2 The Derivative

We are now ready to reap all the hard work we put into studying limits in Chapter 1. Many important real-world problems can be formulated in terms of system of equations using the concept of limits. The next two chapters concern the following specific question:

What is the instantaneous rate of change of certain quantity of interest?

It is the word instantaneous that links to finding limits! These rates of change are ubiquitous that they deserve a special name:

Instantaneous rate of change = Derivative

2.1 Two Problems with One Theme

We study two seemingly unrelated problems: one geometric and the other mechanical.

The tangent line

The tangent line to a curve at a given point P is the line that passes through P in the same direction as the curve. At first sight this definition seems confusing if anything, how does one define the "direction" of a curve at a point? It makes sense to talk about the direction of a line passing through two given points P and Q on the curve, the problem is that we are only given P. How do we go from two points P and Q to one point P? If you happen to read on this while I ask the question in class, please stand up now and say this out loud: *We move the point Q closer and closer to the point P, duhhhhhhhhhh Chee Han duhhhhhhhhhhh*........... This leads us to the notion of **secant line**: a line connecting two points on the curve.

Now that we know what the tangent line should be, a practical question arises: *can we write down the equation of the tangent line through P?* We know the point P, but is that enough? NOOOOOOOOOOOOOOOOOO, we also need the slope of the tangent line, but how on Earth do we measure the slope of a line with just one point? Since we are approximating the tangent line by secant lines through P, why not approximate the slope of the tangent line by the slope of secant lines through P? Let us summarise this discussion using limit notations:

Definition 2.1. The **tangent line** to the curve $y = f(x)$ at a given point $P = (c, f(c))$ is the line passing through P with slope

$$m_{\text{tangent}} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided that this limit exists and is not ∞ or $-\infty$.

The slope of the tangent line is understood to be the limit of the **difference quotient** $\dfrac{f(x) - f(c)}{x - c}$ as x approaches c, if the limit exists. The second limit expression above is obtained using the following change of variables. We set $h = x - c$ and so $x = c + h$. Now as x approaches c, the quantity $h = x - c$ approaches 0. Thus we have the equality

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

Example 2.2. Find the slope of the tangent line to the curve $y = x^2$ at the point $(2, 4)$.

Example 2.3. Find the equation of the tangent line to the graph of $y = \dfrac{1}{x}$ at $x = 3$.

Average velocity and instantaneous velocity

The concept of instantaneous rate of change is familiar. If $d(t)$ is a function that gives your distance travelled in your car as a function of time, then the instantaneous rate of change of $d(t)$ at time $t = t_0$ would merely be the reading of your speedometer at time $t = t_0$. If you were outside of the car and were trying to calculate its velocity, your only option is to see how far the car travelled in some amount of time and then divide by the amount of time. This is the average velocity over the time interval. In general, the **average velocity over the time interval $[t_0, t]$** would be given by

$$\boxed{v_{\text{avg}} = \frac{d(t) - d(t_0)}{t - t_0}.}$$

We would expect that as t approaches t_0 that this average velocity should approach the instantaneous velocity at time $t = t_0$. As above, we could write

$$\boxed{v(t_0) = \lim_{t \to t_0} \frac{d(t) - d(t_0)}{t - t_0} = \lim_{h \to 0} \frac{d(t_0 + h) - d(t_0)}{h}}$$

provided the limit exists and is not ∞ or $-\infty$.

These are the same limits considered in our discussion of the slope of the tangent line, so it appears that the velocity at time $t = t_0$ should be the slope of the tangent line to the function $d(t)$ at $t = t_0$. In other words, **we can think of the slope of the tangent line to a curve at a point P as an instantaneous rate of change of the curve at that point**.

Example 2.4. The distance an object falls is proportional to the time it falls squared. With an accurate experiment, one would find that the cosntant of proportionality is 16 ft/s^2. That is, if $d(t)$ denote the distance in feet an object has fallen after t seconds, then

$$d(t) = -16t^2.$$

This is an approximation as it neglects air resistance. Now, suppose a tennis ball is dropped off of a 144 foot tower.

(a) How long before the tennis ball hits the ground?

(b) Find the average velocity of the tennis ball in the first 2 seconds.

(c) Find the instantaneous velocity of the tennis ball at $t = 1$ second.

(d) Find the instantaneous velocity of the tennis ball at $t = 3$ seconds, right as it hits the ground.

2.2 The Derivative

In the last section, we have seen that *slope of the tangent line to a curve* and *instantaneous rate of change of a function* are manifestations of the same underlying concept. Let us formalise this idea now.

Derivative at a point

The derivative of a function f at $x = c$, denoted by $f'(c)$ (read "f prime of c"), is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

If this limit does exist, *i.e.* $f'(c)$ exists, then we say that f is differentiable at $x = c$.

Example 2.5. Find $f'(1)$ for the function $f(x) = x^2 - x$.

Derivative as a function

Above, we defined how to find the derivative of a function $f(x)$ at a given input $x = c$. But for every value c at which the derivative exists, we get a corresponding value $f'(c)$. This is a function which we call the derivative of $f(x)$.

Definition 2.6. The **derivative** of a function $f(x)$ is the function $f'(x)$ (read "f prime of x") given by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.}$$

We say that f is **differentiable on the open interval (a, b)** if it is differentiable at all points in the interval, *i.e.* $f'(x)$ exists for all $a < x < b$. Finding a derivative is called **differentiation**.

Example 2.7. Find the derivative $f'(x)$ for the following functions.

(a) $f(x) = 5x - 3$

(b) $f(x) = \dfrac{1}{x}$

(c) $f(x) = \sqrt{x}$

Non-differentiable functions

There will be functions that are not differentiable, simply because the derivative is defined in terms of a limit, and it is possible that limits do not exist! Let us examine a number of ways for which a function is not differentiable at a point.

1. ____

   In this case, we cannot even evaluate $f(c)$ and so the expression $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ does not make any sense. For example, the reciprocal function $f(x) = \frac{1}{x}$ is not differentiable at $x = 0$ because it is not even defined at $x = 0$. Note that the function is not continuous at $x = 0$.

2. It seems like the issue above is continuity, so maybe the function would be differentiable if it is continuous? Sadly, this is not true in general.

   (a) The absolute value function $f(x) = |x|$ is continuous everywhere, but we claim that it is not differentiable at $x = 0$.

   The problem is that ____.

   (b) The cube root function $f(x) = \sqrt[3]{x} = x^{1/3}$ is continuous everywhere, but we claim that it is not differentiable at $x = 0$.

   The problem is that ____.

We have seen a few examples where a function is continuous but not differentiable. Can a function be differentiable but not continuous? The answer is no, as the following theorem indicates.

Theorem 2.8. *If f is differentiable at $x = c$, then f is continuous at $x = c$.*

2.3 Rules for Finding Derivatives

The truth is, using the limit definition of a derivative to find the derivative of a function, *i.e.* by evaluating the limit of the difference quotient, is no fun at all; the process can be onerous and time consuming for all we know. Because of this, it seems reasonable that we should develop rules and properties that will allow us to compute derivatives of familiar functions and more importantly, arithmetic combinations of these functions. Before we proceed, we point out that there are four common notations for the derivative of a function $y = f(x)$:

♠ Lagrange's notation: $\boxed{f'(x)}$

♠ Newton's notation: $\boxed{\dot{y}}$

♠ Leibniz's notation: $\boxed{\dfrac{dy}{dx}}$

♠ Euler's notation: $\boxed{D_x f}$

Constant Multiple, Sum, Difference and Power Rules

Suppose f and g are differentiable functions and k is a constant. Then

1. $D_x(k) = 0$, *i.e.* the derivative of any constant function is zero.
2. $D_x\left[kf(x)\right] = kD_x\left[f(x)\right]$, *i.e.* constant pulls out of differentiation.
3. $D_x\left[f(x)+g(x)\right] = D_x\left[f(x)\right] + D_x\left[g(x)\right]$.
4. $D_x\left[f(x)-g(x)\right] = D_x\left[f(x)\right] - D_x\left[g(x)\right]$.
5. $D_x\left[x^n\right] = nx^{n-1}$, where n is any positive integer.

Example 2.9. Use the Power Rule and rules of derivative to find the derivatives of the following functions.

(a) $D_x\left[3x^4 - x^3 + x\right]$

(b) $D_x\left[x^9 + 10x^8 - 2x^3 + 6\right]$

Now with our properties and the Power Rule, we know how to differentiate any polynomial. It turns out that the "Power Rule" above holds for more than just positive integers. We won't prove this now; instead, we will return to this when we have more techniques at our disposal.

Product Rule

Suppose f and g are differentiable functions. Then

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

Example 2.10. Compute the derivative of the following functions using the Product Rule.

(a) $f(x) = (x^2 + x)(x^4 + 5x^2)$

(b) $f(x) = (6 - x^3)(x^2 - 1)$

Quotient Rule

Suppose f and g are differentiable functions with $g(x) \neq 0$. We have that

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}.$$

Remember this with the mnemonic "low D-high minus high D-low, square the bottom and away we go!"

Example 2.11. Use the Quotient Rule to differentiate the function $F(x) = \dfrac{7x^3 - x}{2x^4 + 5}$.

Example 2.12. Use the Quotient Rule to show that the Power Rule holds for negative integer exponents. That is,

$$D_x\left[x^{-n}\right] = -nx^{-n-1}$$

where n is any positive integer.

Example 2.13. Find the equation of the tangent line to the graph of $f(x) = \dfrac{1}{x^2+1}$ at $x = 1$.

This is a good time to return to the idea that the derivative is the slope of the tangent line to the graph at a given point. Specifically, the tangent line to the graph of $y = f(x)$ at $x = c$ is the unique line with slope $f'(c)$ that passes through the point $P = (c, f(c))$. Using point-slope form, the equation of the tangent line has the form

$$\boxed{y - f(c) = f'(c)(x-c) \implies y = f(c) + f'(c)(x-c).}$$

This formula is worth memorising, since we will have occasion to use it often.

2.4 Derivatives of Trigonometric Functions

We now expand the list of functions we can differentiate to include the trigonometric functions. The derivatives of sine and cosine are obtained by using the **special trigonometric limits** we found in Section 1.4, while the derivatives of the other trigonometric functions follow from these using the Quotient Rule. Recall the angle-sum identities

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and the special trigonometric limits

$$\lim_{t\to 0} \frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{t\to 0} \frac{1-\cos t}{t} = 0.$$

Derivatives of Sine and Cosine Functions

The functions $\sin x$ and $\cos x$ are both differentiable on $\mathbb{R}$ and

$$D_x[\sin x] = \cos x \qquad \text{and} \qquad D_x[\cos x] = -\sin x.$$

Let us prove that $D_x[\sin x] = \cos x$ using the limit definition of the derivative.

$$\begin{aligned}
&D_x[\sin x] \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \left[\text{From the angle-sum identity.}\right] \\
&= \lim_{h\to 0} \frac{\sin(x)[\cos h - 1] + \cos x \sin h}{h} && \left[\text{Factor } \sin x \text{ from the first and third term.}\right] \\
&= \lim_{h\to 0} \left\{ \frac{\sin x[\cos h - 1]}{h} + \frac{\cos x \sin h}{h} \right\} && \left[\text{Separating fraction.}\right] \\
&= \lim_{h\to 0} \left\{ \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right\} \\
&= \left(\lim_{h\to 0} \sin x \right) \left(\lim_{h\to 0} \frac{\cos h - 1}{h} \right) + \left(\lim_{h\to 0} \cos x \right) \left(\lim_{h\to 0} \frac{\sin h}{h} \right)
\end{aligned}$$

Since $\sin x$ and $\cos x$ do not depend on h, the limit as $h \longrightarrow 0$ of $\sin x$ and $\cos x$ are just itself, *i.e.*

$$\lim_{h\to 0} (\sin x) = \sin x \quad \text{and} \quad \lim_{h\to 0} (\cos x) = \cos x.$$

The remaining ones are special trigonometric limits:

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{1-\cos h}{h} = -1(0) = 0 \quad \text{and} \quad \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Hence

$$D_x[\sin x] = [\sin x][0] + [\cos x][1] = \cos x.$$

Derivatives of Other Trigonometric Functions

For all x in the domain of the trigonometric function, we have the following:

$D_x[\tan x] =$ $\qquad$ $D_x[\cot x] =$

$D_x[\sec x] =$ $\qquad$ $D_x[\csc x] =$

Example 2.14. Compute the following derivatives.

(a) $D_x(3\sin x - 5\tan x)$

(b) $D_x(x^3 \tan x + \cos x)$

(c) $D_x\left(\dfrac{x\cot x - x^5}{\sin x}\right)$

2.5 The Chain Rule

Imagine trying to find the derivative of $f(x) = (x^2 + x - 3)^{1000}$ or $g(x) = \cos(4x)$. For $f(x)$, we could expand the quadratic function $(x^2 + x - 3)$ 1000 times and then differentiate the resulting polynomial using Power Rule; for $g(x)$, we might be able to use some trigonometric identities and rewrite it in terms of $\sin x$ and $\cos x$. Fortunately, there is a much better way! This is not obvious at all, but the idea, due to Leibniz, is to rewrite the given function as the composition of several functions.

Chain Rule

Suppose g is differentiable at x and f is differentiable at $g(x)$. Then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$D_x\left[f(g(x))\right] = f'(g(x))g'(x).$$

In Leibniz notation, if we set $u = g(x)$ and $y = f(u) = f(g(x))$, then the Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 2.15. Suppose $f(x) = x^2$ and $g(x) = x^3 - x$. Use the Chain Rule to find $D_x\left[f(g(x)\right]$. Compare this to what you get when you first find $f(g(x))$ by substitution then differentiate the resulting function.

Remark 2.16. Actually, the Chain Rule can also be applied to composition of three or more functions! For example, if we consider the composite function $(f \circ g \circ h)(x)$, then

$$D_x\left[f(g(h(x)))\right] = f'(g(h(x)))g'(h(x))h'(x).$$

Example 2.17. Differentiate the following functions using the Chain Rule.

(a) $F(x) = (x^2 + x - 3)^{100}$

(b) $F(x) = (5x^3 - x^2 + 3x - 1)^8$

(c) $F(x) = \dfrac{1}{5 - x^2}$

(d) $F(x) = \cos(x^4 - x^2 + 2)$

(e) $F(x) = \tan^5(x)$

(f) $F(x) = \left(\dfrac{x^2 - 1}{x}\right)^7$

(g) $F(x) = \sin^5\left(\frac{1}{x^3+1}\right)$

(h) $F(x) = \sin^2(x^2 - x)\sec^3(4x)$

(i) $F(x) = \sin\left(\cos(x^2)\right)$

(j) $F(x) = x\cos^2(4x)$

2.6 Higher-Order Derivatives

When we differentiate a function $f(x)$, we get the derivative $f'(x)$ which is a function itself. We could now differentiate $f'(x)$ to get the **second derivative** of f, denoted $f''(x)$. Repeating the same process yields the **third derivative** of f, denoted $f'''(x)$; the **fourth derivative** of f, denoted $f''''(x)$, and so on. These multiple derivatives are referred to as **higher-order derivatives**.

Derivative	Lagrange notation	y' notation	D notation	Leibniz notation
First	$f'(x)$	y'	$D_x(y)$	$\frac{dy}{dx}$
Second	$f''(x)$	y''	$D_x^2(y)$	$\frac{d^2y}{dx^2}$
Third	$f'''(x)$	y'''	$D_x^3(y)$	$\frac{d^3y}{dx^3}$
Fourth	$f^{(4)}(x)$	$y^{(4)}$	$D_x^4(y)$	$\frac{d^4y}{dx^4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth	$f^{(n)}(x)$	$y^{(n)}$	$D_x^n(y)$	$\frac{d^ny}{dx^n}$

Example 2.18. For the function $y = f(x) = x^2 + \cos x + \frac{1}{x}$, find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

Example 2.19. Find $f''(x)$ if $f(x) = \dfrac{x^2+x}{x^3+9}$.

Example 2.20. Compute the first few derivatives of $f(x) = 3\sin(2x)$ and use the pattern you discover to find $f^{(26)}(x)$.

Physics of moving objects

We used the notion of instantaneous velocity to motivate the definition of derivative. Now let us return to this idea. Unless stated otherwise, the word **velocity** refers to **instantaneous velocity**. Suppose $s(t)$ is the position function represents the position of some object at time t. How are the velocity and acceleration of the object related to $s(t)$?

Velocity =

Acceleration =

Note that velocity $v(t)$ could be positive or negative. This reflects the fact that velocity is a vector quantity, *i.e.* it has a direction and magnitude. Essentially, positive velocity corresponds to moving in the direction of increasing position variable (usually right or up); negative velocity corresponds to moving in the direction of decreasing position variable (usually left or down). **Speed** is defined to be the magnitude of velocity, *i.e.* $|v(t)|$; speed does not take direction into account.

Example 2.21. Let $s(t) = \cos(t)$ be the position of a cart (measured in feet) moving back and forth on a track at time t (measured in seconds). What is the velocity of the cart at $t = \pi/2$? What is the speed of the cart at $t = \pi/2$?

Example 2.22. Let the position of a charged particle in a magnetic field be given by

$$s(t) = t^3 - 6t^2 + 9t$$

where s is measured in metres and t in seconds. Assume the particle travels on a number line, with positive values of s to the right.

(a) What is the velocity of the particle at time t?

(b) When is the particle at rest?

(c) When is the particle moving to the right? When is the particle moving to the left?

(d) Find the acceleration of the particle at time t. When is the particle undergoing no acceleration?

(e) At what time is the particle moving fastest to the left?

2.7 Implicit Differentiation

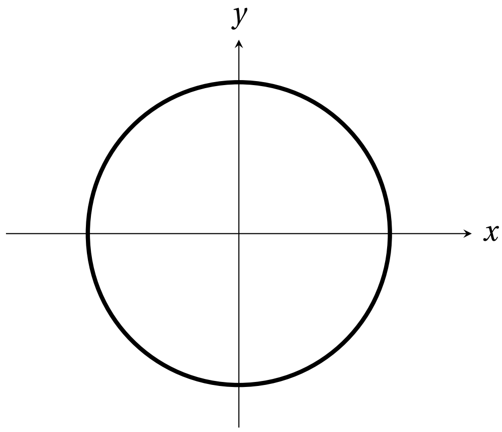
Recall that the slope of the tangent line to the graph of a function f at $x = c$ is given by $f'(c)$, but the truth is that any suitable smooth curve in the xy-plane could have a tangent line at a point on the curve. If the curve happens to have an explicit equation of the form $y = f(x)$, great, we find $f'(c)$ and move on.

Not all curves in the xy-plane
have explicit equations. What then?

Example 2.23. A classical example is the unit circle which has the equation

$$x^2 + y^2 = 1.$$

The unit circle is not the graph of a function because it fails to pass the Vertical Line Test. Observe that if we remove some parts of the circle, say the lower half of the circle, then the remaining curve becomes the graph of a function. This function is called an **implicit function**, and *finding the derivatives of implicit functions is the focus of this section.*



Let us try to solve $x^2 + y^2 = 1$ for y:

The problem is apparent: we have two implicit functions..........

1. The graph of the positive square root function $y = g(x) =$ ____________________

   corresponds to the ________________________ of the circle.

2. The graph of the negative square root function $y = h(x) =$ ____________________

   corresponds to the ________________________ of the circle.

Together, the graphs of $g(x)$ and $h(x)$ constitute the entire unit circle. The problem of which implicit functions to choose is dictated by the given point (x, y) on the unit circle.

♠ The implicit function of $x^2 + y^2 = 1$ at the point $(0, 1)$ is

♠ The implicit function of $x^2 + y^2 = 1$ at the point $(0, -1)$ is

■■ The implicit function of $x^2 + y^2 = 1$ at the point $(1, 0)$ is

Although the equation $x^2 + y^2 = 1$ does not represent y as a function of x, it is still "nice" in the sense that we are able to solve for y in terms of x. As you can see, we actually solve the problem of finding the slope of the tangent line of a given point (except $(\pm 1, 0)$) on the unit circle, because $g(x)$ and $h(x)$ are both explicit functions of x and so we could compute their derivatives. Indeed, *assuming the Power Rule holds for rational exponents,*

Surprisingly, it is not necessary to find the formula for an implicit function in order to find its derivative. The idea behind this technique, called **implicit differentiation**, is to treat y as a function of x, *i.e.* we replace y with $y(x)$. For the equation of unit circle,

$$x^2 + \left[y(x)\right]^2 = 1.$$

We differentiate both sides of the equation with respect to x, in particular we use the Chain Rule to differentiate the term $\left[y(x)\right]^2$.

Example 2.24. It seems like implicit differentiation is unnecessary if we can find implicit functions, but don't be deceived by the unit circle example! Indeed, consider the equation

$$y^3 + 7y = x.$$

Clearly we cannot solve for y in terms of x. Nevertheless, one can show that for every x-value, there is exactly one corresponding y-value. We say that the equation $y^3 + 7y = x$ defines y as an **implicit function** of x, even when we cannot write down an explicit formula for y. To find $\frac{dy}{dx}$ using implicit differentiation, we rewrite the equation by replacing y with $y(x)$ again:

We differentiate both sides and apply the Chain Rule to differentiate terms involving $y(x)$.

The technique of implicit differentiation can be summarised as follows:

1. We are given an equation in x and y.
2. Differentiate both sides of the equation with respect to x. Treat y as a "mystery" function of x; hence, use the Chain Rule to differentiate expressions involving y. This will result in an equation involving x, y, and $\frac{dy}{dx}$.
3. Either solve for $\frac{dy}{dx}$ in terms of x and y, or solve for $\frac{dy}{dx}$ at a specific point by plugging in for x and y.

Example 2.25. Find the value of $\dfrac{dy}{dx}$ at the point $(1,-1)$, where y is given by the equation

$$x^2 - y^3 = 2x.$$

Example 2.26. Find the equation of the tangent line to the curve

$$5x^2 - 7xy^2 = x^2y^4 - 3$$

at the point $P = (1,1)$.

Example 2.27. At what values of x is the tangent line to the curve determined by

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

horizontal? This curve is called the "bouncing wagon" for obvious reasons.

Implicit differentiation can be used to show that the Power Rule holds not only for integer exponents, but rational exponents as well. We refer the interested readers to the textbook for the proof.

Power Rule, Again

For any nonzero rational number n,

$$D_x\left[x^n\right] = nx^{n-1}.$$

Example 2.28. Differentiate the following functions.

(a) $f(x) = x^{2/5} - 5x^{-3/2}$

(b) $f(x) = \sqrt[3]{x^2+9} + \cos\left(\sqrt[5]{x}\right)$

2.8 Related Rates

If a quantity y depends on time t, then its derivative $\frac{dy}{dt}$ is called a **time rate of change**. As before, if y has an explicit expression in terms of t, we simply differentiate to obtain $\frac{dy}{dt}$ and evaluate the derivative at the given time. However, it might very well be the case that we know a relation that connects y and another quantity x that also depends on time instead, *i.e.* we have two changing quantities that, for all time t, are related by an equation.

Differentiating this equation with respect to t gives us a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$, and we called these related rates.

Example 2.29. Suppose we have a circle whose radius is changing as a function of time. As the radius of the circle changes, so does the area, and we are interested in knowing how fast the area of the circle is changing at a given time.

1. First, we need an equation that relates area and radius of a circle as a function of time:

   Note that this equation holds true for all time t even though A and r are changing in time.

2. We can differentiate the relation using the Chain Rule:

3. If we know that at a particular time t^*, the radius is 3 metres and it is growing at a rate of 1.5 metres per second, then

Rule of thumb for solving related rates problems

Example 2.30. Suppose a balloon is being inflated from an air tank which expels 0.2π ft^3 of air per second. For simplicity, assume that the balloon is always perfectly spherical. How fast is the radius increasing when there is $\dfrac{4\pi}{3}$ ft^3 of air in the balloon?

Example 2.31. A 13 foot ladder is leaning against a vertical wall. Suppose the base of the ladder starts sliding away from the wall at a constant rate of 0.4 feet per second. How fast is the top of the ladder sliding down the wall when the base of the ladder is 5 feet away from the wall ?

Example 2.32. A person stands on a lighthouse that is 30 metres tall. At time $t = 0$, a ship approaches at a speed of 2 metres per second travelling directly towards the lighthouse from 160 metres away (from the base of the lighthouse).

(a) How fast is the distance between the person and the ship decreasing one minute after the ship leaves?

(b) How fast is the angle of elevation changing when the ship is 30 metres from the lighthouse?

2.9 Differentials and Approximations

This final section is to tie up any loose ends on tangent line, in particular why do we care so much about tangent line. Consider a point $P = \big(a, f(a)\big)$ on the graph of a function f. The intuition is that, if we zoom in the point P close enough, then the graph of $f(x)$ looks approximately like the tangent line at point P; this should not be too surprising since the slope of the tangent line, which is the derivative $f'(x)$, is defined in terms of limits.

Linear Approximation

If f is a differentiable function at $x = a$, then the function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linear approximation of f at $x = a$**. The function $L(x)$ is also called the linearisation of f at $x = a$ and is simply the equation of the tangent line to the graph of f at $x = a$.

Example 2.33. Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 1$, and use this to approximate $\sqrt{1.02}$ and $\sqrt{0.97}$. How do these approximations compare to the true values?

Example 2.34. Find the linearisation of $f(x) = \tan x$ at $x = \dfrac{\pi}{4}$, and use this to estimate $\tan\left(\dfrac{\pi}{3}\right)$.

Differentials

The same logic in terms of approximating a function by its tangent line can be used to derive a method for approximating changes in functions. Suppose we are given a function $y = f(x)$ and an x-value $x = a$. As discussed above, when $x \approx a$, we have

As x gets closer to a (or $\Delta x \longrightarrow 0$), this approximation gets better and better and is exactly equal in the limit.

The equation ____________________ is called the ____
This viewpoint is beneficial when we are concerned with how the output of the function f changes near a given input $x = a$. If we think of d as representing change, then

$dx =$

$\Delta y =$

$dy =$

Example 2.35. Consider the function $f(x) = x^2 + x$.

(a) Use differentials to estimate how $f(x)$ changes as x changes from $x = 4$ to $x = 4.1$.

(b) Find a good approximation to $(4.1)^2 + 4.1$.

Example 2.36. Suppose a sphere 1 metre in radius is given a coat of paint 2 millimetres thick. Use differentials to estimate the volume of the paint used.

Error Estimation

Error estimation is ubiquitous in science. When scientists conduct experiment, they measure the input x_0 that has a possible error of size $\pm\Delta x$; the value x_0 is then used to calculate the corresponding output y_0 that is contaminated by the error in x. An important task is to estimate the magnitude and quantify the size of the error of y_0. This can be achieved by means of differentials.

Definition 2.37. Given a function $y = f(x)$,

1. the **absolute error** is the change in y for a small change in x:

2. the **relative error** is the ratio of the absolute error and the quantity:

We often express the relative error as a percentage, and refer to that as *percent error*.

Example 2.38. You are building a container that will be used to carry water. The container is to be a cube with no lid, and it must carry 27 cm^3 of water with a possible 2% error.

(a) What is the absolute error for the length of the sides of the container?

(b) What is the absolute error for the surface area of the container?