3 Applications of the Derivative

3.1 Maxima and Minima

Definition 3.1. Given a function $f(x)$ defined on the set S (could be intervals or $\mathbb{R}$), we say that

1. $f(x)$ has a **global (absolute) maximum** at $x = c$ on the set S if $f(x) \leq f(c)$ for every x in S. The number $f(c)$ is called the **maximum value** of $f(x)$ on S.
2. $f(x)$ has a **global (absolute) minimum** at $x = c$ on the set S if $f(x) \geq f(c)$ for every x in S. The number $f(c)$ is called the **minimum value** of $f(x)$ on S.
3. Together, the maximum and minimum values of $f(x)$ on S are referred to as the **extreme values** of $f(x)$ on S.

The *existence of extreme values* depend on the function $f(x)$ and the underlying set S.

Example 3.2. Consider the function $f(x) = \dfrac{1}{1+x^2}$. Determine, if any, the extreme values of f on the following sets:

(a) $S = \mathbb{R}$

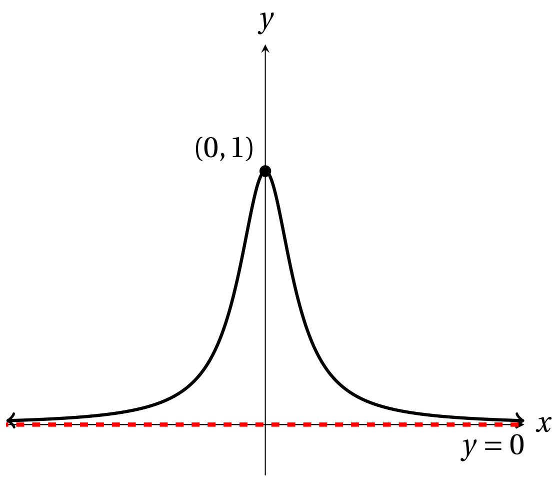
(b) $S = [-2, 1]$

(c) $S = (-2, 0)$

(d) $S = (-3, 2]$

(e) $S = (1, 4]$

Solution: The given function $f(x) = \dfrac{1}{1+x^2}$ is a rational function and it is relatively easy to graph it. The graph is symmetric with respect to the y-axis, has no vertical asymptote and x-intercept, has $(0, 1)$ as y-intercept and horizontal asymptote $y = 0$.



(a) $S = \mathbb{R}$: The maximum value is $y = 1$ and there is no minimum value.

(b) $S = [-2, 1]$: The maximum value is $y = 1$ and the minimum value is $y = f(-2) = 1/5$.

(c) $S = (-2, 0)$: There are no maximum values and minimum values. The maximum value seems to be $y = 1$ again, but the point where this occurs, $x = 0$, is not in the set $S = (-2, 0)$.

(d) $S = (-3, 2]$: The maximum value is $y = 1$ and the is no minimum value.

(e) $S = (1, 4]$: There is no maximum value and the minimum value is $y = f(4) = 1/17$.

The last example provides a way of finding extreme values: graph the function. The (recurring) fundamental fact of life is that there are functions whose graphs cannot be sketched accurately by hand, and even if we could, it may very well be onerous. This prompts a practical question:

Is there an algebraic way to find extreme values of
any given functions, *i.e.* without graphing functions?

Before we explore this problem, we state a beautiful theorem that guarantees the existence of extreme values, though it doesn't really tell us what the extreme values are.

Extreme Value Theorem (EVT)
Suppose f is continuous on the closed interval $[a,b]$. Then f attains both a maximum value and a minimum value at some points in the interval $[a,b]$.

Remarks 3.3.

1. The theorem is false if
   (a) f is not continuous;
   (b) $[a,b]$ is replaced with either $(a,b]$ or $[a,b)$ or (a,b).

2. If f fails to be continuous and/or is not defined on a closed interval $[a,b]$, then all bets are off, *i.e.* we might or might not have extreme values. Below are examples of functions that do not satisfy the condition of EVT, yet they still have maximum and minimum values.

Where do extreme values occur than? Let us draw some graphs.

Critical Point Theorem (CPT)

Let f be defined on an interval I containing the point $x = c$. If $f(c)$ is an extreme value of f on I, then $x = c$ must be a **critical point**. That is, $x = c$ is either

1. an endpoint of the interval I; or
2. a **stationary point** of f in the interval I, that is, a point where $f'(c) = 0$; or
3. a **singular point** of f in the interval I, that is, a point where $f'(c)$ DNE.

The Critical Point Theorem asserts that if a function has an extreme value, then it must happen at one of the critical points of f. This means that to find extreme values, we only need to look for critical points and carefully examine if any of the corresponding y-values are extreme values.

In view of both EVT and CPT, we can write down a straightforward procedure for finding extreme values of a continuous function on a closed interval I:

1. Find the derivative $f'(x)$.
2. Find all the critical points of f that belong to I.
3. Evaluate f at each of these critical points. The largest of these values is the maximum value; the smallest is the minimum value.

Example 3.4. Identify the critical points and find the extreme values of the function $f(x) = x^4 - 2x^2 + 3$ on the interval $[-2, 3]$.

Example 3.5. Identify the critical points and find the extreme values of the function $f(x) = x^{2/3}(x-1)$ on the interval $[-1, 2]$.

3.6 The Mean Value Theorem for Derivatives

Consider some "nice enough" function on a closed interval $[a, b]$, and draw the secant line between the points $(a, f(a))$ and $(b, f(b))$. Geometrically, we can clearly see that we can always find at least a point $x = c$ in the interval (a, b) such that $f'(c)$ equals to the slope of the secant line. This result is known as the Mean Value Theorem.

Mean Value Theorem (MVT)

Suppose f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). There there is at least one number $x = c$ in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remarks 3.6. 1. Similar to EVT, the theorem is false if

(a) f is not continuous on $[a, b]$;

(b) f is not differentiable on (a, b).

2. Similar to EVT, MVT only tells us there is some point $x = c$ satisfying $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ but it doesn't give the actual value of that c.

3. MVT is similar in spirit to the Intermediate Value Theorem (see Section 1.6). The main difference is that MVT deals with the derivative $f'(x)$ instead of $f(x)$ itself.

Example 3.7. Find the value(s) of c guaranteed by the Mean Value Theorem for the following functions on their corresponding intervals.

(a) $f(x) = x^3 - x$ on $[-2, 2]$

(b) $f(x) = x^4 - x - 5$ on $[-1, 2]$

Example 3.8. Show that the function $f(x) = 1 - x^{2/3}$ satisfies $f(1) = f(-1)$ and there is no number c in the interval $(-1,1)$ such that $f'(c) = 0$. Why does this not contradict the Mean Value Theorem?

Solution: We see that

$$\begin{aligned} f(1) &= 1 - (1)^{2/3} = 0 \\ f(-1) &= 1 - (-1)^{2/3} = 1 - \left[(-1)^2\right]^{1/3} = 0. \end{aligned}$$

The derivative of f is

$$f'(x) = -\frac{2}{3}x^{-1/3} = -\frac{2}{3x^{1/3}} = -\frac{2}{3\sqrt[3]{x}}$$

and clearly $f'(x) \neq 0$ for every x in $(-1,0) \cup (0,1)$ and $f'(0)$ DNE. However, this doesn't contradict the Mean Value Theorem since f is not differentiable at the point $x = 0$, which is in the interval $(-1,1)$.

Yes, the Mean Value Theorem is a beautiful geometrical result but why should we care if we can find a point where its derivative equals to the slope of the secant line? As it turns out, MVT is often used to gain knowledge of the function itself from information about the derivative of the function. Essentially, it tells us how f' affects f.

Example 3.9. Let f be a differentiable function. Suppose $f(0) = 2$ and $1 \leq f'(x) \leq 3$ for all x in $\mathbb{R}$. What are the maximum and minimum possible value of $f(3)$ and $f(-5)$?

Proposition 3.10. *Suppose f and g are differentiable functions such that $f(a) = g(a)$ and $f'(x) \geq g(x)$ for all $x \geq a$. Then $f(x) \geq g(x)$ for all $x \geq a$.*

Proof. Let $h(x) = f(x) - g(x)$. Then $h(a) = f(a) - g(a) = 0$ by assumption. The derivative of h is

$$h'(x) = f'(x) - g'(x) \geq 0 \quad \text{for all } x \geq a.$$

This says that the instantaneous rate of change of $f(x)$ is always nonnegative for all $x \geq a$, and we deduce that h is a nondecreasing function on $[a, \infty)$, *i.e.*

$$h(x) \geq h(a) = 0 \quad \text{for all } x \geq a.$$

It follows that $f(x) - g(x) \geq 0$ or $f(x) \geq g(x)$ for all $x \geq a$.

□

Example 3.11. Use Proposition **??** to show that for all $x \geq 1$, the graph of $f(x) = \sqrt{x}$ lies below the tangent line of $f(x)$ at $x = 1$.

Finally, MVT has important corollaries which are relevant to the concept of antiderivatives; see Section 3.8 later.

1. Suppose $f'(x) = 0$ for all x in the interval (a, b). Then $f(x)$ is constant on (a, b).
2. Suppose $f'(x) = g'(x)$ for all x in the interval (a, b). Then $f(x) = g(x) + C$ on (a, b) for some constant C.

3.2 Monotonicity and Concavity

Definition 3.12.

Let f be defined on an interval I (could be open, closed, or neither). We say that

1. f is **increasing** on I if
$$x < y \implies f(x) < f(y) \qquad \text{for every } x, y \text{ in } I.$$
2. f is **decreasing** on I if
$$x < y \implies f(x) > f(y) \qquad \text{for every } x, y \text{in } I.$$
3. f is **strictly monotonic** on I if it is either increasing or decreasing on I but not both.

Given a function f, we can clearly tell where is f increasing or decreasing if the graph of f is given, but is it possible to find this information without having to graph the function? The answer is, of course, YES, and the main idea is to track the tangent line as we move to the right.

Monotonicity Theorem

Let f be continuous on an interval $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

Proof. Take any 2 points, say x and y in an interval (a, b), such that $x < y$. MVT asserts that there is some point c between x and y such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

or

$$f(y) - f(x) = f'(c)(y - x).$$

There are two cases to consider.

1. If $f'(c) > 0$, then $f(y) - f(x) = (+)(+) = (+)$ which means that $f(y) - f(x) > 0$.
2. If $f'(c) < 0$, then $f(y) - f(x) = (-)(+) = (-)$ which means that $f(y) - f(x) < 0$.

$\square$

Example 3.13. Find the intervals where $f(x) = x^4 - 2x^2 + 3$ is increasing or decreasing.

Example 3.14. Find the intervals where $g(x) = \dfrac{x}{x^2 + 2x + 3}$ is increasing or decreasing.

Remark 3.15. Observe that the Monotonicity Theorem applies if $f'(x) > 0$ or $f'(x) < 0$. Sometimes, stationary points, *i.e.* points where $f'(c) = 0$, should be included in the intervals of increasing or decreasing. This can be determined by examining the sign of $f'(x)$ near $x = c$.

1. Consider the function $f(x) = x^3$.

2. Consider the function $f(x) = x^4$.

What does $f''(x)$ tells us about $f(x)$ then? From the Monotonicity Theorem, we see that

1. if $f''(x) > 0$ on some interval (a, b), then $f'(x)$ is increasing on (a, b), *i.e.* slope of tangent lines is getting larger to the right;

2. if $f''(x) < 0$ on some interval (a, b), then $f'(x)$ is decreasing on (a, b), *i.e.* slope of tangent lines is getting smaller to the right.

Roughly speaking, $f''(x)$ tells us about the “wiggliness” or “curviness” of the function itself.

Concavity Theorem

Suppose f is twice differentiable on an open interval I, *i.e.* $f''(x)$ exists for all x in I.

1. If $f''(x) > 0$ for every $x \in I$, then $f(x)$ is **concave up** on I.
2. If $f''(x) < 0$ for every $x \in I$, then $f(x)$ is **concave down** on I.

Definition 3.16. A point $(c, f(c))$ on the graph of $y = f(x)$ where the function changes concavity, *i.e.* f'' changes sign, is called an **inflection point**.

Example 3.17. Consider the function $f(x) = 4x^3 + 3x^2 - 6x + 1$.

(a) Determine the intervals where $f(x)$ is increasing or decreasing.

(b) Determine the intervals where $f(x)$ is concave up or concave down. List, if any, inflection points of $f(x)$.

Example 3.18. Consider the function $f(x) = \dfrac{x}{4+x^2}$.

(a) Determine the intervals where $f(x)$ is increasing or decreasing.

(b) Determine the intervals where $f(x)$ is concave up or concave down. List, if any, inflection points of $f(x)$.

The last two examples demonstrate that if there are inflection points, then they must be located at points where $f''(c) = 0$. The next example shows that the converse is false, *i.e.* there exists a point $x = c$ where $f''(c) = 0$ but $(c, f(c))$ is not an inflection point.

Example 3.19. Consider the function $f(x) = x^4$.

Example 3.20 ($f'' \neq 0$ but there is an inflection point)**.** Consider the function $f(x) = x^{1/3}$.

In a nutshell, points where $f''(c) = 0$ or $f''(c)$ DNE are "candidates" for inflection points. We need to check if the concavity actually changes at these points before we can decide if it is really an inflection point!

3.3 Local Extrema and Extrema on Open Intervals

- A function f has a **local maximum** at $x = c$ if $f(x)$ is defined in a neighbourhood of $x = c$ and $f(x) \leq f(c)$ for all x near c, in which case $f(c)$ is the **local maximum value**.
- A function f has a **local minimum** at $x = c$ if $f(x)$ is defined in a neighbourhood of $x = c$ and $f(x) \geq f(c)$ for all x near c, in which case $f(c)$ is the **local minimum value**.
- A function f has a local extrema at $x = c$ if it has either a local maximum or a local minimum but not both. The number $f(c)$ is then called the local extreme value.

First Derivative Test

Suppose $x = c$ is a critical point of a continuous function f.

1. If $f'(x)$ is positive to the left of c and negative to the right of c, then f has a local maximum at $x = c$.
2. If $f'(x)$ is negative to the left of c and positive to the right of c, then f has a local minimum at $x = c$.
3. If $f'(x)$ has the same sign on both sides of c, then f has neither a local maximum or a local minimum at $x = c$.

Example 3.21. Find all the critical points and use the First Derivative Test to decide which of the critical points give a local maximum and which give a local minimum.

(a) $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 18x + 5$

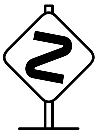
(b) $f(x) = 3x^4 - 4x^3 - 2$

(c) $f(x) = \cos^2 x - 2\sin x$ on the interval $(0, 2\pi)$

Second Derivative Test

Suppose f'' is continuous near $x = c$.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.



If $f''(c) = 0$, then the Second Derivative Test is inconclusive. This means that we need to use the First Derivative Test to decide.

Example 3.22. Find all the critical points and use the Second Derivative Test (whenever possible) to determine which of the critical points give a local maximum and which give a local minimum.

(a) $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 18x + 5$

(b) $f(x) = x^4 - 4x^3$

Example 3.23. Find the global extreme values of $f(x) = x^2 + \dfrac{1}{x^2}$.

3.5 Graphing Functions Using Calculus

This section is all about applying the techniques we have learned in the past few sections to graphing functions. To most accurately sketch the graph of a function f, we find all the following:

1. **Domain and Range:**

2. **Symmetry and Periodicity:** Symmetry and periodicity allows you to construct the whole graph from a smaller piece.

3. **Intercepts:** Where are the x-intercepts and the y-intercept?

4. **Asymptotes:**

5. **Intervals of increasing/decreasing:** These are determined by the sign of $f'(x)$.

6. **Local maximum/minimum:** Critical points (ignoring endpoints) are the candidates for local maximum/minimum.

7. **Concavity and inflection points:** These are determined by the sign of $f''(x)$.

Example 3.24. Sketch a graph of the function $f(x) = \dfrac{1}{9}x^3 - 3x$.

Domain:

Range:

Symmetry:

Periodicity:

x-intercept(s):

y-intercept:

VA:

HA or SA:

Increasing:

Decreasing:

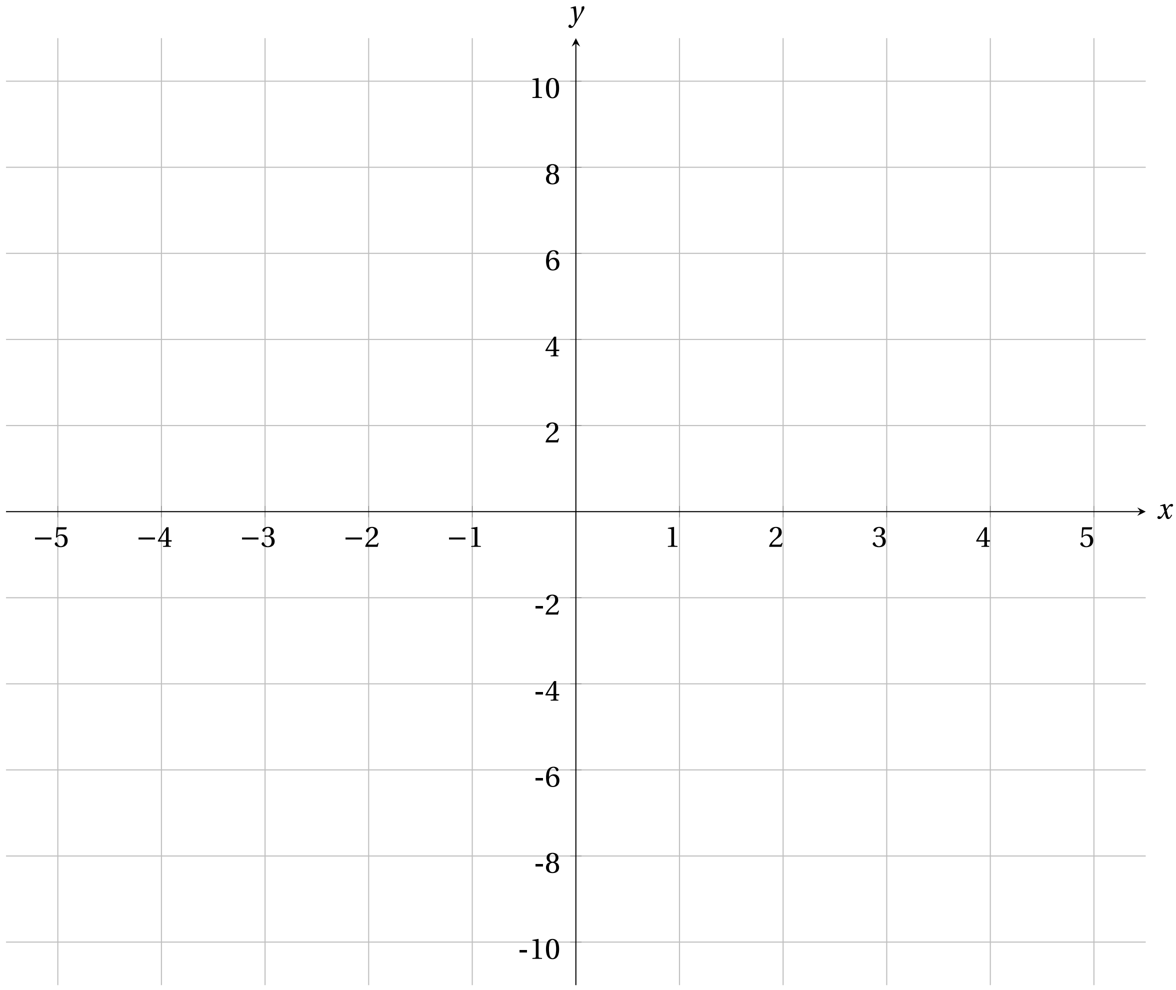
Concave up:

Concave down:

Local maximum(s):

Local minimum(s):

Inflection point(s):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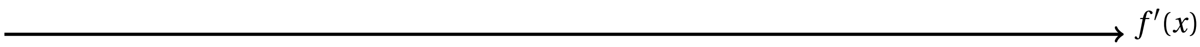

$f'(x)$


$f''(x)$

Example 3.25. Sketch a graph of the function $f(x) = \dfrac{3x+4}{x+1}$.

Domain:

Range:

Symmetry:

Periodicity:

x-intercept(s):

y-intercept:

VA:

HA or SA:

Increasing:

Decreasing:

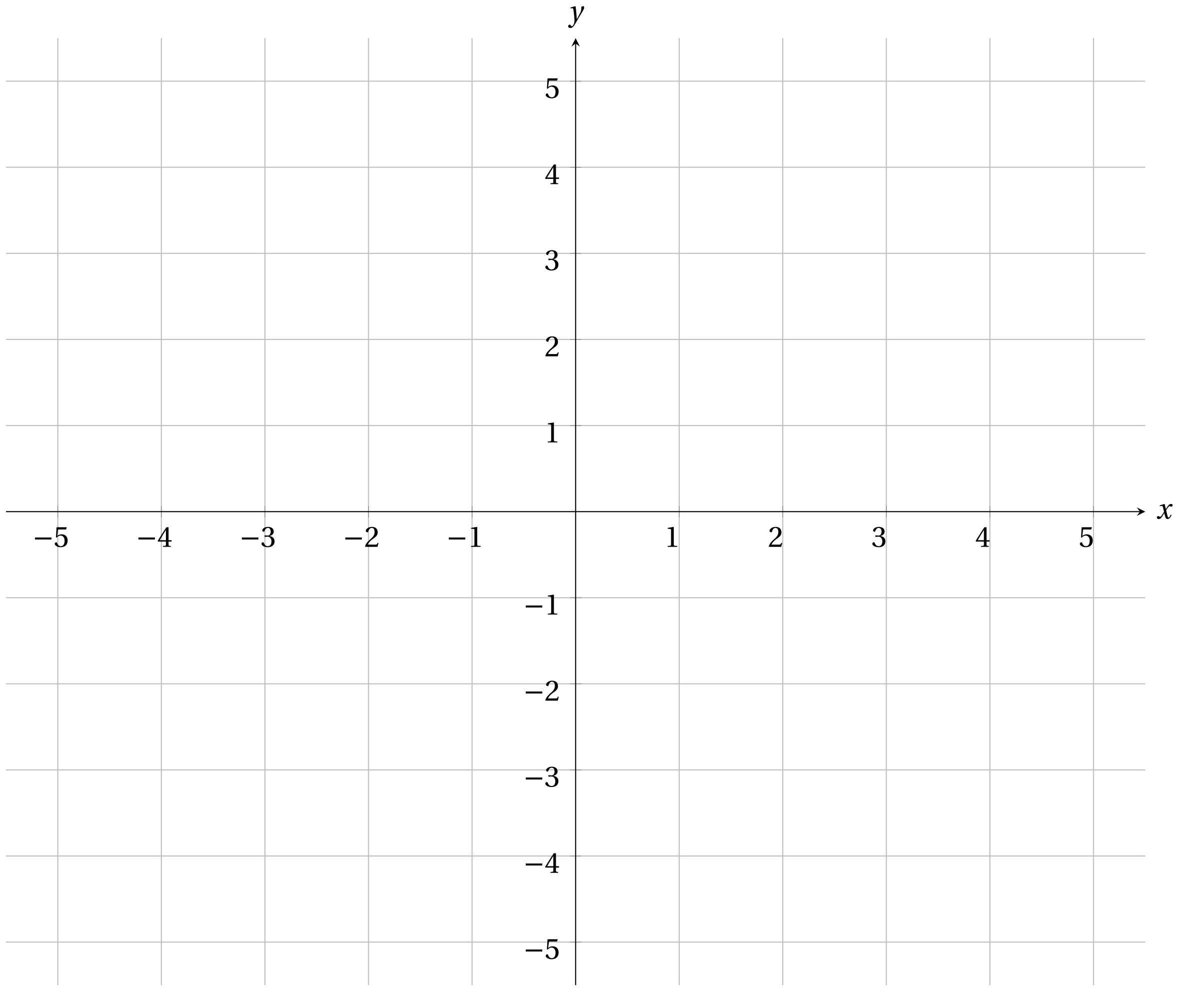
Concave up:

Concave down:

Local maximum(s):

Local minimum(s):

Inflection point(s):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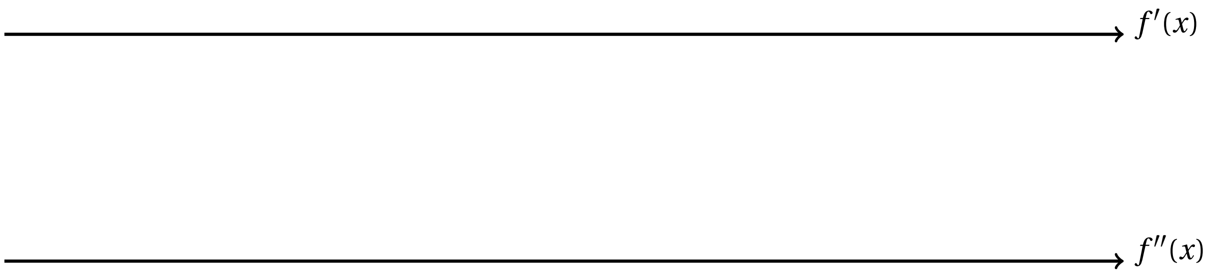

$f'(x)$
$f''(x)$

Example 3.26. Sketch a graph of the function $f(x) = \sqrt{x}(1-x)^2$.

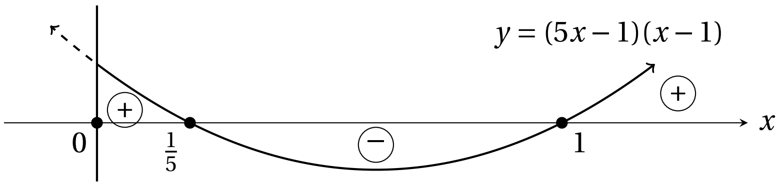
Solution: The algebraic trick here is to rewrite $f(x)$ by expanding $(1-x)^2$:

$$f(x) = \sqrt{x}(1-2x+x^2) = x^{1/2} - 2x^{3/2} + x^{5/2}.$$

The first derivative of f is

$$\begin{aligned}
f'(x) &= \frac{1}{2}x^{-1/2} - 3x^{1/2} + \frac{5}{2}x^{3/2} \\
&= \frac{1}{2x^{1/2}} - 3x^{1/2} + \frac{5}{2}x^{3/2} \\
&= \frac{1}{2x^{1/2}}\left[1 - \left(3x^{1/2}\right)\left(2x^{1/2}\right) + \left(\frac{5}{2}x^{3/2}\right)\left(2x^{1/2}\right)\right] && \left[\text{Factor out } \frac{1}{2x^{1/2}} \text{ from each term.}\right] \\
&= \frac{1}{2x^{1/2}}\left[1 - 6x + 5x^2\right] \\
&= \frac{1}{2x^{1/2}}\left[(5x-1)(x-1)\right]. && \left[\text{Factor the quadratic equation.}\right]
\end{aligned}$$

Thus the critical points of f are $x = 0, \frac{1}{5}, 1$. Since the denominator of f' is always positive for $x > 0$, $f'(x)$ has the same sign as the numerator $(5x-1)(x-1)$, whose graph is a parabola opening-upward, with roots $x = 1/5$ and $x = 1$:

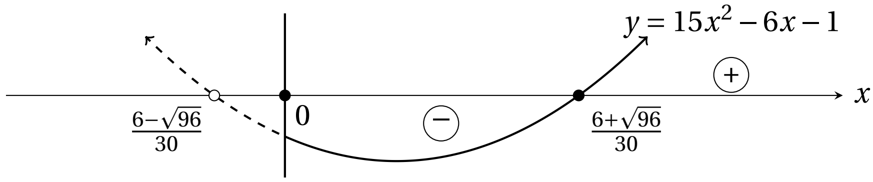


From the graph, we deduce that f is increasing on $\left(0, \frac{1}{5}\right) \cup (1, \infty)$ and decreasing on $\left(\frac{1}{5}, 1\right)$. Using the First Derivative Test, we deduce immediately that f has a local maximum at $x = 1/5$ and a local minimum at $x = 1$. Now, the second derivative of f is

$$\begin{aligned}
f''(x) &= -\frac{1}{4}x^{-3/2} - \frac{3}{2}x^{-1/2} + \frac{15}{4}x^{1/2} \\
&= -\frac{1}{4x^{3/2}} - \frac{3}{2x^{1/2}} + \frac{15}{4}x^{1/2} \\
&= \frac{1}{4x^{3/2}}\left[-1 - \left(\frac{3}{2x^{1/2}}\right)\left(4x^{3/2}\right) + \left(\frac{15}{4}x^{1/2}\right)\left(4x^{3/2}\right)\right] && \left[\text{Factor out } \frac{1}{4x^{3/2}} \text{ from each term.}\right] \\
&= \frac{1}{4x^{3/2}}\left[-1 - 6x + 15x^2\right].
\end{aligned}$$

Since the denominator of f'' is always positive for $x > 0$, $f''(x)$ has the same sign as the numerator $15x^2 - 6x - 1$, whose roots we need to find using the quadratic formula:

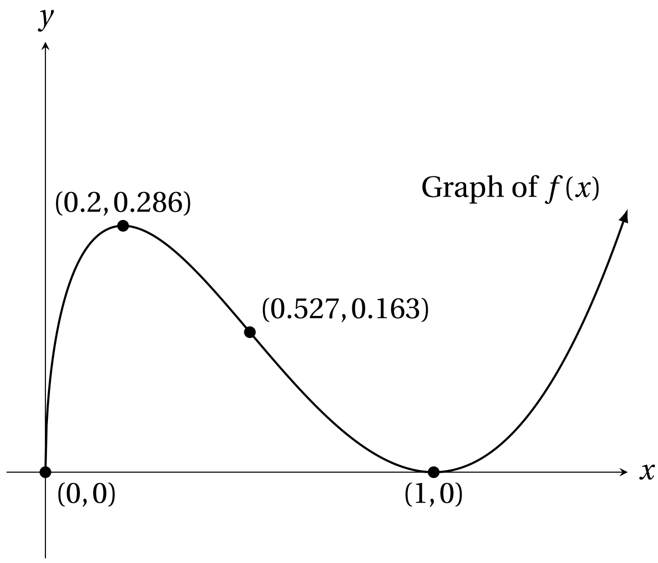
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(15)(-1)}}{2(15)} = \frac{6 \pm \sqrt{96}}{30}.$$

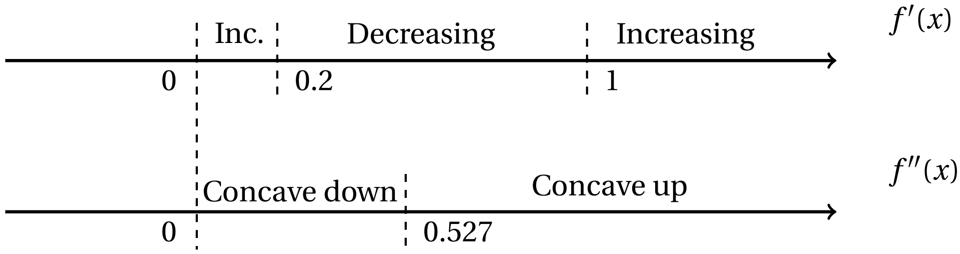


From the graph, we deduce that f is concave up on $\left(\frac{6+\sqrt{96}}{30}, \infty\right) \approx (0.527, \infty)$ and concave down on $\left(0, \frac{6+\sqrt{96}}{30}\right) \approx (0, 0.527)$. Moreover, f has an inflection point at $\left(\frac{6+\sqrt{96}}{30}, f\left(\frac{6+\sqrt{96}}{30}\right)\right) \approx (0.527, 0.163)$.

Domain: $[0, \infty)$.
Range: $[0, \infty)$.
Symmetry: None.
Periodicity: None.
x-intercepts: $(0,0)$ and $(1,0)$.
y-intercept: $(0,0)$.
VA: None.
HA or SA: None.

Increasing: $(0, 0.2) \cup (1, \infty)$.
Decreasing: $(0.2, 1)$.
Concave up: $\approx (0.527, \infty)$.
Concave down: $\approx (0, 0.527)$.
Local maximum: $f(0.2) \approx 0.286$.
Local minimum: $f(1) = 0$.
Inflection point: $\approx (0.527, 0.163)$.





3.4 Practical Problems

This section is all about optimisation problems, that is, problems concerned with minimising or maximising certain quantities of interest. There are no new calculus techniques introduced in this section. Instead the difficulty lies in correctly translating the problem description into mathematics. The easiest way to get comfortable with solving these optimisation problems is to do examples. Before we do this, let us write down a general step-by-step procedures for tackling optimisation problems:

1. Draw a sketch describing the problem. Determine useful information and assign variables to relevant information.
2. Write down the objective function, *i.e.* a formula describing all the relevant quantities.
3. Occasionally the objective function might have more than one independent variable. In this case, eliminate all but one independent variable; generally this is achieved using the constraint function.
4. Find critical points.
5. Determine which of the critical points give the global minimum and/or maximum.

These are typically physical problems, so you should always stare at your answer and determine if it makes physical sense.

Example 3.27. Farmer Eli wants to build a rectangular pen for his chickens. He wants the pen to be as large as possible (in area), but he has only 100 feet of fence. What should the dimensions of his pen be?

Example 3.28. Farmer Eli change his mind and wants 4 rectangular pens (side by side) of the same size out of the 100 feet of fence that he has. What should the dimensions be now to maximise the area?

Example 3.29. A theater found that when it priced its tickets at \$20 a piece, it sold 200 tickets on average. After it lowered its prices to \$18, it sold 240 tickets on average. Assuming that the demand function (the number of tickets sold at a given price) is linear, how should the theater price its tickets in order to maximise revenue?

Example 3.30. 240 square inches of printed material is to appear on a poster with top and bottom margins of 5 inches and side margins of 3 inches. Find the dimensions of the poster that uses the least paper, that is, has the least area.

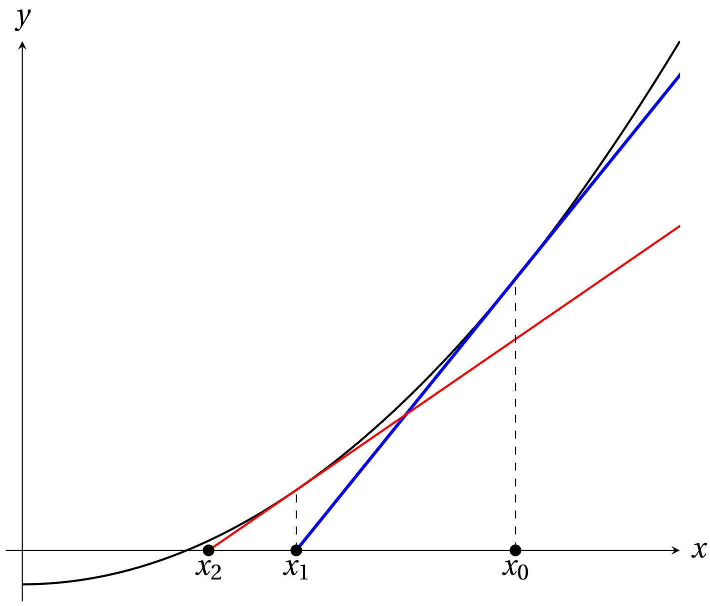
Example 3.31. Find the point on the graph of the parabola $y = x^2 + 1$ which is closest to the point $\left(1, \dfrac{3}{2}\right)$.

3.7 Solving Equations Numerically

In this section, we discuss numerical methods for approximating solutions to equations. The textbook presents three numerical methods for approximating solutions to equations: **Bisection Method**, **Newton's Method** and **Fixed-Point Algorithm**. These numerical methods are important for the simple reason that explicit solutions are just not possible in general and one must be contented with being able to numerically approximate a solution to any specified degree of accuracy. Due to time constraint, we only study Newton's method below.

Newton's method

Albeit some limitations, Newton's method is an extremely powerful root-finding algorithm since it converges quadratically in general. The geometrical idea of Newton's method is presented in the following figure:



Given some function $f(x)$, consider an initial guess x_0 which we may assumed to be sufficiently close to a root r of f. The idea is to represent $f(x)$ near $x = x_0$ with its linear approximation, *i.e.*

$$f(x) \approx L(x; x_0) = f(x_0) + f'(x_0)(x - x_0).$$

We use the root of this tangent line, denoted by x_1, to approximate r. Let us now solve for x_1:

This procedure can be repeated and provided ________________________ for each $n \geq 0$, we obtain the recursive relation:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0}$$

We repeat the process for $n = 0, 1, \ldots$ until $|x_{n+1} - x_n| < E$ for some specificed tolerance E.

Example 3.32. Use Newton's method to find the fourth approximation, x_3, to the solution of

$$f(x) = x^3 - x - 1 = 0,$$

with $x_0 = 1$. **Note:** This is a reasonable first guess since $f(1) = -1$ and $f(2) = 5$, and so the Intermediate Value Theorem says that there is a root somewhere between $x = 1$ and $x = 2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				

Example 3.33. Use Newton's method to estimate $\sqrt{2}$ correctly to 5 decimal places.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				
3				

3.8 Antiderivatives

Antiderivatives = Inverse of Derivatives

Definition 3.34. A function $F(x)$ is an **antiderivative** of $f(x)$ on the interval I if $F'(x) = f(x)$ for all x in I. When an interval is not specified, it is assumed that $I = (-\infty, \infty)$ or the domain of f.

Theorem 3.35. *If $F(x)$ is an antiderivative of $f(x)$ on I, then any other antiderivative of $f(x)$ on I is of the form $F(x)+C$ for some constant C. We call this family of functions $F(x)+C$ the* **general antiderivative** *of f.*

Proof. This follows immediately from the fact that the derivative of any constant is zero:

$$D_x[F(x)+C] = D_x[F(x)] + D_x[C] = F'(x) + 0 = F'(x) = f(x).$$

□

Note the following notation:

$$= \int f(x)\,dx =$$

1. This is also called the **indefinite integral** of $f(x)$.
2. $\int$ is called the **integral sign** and $f(x)$ is referred to as the **integrand**.

So the two terms are basically interchangeable:

General Antiderivative $\Longleftrightarrow$ Indefinite Integral

Example 3.36. Find the following general antiderivatives. Verify your answer by taking the derivative.

(a) $\displaystyle\int 7\,dx$

(b) $\displaystyle\int x^2\,dx$

(c) $\displaystyle\int \sin(4x)\,dx$

Integral is a Linear Operator

The indefinite integral is linear. That is, if $F(x)$ is an antiderivative of $f(x)$, $G(x)$ is an antiderivative of $g(x)$, and k a real number, then

$$\int kf(x)\,dx = k\int f(x)\,dx =$$
$$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx =$$
$$\int f(x)-g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx =$$

Let us recall the Power Rule for derivatives. If $f(x) = x^n$, then its derivative $f'(x) = nx^{n-1}$ is found by:

1. Multiply the exponent;
2. Decrease (lower) the exponent by 1.

Because antiderivative is the inverse of derivative, we simply reverse the above steps to find the antiderivative of $f(x) = x^n$:

1.

2.

We arrive at the first rule for finding antiderivative, the **Power Rule**:

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for any } n \neq -1.}$$

Example 3.37. Find the following indefinite integrals.

(a) $\int 5x^2 + 3x^{2018}\,dx$

(b) $\displaystyle\int x^4 - 4x^2 - \frac{1}{x^2}\,dx$

(c) $\displaystyle\int \sqrt{x} - 6\sqrt[3]{x}\,dx$

The Power Rule, along with basic trigonometric differentiation rules gives us the following list of functions f and their general antiderivatives F:

General Antiderivative, $F(x)$	$f(x)$	Derivative, $f'(x)$
	k constant	
	x^n	
	$\sin x$	
	$\cos x$	
	$\tan x$	
	$\sec^2 x$	

Generalised Power Rule

Suppose $g(x)$ is a differentiable function and $n \neq -1$. Then

$$\int g(x)^n g'(x)\, dx = \frac{g(x)^{n+1}}{n+1} + C.$$

Example 3.38. Find the following indefinite integrals.

(a) $\displaystyle\int (3x^2+1)^4(6x)\, dx$

(b) $\displaystyle\int \sin^6 x \cos x\, dx$

(c) $\displaystyle\int \left(x^3+9\right)^7 (x^2)\, dx$