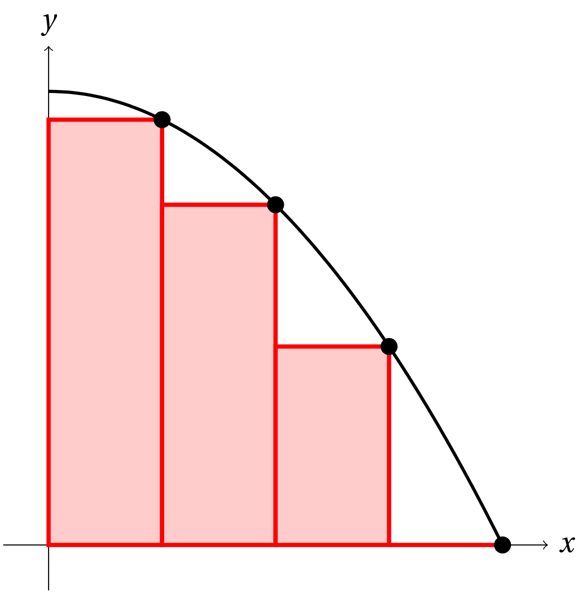
The idea of the definite integral arose from the problems of calculating lengths, areas, and volumes of curvilinear geometric figures, *i.e.* objects with a curved boundary. Consider the graph of the function $f(x) = 1 - x^2$ between $x = 0$ and $x = 1$. There is no formula for finding the exact area underneath the graph of f and above the x-axis, from $x = 0$ to $x = 1$. Let us denote this region by R. We actually discussed about how to approach this problem the first day in class: **cover the region *R* with familiar shapes whose areas can be found easily**. Well, I think it is unanimous that rectangles are the easiest one among all.

Let us demonstrate how to do these approximations with 4 rectangles of equal width. To this end, we first divide the interval $[0, 1]$ into 4 equally sized subintervals:

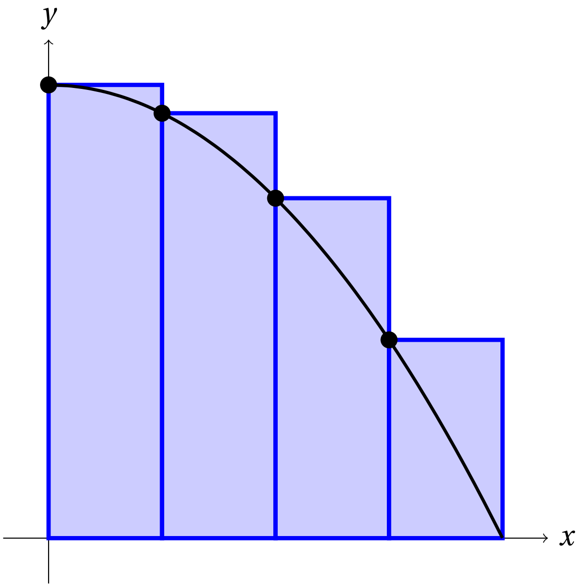
$$\left[0, \frac{1}{4}\right] \bigcup \left[\frac{1}{4}, \frac{1}{2}\right] \bigcup \left[\frac{1}{2}, \frac{3}{4}\right] \bigcup \left[\frac{3}{4}, 1\right].$$

We can clearly tell that each rectangle has width $1/4$. There are certainly many choices in terms of how to arrange these rectangles to cover the region R, but let us look at two particular choices.

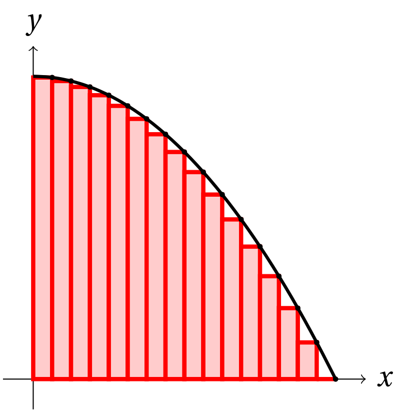
1. We **inscribe** rectangles in the region R. In this case, the height of each inscribed rectangle is given by the value of f at the right-endpoint of the subinterval.

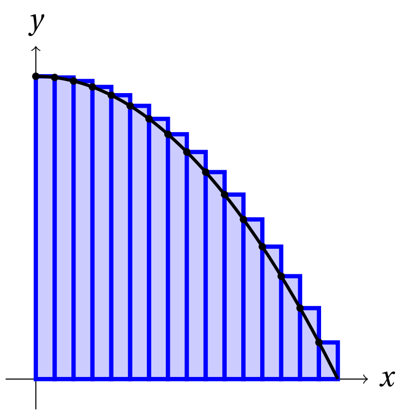


2. We **circumscribe** the region R by rectangles. In this case, the height of each circumscribed rectangle is given by the value of f at the left-endpoint of the subinterval.



The figures below are the analogous approximations with 16 rectangles.





We would expect our approximations to get better and better as the number of rectangles increase. This suggests a reasonable method to compute the area between the graph of $f(x)$, the x-axis and the lines $x = a$ and $x = b$:

1.

2.

3.

Sigma notation

The Greek letter sigma Σ (which is the Greek "S") is used with an index to represent a sum of a given list of numbers. More precisely, given a list of numbers $a_1, a_2, \ldots, a_n$, its sum is denoted by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n, \quad \text{where } i = \text{indexing variable (dummy)},$$

$$1 = \text{lower index},$$

$$n = \text{upper index}.$$

Example 4.1. Compute the following sums by writing out all of the terms.

(a) $\sum_{i=1}^{5} (2i+3) =$

(b) $\sum_{k=1}^{5} (k^2-1) =$

Special Sum Formulas

Let n be a positive integer and c be any constant. Then

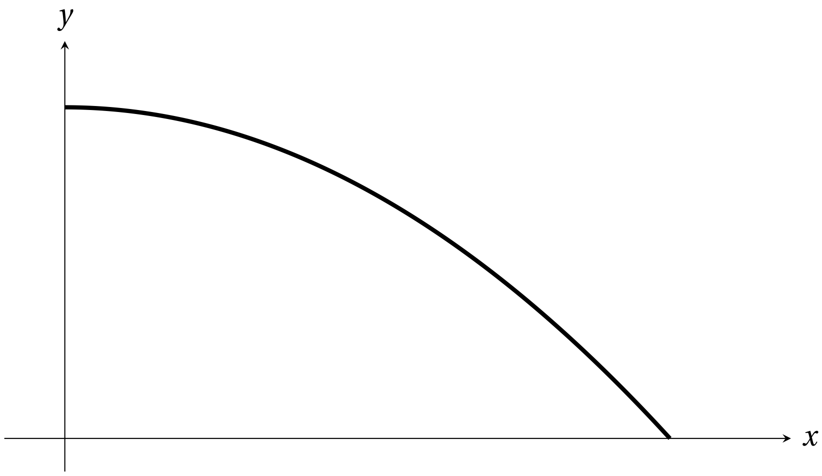
1. $\sum_{i=1}^{n} c =$
2. $\sum_{i=1}^{n} i =$
3. $\sum_{i=1}^{n} i^2 =$
4. $\sum_{i=1}^{n} i^3 =$

Example 4.2. Use the Special Sum Formulas to re-evaluate the sums in Example 4.1.

(a) $\sum_{i=1}^{5}(2i+3) =$

(b) $\sum_{k=1}^{5}(k^2-1) =$

Example 4.3. Find the area of the region underneath the graph of $f(x) = 1 - x^2$ and above the x-axis between $x = 0$ and $x = 1$ by approximating it with inscribed rectangles and then taking a limit.



1. Divide the interval $[0,1]$ into n subintervals of equal length.
   - Label the endpoints of successive subintervals as $x_0, x_1, \ldots, x_n$. These points are given by the formula:
   - Label the inscribed rectangles as $R_1, R_2, \ldots, R_n$.
   - Compute Δx, the length of each of these subintervals.

2. **Now listen to me:** Look only at a particular rectangle R_i. The height of R_i is given by the value of $f(x)$ at the right-endpoint of the corresponding subinterval.

   - This right endpoint is:

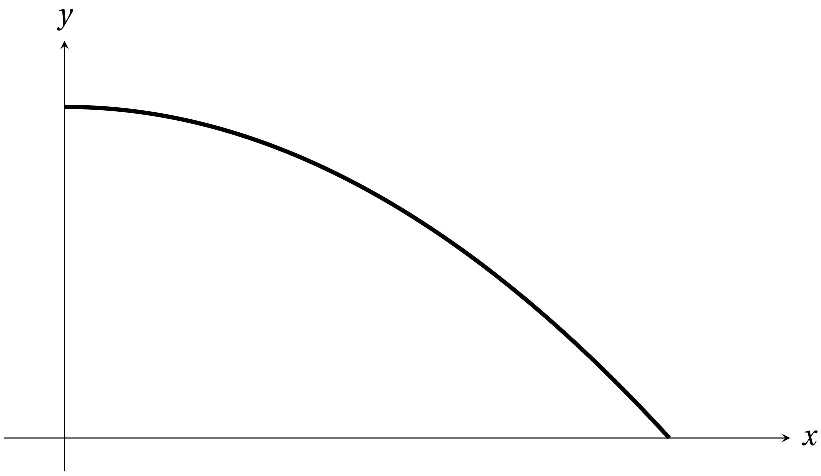
   - Consequently, the height of the rectangle R_i is

   - Finally, the area of the rectangle R_i is

3. Write an expression using the sigma notation for the total area of the n inscribed rectangles and then evaluate this sum. The answer should be a function of n only.

4. Take the limit of your answer from part (c) as $n \longrightarrow \infty$ to find the actual area.

Example 4.4. Repeat Example 4.3, but with circumscribed rectangles instead.



Riemann sum

The type of sums that arises using rectangles approximation are called **Riemann sums**. The usual procedure is as follows:

1. Break the interval $[a, b]$ into n equally-spaced subintervals, each of length $\Delta x =$
2. Pick a sample point $\overline{x_i}$ from the ith subinterval and set $f(\overline{x_i})$ to be the height of the rectangle R_i on that subinterval.
3. The area of all n rectangles, which we call a Riemann sum, is

The sample point can be any point in the subinterval, including endpoints as well. Below we list three common choices. Suppose $f(x)$ is a continuous function defined on the interval $[a, b]$. Let n be any positive integer and set $\Delta x = \dfrac{b-a}{n}$.

1. The **left-endpoint approximation** to the area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$\boxed{L_n := \sum_{i=1}^{n} f(a + (i-1)\Delta x)\,\Delta x}$$

2. The **right-endpoint approximation** to the area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$\boxed{R_n := \sum_{i=1}^{n} f(a + i\Delta x)\Delta x}$$

3. The **midpoint approximation** to the area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$\boxed{M_n := \sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)\Delta x}$$

Example 4.5. Compute the left-endpoint, right-endpoint, and midpoint approximations to the area under the graph of $f(x) = x^2$ between $x = 0$ and $x = 3$ with $n = 3$.

Definite Integral

Suppose f is a function defined on the interval $[a,b]$. The definite integral of f from $x=a$ to $x=b$ is given as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(\overline{x_i})\Delta x,$$

provided this limit exists. If it does exist, then $f(x)$ is said to be **integrable**.

Remark 4.6. If $f(x)$ is integrable, then the limit of the Riemann sum will be the same regardless of the sample points $\overline{x_i}$ chosen on each subinterval.

For $f(x)\geq 0$, a Riemann sum approximates the area under the graph of $f(x)$ and above the x-axis. However, it is possible that the terms $f(\overline{x_i})\Delta x$ in a Riemann sum is negative, which occurs when $f(\overline{x_i})<0$. *WAITTTTTTTT, we know damn well that area cannot be negative, so does this mean we just break math..........???* The precise geometric meaning is that,

The definite integral gives the signed area of
the region between the graph of $f(x)$ and the x-axis.

$$\int_a^b f(x)\,dx =$$

Example 4.7. This geometrical interpretation of the integral as a signed area allows us to compute certain definite integrals geometrically. Evaluate both $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ and $\int_{-2}^{1} x+1\,dx$.

Example 4.8. Find $\int_0^4 (5x+3)\,dx$ by taking the limit of the left-endpoint approximations.

Example 4.9. Find $\int_1^4 (x^2 - x)\,dx$ by taking the limit of the right-endpoint approximations.

Solution: The width is $\Delta x = \frac{4-1}{n} = \frac{3}{n}$, and the points x_i has the formula

$$x_i = 1 + i\Delta x = 1 + \frac{3i}{n}, \quad i = 0, 1, \ldots, n.$$

Note that $x_0 = 1 + \frac{0}{n} = 0$ and $x_n = 1 + \frac{3n}{n} = 1 + 3 = 4$. The right-endpoint approximation is

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} \left[\left(1 + \frac{3i}{n}\right)^2 - \left(1 + \frac{3i}{n}\right)\right]\left(\frac{3}{n}\right) && \left[\text{Since } f(x) = x^2 - x.\right] \\
&= \sum_{i=1}^{n} \left[\left(1 + \frac{3i}{n}\right)\left(1 + \frac{3i}{n} - 1\right)\right]\left(\frac{3}{n}\right) && \left[\text{Factor out } \left(1 + \frac{3i}{n}\right).\right] \\
&= \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)\left(\frac{9i}{n^2}\right) \\
&= \sum_{i=1}^{n} \left[\frac{9i}{n^2} + \frac{27i^2}{n^3}\right] && \left[\text{Multiply } \frac{9i}{n^2} \text{ into the first parenthesis.}\right] \\
&= \frac{9}{n^2}\left(\sum_{i=1}^{n} i\right) + \frac{27}{n^3}\left(\sum_{i=1}^{n} i^2\right) && \left[\text{Factor out the constants } \frac{9}{n^2} \text{ and } \frac{27}{n^3}.\right] \\
&= \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) && \left[\text{Use the Special Sum Formulas.}\right] \\
&= \frac{9(n+1)}{2n} + \frac{27(n+1)(2n+1)}{6n^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_1^4 (x^2 - x)\,dx = \lim_{n\to\infty} R_n &= \lim_{n\to\infty}\left[\frac{9(n+1)}{2n} + \frac{27(n+1)(2n+1)}{6n^2}\right] \\
&= \left(\lim_{n\to\infty} \frac{9n+9}{2n}\right) + \left(\lim_{n\to\infty} \frac{27(2n^2+3n+1)}{6n^2}\right) \\
&= \left(\lim_{n\to\infty} \frac{9n}{2n}\right) + \left(\lim_{n\to\infty} \frac{54n^2}{6n^2}\right) \\
&= \frac{9}{2} + \frac{54}{6} = \frac{27}{2}.
\end{aligned}$$

Terminology and properties of definite integrals

Below we list some properties of definite integrals.

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. If f is integrable on an interval containing the points a, b, c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

   no matter what the order of a, b, c is.

4. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Example 4.10. Not all functions are integrable! Consider the following function on the interval $[0,1]$, defined by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$

4.3 The First Fundamental Theorem of Calculus

Similar to finding derivatives, we want to develop rules that will allow us to compute definite integrals without having to take limits of Riemann sums every single time. In the next two sections, we will see that definite integrals can be easily evaluated once we have found an antiderivative of the integrand. This fundamental connection is captured in a theorem called the Fundamental Theorem of Calculus (FTC). We will have two different but equivalent versions of the FTC.

Theorem 4.11 (Comparison and Linearity Properties of the Definite Integral)**.** *Suppose f and g are integrable on $[a,b]$.*

1. *If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.*
2. *If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.*
3. *If $m \leq f(x) \leq M$ on $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.*
4. *If c and k are constants, then*
$$\int_a^b cf(x) + kg(x)\,dx = c\int_a^b f(x)\,dx + k\int_a^b g(x)\,dx.$$

First Fundamental Theorem of Calculus (FTC1)

Suppose f is continuous on $[a,b]$. Then the **accumulation function**

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$, differentiable on (a,b), and $g'(x) = f(x)$ for all x in (a,b). That is,

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Example 4.12. Find the following derivatives.

(a) $\dfrac{d}{dx}\left(\displaystyle\int_0^x t^2\,dt\right)$

(b) $\dfrac{d}{dx}\left(\displaystyle\int_1^x \frac{\sin t}{1+t}\,dt\right)$

(c) $\dfrac{d}{dx}\left(\displaystyle\int_x^4 \sqrt{u}\,du\right)$

(d) $\dfrac{d}{dx}\left(\displaystyle\int_2^{x^3} \frac{t}{t^4+1}\,dt\right)$

(e) $\dfrac{d}{dx}\left(\int_{\cos x}^{3} t^5\,dt\right)$

Example 4.13. Find a formula for

$$g(x) = \int_1^x t^4\,dt.$$

Hint: Note that FTC1 gives us $g'(x)$. What is $g(1)$?

Solution: Applying FTC1 to $g(x)$ gives

$$g'(x) = \frac{d}{dx}\left(\int_1^x t^4\,dt\right) = x^4.$$

This says that $g(x)$ is the general antiderivative of x^4, *i.e.*

$$g(x) = \int g'(x)\,dx = \int x^4\,dx = \frac{x^5}{5} + C.$$

For $x = 1$,

$$g(1) = \int_1^1 t^4\,dt = 0.$$

This means that

$$0 = g(1) = \frac{1}{5} + C \implies C = -\frac{1}{5}.$$

Hence

$$g(x) = \frac{x^5}{5} - \frac{1}{5}.$$

We just demonstrated that it is possible to compute definite integrals using antiderivatives.

4.4 The Second Fundamental Theorem of Calculus and the Method of Substitution

The following is convincingly the most important and powerful theorem in terms of evaluating definite integrals!

Second Fundamental Theorem of Calculus (FTC2)

Suppose $f(x)$ is continuous on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of f.

Example 4.14. Evaluate the following definite integrals using FTC2.

(a) $\displaystyle\int_0^1 1 - x^2\,dx$

(b) $\displaystyle\int_0^4 5x + 3\,dx$

(c) $\displaystyle\int_{-1}^2 x^3 - \frac{1}{3}x^5 + 1\,dx$

(d) $\displaystyle\int_0^{\pi/4} 2\cos(2x)\,dx$

The following is merely a restatement of the FTC2. It has the benefit of being phrased in a way that is useful in applications in the physical and natural sciences.

Net Change Theorem

The integral of a rate of change is the net change:

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Example 4.15. If an object moves along a straight line with position function $s(t)$ and velocity $v(t) = s'(t)$, then the integral of the velocity is the change in position, *i.e.*

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(b) - s(a).$$

Suppose a particle moves along a straight line with velocity given by $v(t) = t^3 - 5t^2 + 6t$. If $s(0) = 2$, where is the particle at $t = 3$?

The substitution rule

Essentially, FTC2 tells us that the key to evaluating definite integrals is finding an antiderivative of the integrand. So far we have only looked at fairly simple functions, what about scary, complicated functions? The integration technique of substitution is "undoing" the Chain Rule in disguise.

Substitution Rule

Suppose $g(x)$ is a differentiable function whose range is an interval I and $f(x)$ is continuous on I. If F is an antiderivative of f, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

Example 4.16. Find the following indefinite integrals.

(a) $\int \sqrt{3x+1}\,dx$

(b) $\int \frac{\cos\theta}{\sin^3\theta}\,d\theta$

Example 4.17. Find the following indefinite integrals.

(a) $\displaystyle\int \frac{z\cos\left(\sqrt[3]{z^2+3}\right)}{\left(\sqrt[3]{z^2+3}\right)^2}\,dz$

(b) $\displaystyle\int x^6 \sin(3x^7+9)\sqrt[3]{\cos(3x^7+9)}\,dx$

Solution: Let $u(x) = \cos(3x^7+9)$. Then

$$du = \left[-\sin(3x^7+9)\right]\left[21x^6\right]dx = -21x^6\sin(3x^7+9)\,dx,$$

or

$$-\frac{1}{21}\,du = x^6\sin(3x^7+9)\,dx.$$

Thus

$$\begin{aligned}\int x^6\sin(3x^7+9)\sqrt[3]{\cos(3x^7+9)}\,dx = \int -\frac{1}{21}\sqrt[3]{u}\,du &= -\frac{1}{21}\int u^{1/3}\,du\\ &= -\frac{1}{21}\left(\frac{u^{4/3}}{\frac{4}{3}}\right)+C\\ &= -\frac{1}{21}\left(\frac{3}{4}\right)\left[\cos(3x^7+9)\right]^{4/3}+C\\ &= -\frac{1}{28}\left[\cos(3x^7+9)\right]^{4/3}+C.\end{aligned}$$

When using the Substitution Rule to evaluate a definite integral, we can

1. either use substitution to find the indefinite integral, *i.e.* the antiderivative, then evaluate this antiderivative at the given endpoints, or
2. we can use the substitution $u = g(x)$ to change the limits of integration, *i.e.*

$$\int_a^b f(g(x))g'(x)\,d = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 4.18. Evaluate the following definite integrals.

(a) $\int_0^1 x^2(2+x^3)^5\,dx$

(b) $\int_{0}^{\sqrt{\pi}} \theta \sin(\theta^2)\, d\theta$

(c) $\int_{1}^{3} \frac{x^2+1}{\sqrt{x^3+3x}}\, dx$

4.5 The Mean Value Theorem for Integrals and the Use of Symmetry

Definition 4.19. If f is an integrable function on the interval $[a,b]$, then the **average value** of f on $[a,b]$ is given by

$$\boxed{f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

This actually arises from the idea of Riemann sum. Divide the interval $[a,b]$ into n equally-spaced subintervals, each of width $\Delta x = (b-a)/n$. Pick a sample point $\overline{x_i}$ from the ith subinterval, $i = 1,2,\ldots,n$. This gives the set of points $\{f(\overline{x_1}), f(\overline{x_2}),\ldots, f(\overline{x_n})\}$ and the average value of this n numbers is

$$\frac{f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})}{n} = \frac{1}{n}\sum_{i=1}^{n} f(\overline{x_i}) = \frac{1}{b-a}\sum_{i=1}^{n} f(\overline{x_i})\Delta x.$$

Taking the limit as $n \longrightarrow \infty$, we obtain

$$\begin{aligned} f_{\text{avg}} = \lim_{n\to\infty}\left(\frac{1}{b-a}\sum_{i=1}^{n} f(\overline{x_i})\Delta x\right) &= \frac{1}{b-a}\left(\lim_{n\to\infty}\sum_{i=1}^{n} f(\overline{x_i})\Delta x\right) \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

There is a nice geometric interpretation of what the average value is for functions $f(x) \geq 0$ on $[a,b]$. We can rewrite the above as

$$f_{\text{avg}}(b-a) = \int_a^b f(x)\,dx.$$

Example 4.20. Find the average values of the function on the given interval.

(a) $f(x) = \sin x$ on $[0, \pi]$

(b) $f(x) = \sin^2 x \cos x$ on $[0, \pi/2]$

Mean Value Theorem for Integrals

Suppose f is continuous on $[a, b]$. Then there is a number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Example 4.21. Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function on the given interval.

(a) $f(x) = 3x^2 - 2$ on $[1, 3]$

(b) $f(x) = \dfrac{1}{(3x+2)^2}$ on $[0, 5]$

Symmetry Theorem

Suppose f is continuous on $[-a, a]$. If f is odd, then

$$\int_{-a}^{a} f(x)\, dx = 0.$$

If f is even, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

Example 4.22. Evaluate the following definite integrals.

(a) $\displaystyle\int_{-2}^{2} |x|\, dx$

(b) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \frac{\tan x}{1+\cos x + x^6}\, dx$

Periodicity Theorem

Suppose $f(x)$ is periodic with period p. Then for any a, b,

$$\int_a^b f(x)\,dx = \int_{a+p}^{b+p} f(x)\,dx.$$

Example 4.23. Evaluate $\int_0^{2\pi} |\sin x|\,dx$.

4.6 Numerical Integration

Recall that if a function f is continuous on $[a,b]$, then it is integrable on $[a,b]$, *i.e.* the definite integral $\int_a^b f(x)\,dx$ exists. Consider the following two integrals

$$\int_0^{\pi} \sin(x^2)\,dx \quad \text{and} \quad \int_0^1 \sqrt{1-x^4}\,dx.$$

These integrals exist because the integrands are continuous over respective intervals, yet our limited integration machineries don't really tell us how to evaluate them. In fact, there are functions which simply do not have an antiderivative. Therefore we will numerically integrate these functions, *i.e.* we will approximate the definite integral using "simple geometry".

We know one technique for approximating integrals already: Riemann sums. Specifically, for any fixed n, especially n large,

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x,$$

where

$$\begin{aligned}
\Delta x &= \frac{b-a}{n} = \text{ the width of rectangles}\\
x_i &= a + i\Delta x, \quad i = 0, 1, \ldots, n\\
\overline{x_i} &= \text{ any sample point in the } i\text{th subinterval } [x_{i-1}, x_i].
\end{aligned}$$

We generally choose $\overline{x_i}$ to be left-endpoints, right-endpoints or midpoints.

1. The **left-endpoint approximation** to the area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$\boxed{L_n := \sum_{i=1}^{n} f(x_{i-1})\,\Delta x}$$

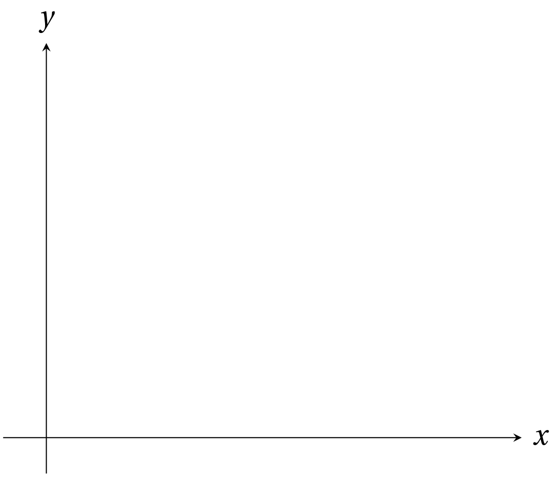
2. The **right-endpoint approximation** to the area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$\boxed{R_n := \sum_{i=1}^{n} f(x_i)\Delta x}$$

3. The **midpoint approximation** to the area under the graph of $f(x)$ between $x = a$ and $x = b$ is given by

$$\boxed{M_n := \sum_{i=1}^{n} f\left(x_{i-\frac{1}{2}}\right)\Delta x}$$

It turns out that there are other, sometimes better, methods for approximating integrals than Riemann sums. With a Riemann sum, we are approximating the (signed) area under the graph by rectangles. If we use different shapes that fit "tighter" to the curve, then we should get more accurate approximation. We are only going to try trapezoids but one could also try parabolas; see textbook for more details.



Trapezoidal Rule

Let $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x, i = 0, 1, \ldots, n$. The trapezoidal rule is

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right].$$

Example 4.24. Find the Trapezoidal Rule with $n = 4$ to approximate $\displaystyle\int_1^5 \frac{1}{x}\,dx$.