**Section II (Separation Theorems)**

Part III of this text is concerned w/ duality correspondences, Rockafellar states that separation theorems are the foundations for duality correspondences. Almost everything that follows in this chapter is based on the fact that a hyperplane in \( \mathbb{R}^n \) divides \( \mathbb{R}^n \) “evenly in two.”

**Definition**

- A hyperplane \( H \) is said to separate \( C_1 \) & \( C_2 \) if \( C_1 \) is contained in one of the closed half-spaces assoc. w/ \( H \) and \( C_2 \) lies in the opposite closed half-space.
- \( H \) separates \( C_1 \) & \( C_2 \) properly if \( C_1 \) & \( C_2 \) are not both contained in \( H \) itself.
- \( H \) separates \( C_1 \) & \( C_2 \) strongly if \( \exists \ E > 0 \) s.t. \( C_1 + E \delta \) is contained in one of the assoc. open half-spaces & \( C_2 + E \delta \) is contained in the opposite open half space.
- Other kinds of separation are considered in general, e.g. strict separation (\( C_1 \) & \( C_2 \) must simply belong to opposing open h.s.), however proper & strong separation are most useful for our purposes.
Theorem 11.1

Let \( C_1, C_2 \subset \mathbb{R}^n \) be non-empty sets. \( \exists \) a hyperplane separating \( C_1 \) & \( C_2 \) properly iff \( \exists \) a vector \( b \) s.t.

\[
\begin{align*}
(i) & \quad \inf \{ \langle x, b \rangle \mid x \in C_1 \} \geq \sup \{ \langle x, b \rangle \mid x \in C_2 \}, \\
(ii) & \quad \sup \{ \langle x, b \rangle \mid x \in C_1 \} > \inf \{ \langle x, b \rangle \mid x \in C_2 \}.
\end{align*}
\]

Additionally, \( \exists \) a hyperplane separating \( C_1 \) & \( C_2 \) strongly iff \( \exists \) a vector \( b \) s.t.

\[
(i) \quad \inf \{ \langle x, b \rangle \mid x \in C_1 \} > \sup \{ \langle x, b \rangle \mid x \in C_2 \}.
\]

Proof

(i) \( \iff \) Suppose \( \exists b \) satisfying (a) \& (b) \& let \( \beta \) be s.t.

\[
\inf \{ \langle x, b \rangle \mid x \in C_1 \} \geq \beta \geq \sup \{ \langle x, b \rangle \mid x \in C_2 \}.
\]

\( b \neq 0 \) \& \( b \in \mathbb{R} \Rightarrow H = \{ x \mid \langle x, b \rangle = \beta \} \) is a hyperplane (Thm 1.3).
Then \( C_1 \subset \{ x \mid \langle x, b \rangle \geq \beta \} \) \& \( C_2 \subset \{ x \mid \langle x, b \rangle \leq \beta \} \).
Condition (b) implies \( C_1 \) \& \( C_2 \) are not both contained in \( H \), i.e. \( H \) separates \( C_1 \) \& \( C_2 \) properly. \( \checkmark \)

(\( \Rightarrow \)) Assume \( C_1 \) \& \( C_2 \) can be separated properly, then there exists \( b \) \& \( \beta \) s.t. \( H = \{ x \mid \langle x, b \rangle = \beta \} \) \& \( x \in C_1 \Rightarrow \langle x, b \rangle \geq \beta \) \& \( x \in C_2 \Rightarrow \langle x, b \rangle \leq \beta \) w/ one \( x \in C_1 \) or \( x \in C_2 \), therefore \( b \) satisfies (a) \& (b). \( \square \)

(ii) \( \iff \) If \( b \) satisfies (c) we can choose \( \epsilon \in \mathbb{R} \) \& \( \delta > 0 \) s.t. \( \langle x, b \rangle \geq \beta + \delta \) \( \forall x \in C_1 \) \& \( \langle x, b \rangle \leq \beta - \delta \) \( \forall x \in C_2 \).
Then \( \exists \epsilon > 0 \) such that \( |\langle y, b \rangle| < \delta \neq y \in E \). Let \( x \in C_1 \) \& \( y \in E \) then

\[
\langle x, b \rangle > \beta + \delta \text{ or } \langle y, b \rangle < \beta - \delta.
\]
\[
\langle x+y, b \rangle = \langle x, b \rangle + \langle y, b \rangle > B+\delta-S-B \Rightarrow C_1+\mathbb{B} \leq \{x | \langle x, b \rangle > B\}. \\
\]
Similarly, we have \( C_2+\mathbb{B} \leq \{x | \langle x, b \rangle > B\}. \) Therefore, 
\( H= \{x | \langle x, b \rangle = B\} \) separates \( C_1 \) \& \( C_2 \) strongly. \( \square \)

(\Rightarrow) Assume \( C_1 \) \& \( C_2 \) can be separated strongly, then
\[
\exists b \notin B s.t
\]
\[
\inf_{x \in C_1} \langle x, b \rangle > \inf_{y \in B} \langle x, b \rangle + \varepsilon \langle y, b \rangle \geq B \geq \sup_{x \in C_2} \langle x, b \rangle + \varepsilon \langle y, b \rangle > \sup_{y \in B} \langle x, b \rangle.
\]

Now we turn to the existence question which is:
given two sets can we find a separating hyperplane?

**Theorem 11.2**

Let \( C \in \mathbb{R}^n \) be a non-empty relatively open convex set, and let \( M \in \mathbb{R}^n \) be a non-empty affine set not meeting \( C \), i.e., \( M \cap C = \emptyset \). Then \( \exists \) a hyperplane \( H \) containing \( M \), s.t. one of the open half-spaces assoc. w/ \( H \) contains \( C \).

**Proof**

If \( M \) is a hyperplane (i.e., \( M \) is \( n-1 \) dimensional) then we are done, \( M \) separates \( C \) otherwise \( M \cap C = \emptyset \).

Therefore, WLOG assume \( M \) is not a hyperplane.

Set up

1) we will construct an affine set \( M' \) one dimension higher than \( M \), s.t. again \( M' \cap C = \emptyset \).

2) In \( n \) steps or less this will determine a hyperplane \( H (H \cap C = \emptyset) \) which will prove the theorem.
WLOG assume $O \in M$, so that $M$ is a subspace. 
Of $C$ then $C \cap (C-M)$ but $O \notin C-M$ ($M \cap C = \emptyset$). $M$ is not a hyperplane $\Rightarrow$ the subspace $M'$ contains a two-dimensional subspace $P$. Define $C' = P \cap (C-M)$, $C'$ is a relatively open convex set in $P$ (Cor 6.5.1: $r_i(P \cap (C-M)) = r_i(P \cap (C-M))$)
and (Cor 6.6.2: $r_i(C-M) = r_i(C) - r_i(M) = C-M$) \& $O \notin C$. Now we want to find a line $L$ through $O$ in $P$ not meeting $C'$ then $M'_1 = M \cup L$ is one dimension higher than $M$ & $M \cap C = \emptyset$ (if $(M+L) \cap C = \emptyset$ then $(M+L) \cap C = \emptyset$), which contradicts $C \cap C' = \emptyset$ by construction). WLOG identify $P \equiv \mathbb{R}^2$.

Cases.
(i) $C'$ is empty or zero dimensional then finding $L$ is trivial.
(ii) aff $C'$ is a line not containing $O$, take $L$ to be the parallel line through $O$.
(iii) aff $C'$ is a line containing $O$, take $L$ to be the perpendicular line through $O$.
(iv) $C'$ is two-dimensional and hence open, then $K = \text{conv} C'$ is the smallest convex cone containing $C'$ (Cor 2.6.3).
Note that $K$ is open and does not contain $O$.
Therefore $K$ is an open sector of $\mathbb{R}^2$ corresponding to an angle no greater than $\pi$. Take $L$ to be the line extending one of the two boundary rays of the sector. \[\square\]
**Theorem 11.3 (Main Separation Theorem)**

Let \( C_1, C_2 \subseteq \mathbb{R}^n \) be non-empty convex sets. In order that there exists a hyperplane separating \( C_1 \) and \( C_2 \) properly, it is necessary and sufficient that \( \text{ri} C_1 \cap \text{ri} C_2 \) have no point in common.

**Proof**

Consider the convex set \( C = C_1 - C_2 \), then \( \text{ri} C = \text{ri} C_1 - \text{ri} C_2 \) (Thm 6.6.2). So \( 0 \notin \text{ri} C \iff \text{ri} C \cap \text{ri} C = \emptyset \). Since \( 0 \notin \text{ri} C \) Thm 11.2 guarantees the existence of a hyperplane containing \( M = \{0\} \subseteq \mathbb{R}^n \) s.t. \( \text{ri} C \) is contained in one of the associated half-spaces. Therefore, if \( 0 \notin \text{ri} (C) \)

\[ \exists \ b \text{ s.t.} \]

\[ 0 \leq \inf_{x \in C} \langle x, b \rangle = \inf_{x \in C_1} \langle x, b \rangle - \sup_{x \in C_2} \langle x, b \rangle \]

\[ 0 \leq \sup_{x \in C} \langle x, b \rangle = \sup_{x \in C_1} \langle x, b \rangle - \inf_{x \in C_2} \langle x, b \rangle \]

Now Thm 11.1 implies \( C_1 + C_2 \) can be separated properly. Conditions in 11.1 imply \( 0 \notin \text{ri} (C) \) since they assert the existence of a h.s. \( D = \{ x \mid \langle x, b \rangle \geq 3 \} \) containing \( C \) where \( \text{ri} D \cap C = \emptyset \Rightarrow \text{ri} C \cap \text{ri} D \) (Cor 6.5.2). □

**Example.** Why does 11.5 only give proper separation?

Consider

\[ C_1 = \{(y_1, y_2) \mid y_1 > 0, y_2 \geq 0, y_1 + y_2 \geq 3\}, \quad C_2 = \{(y_1, 0) \mid y_1 \geq 3, 0 \geq 3\}. \]

Then \( C_1 \cap C_2 = \emptyset \) but the only separating hyperplane is the \( y_1 \)-axis which contains all of \( C_2 \). That is, \( C_1 \) and \( C_2 \) are properly separated but NOT strongly separated.
**Theorem 11.4**

Let $C_1, C_2 \in \mathbb{R}$ be non-empty, convex sets. In order that there exists a hyperplane separating $C_1 \subseteq C_2$ strongly, it is necessary and sufficient that

$$\inf \{ \| x_1 - x_2 \| : x_1 \in C_1, x_2 \in C_2 \} > 0,$$

or in other words that $0 \notin \text{cl}(C_1 - C_2)$.

**Proof**

By definition of strong separation exists $E > 0$ such that $(C_1 + E) \cap (C_2 + E) = \emptyset$. Considering $(C_1 + E) \cap (C_2 + E) = \emptyset$ then it implies $C_1 + E \cap C_2 + E$ can be separated properly, i.e., $C_1 + \frac{E}{2} \cap C_2 + \frac{E}{2}$ belong to opposite (closed) half-spaces so that $C_1 + \frac{E}{2} \cap C_2 + \frac{E}{2}$ belong to opposite open half-spaces. That is $C_1$ and $C_2$ can be separated strongly if and only if for some $E > 0$ the origin is not contained in

$$(C_1 + E) - (C_2 + E) = C_1 - C_2 - 2E$$

i.e., $2E = |C_1 - C_2|$ for some $E > 0$ which says $0 \notin \text{cl}(C_1 - C_2)$. \(\Box\)

Skipping Corollary 11.4.1 b/c it depends on recession.

**Corollary 11.4.2**

Let $C_1, C_2 \in \mathbb{R}^n$ be non-empty, convex sets whose closures are disjoint. If either set is bounded then a hyperplane separating $C_1 \subseteq C_2$ strongly.

**Proof**

See book1, depends on recession & Corollary 11.4.1.
Theorem 11.5 (updated in 18.8)
A closed convex set $C$ is the intersection of the closed half-spaces which contain it.

Proof
Assume $0 \notin C \subseteq \mathbb{R}^n$ (otherwise it is trivial). Consider $a \in C$, then $C = \{a\} \cup C$, $C$ satisfy 11.4 implying $\exists$ hyperplane $H$ separating $\{a\}$ and $C$ strongly. One of the closed half-spaces contains $C$ but not $\{a\}$, therefore the intersection of all closed half-spaces containing $C$ contains no points other than those in $C$.

Corollary 11.5.1
Let $S$ be any subset of $\mathbb{R}^n$. The $\text{cl}(\text{conv} S)$ is the intersection of all the closed half spaces containing $S$.

Corollary 11.5.2
Let $C$ be a convex subset of $\mathbb{R}^n$ other than $\mathbb{R}^n$ itself. Then $\exists$ a closed half-space containing $C$, i.e., there exists some $b \in \mathbb{R}^n$ s.t. the linear $\langle \cdot , b \rangle$ is bounded above on $C$.

Definition
* A supporting half-space to a convex set $C \subseteq \mathbb{R}^n$ is a closed half-space containing $C$ has a point of $C$ in its boundary.
* A supporting hyperplane to a convex set $C$ is a hyperplane which is the boundary of supporting half-spaces to $C$. 
Supporting hyperplanes are \( H = \{ x \in \mathbb{R}^d : \langle x, b \rangle = \beta \} \), \( b \neq 0 \), \( \langle x, b \rangle \leq \beta \) \( \forall x \in C \) \& \( \exists \in C \ s.t. \ \langle x, b \rangle = \beta \), i.e., a supporting hyperplane is assoc. w/ a linear funct which achieves its maximum on \( C \). We consider non-trivial supporting hyperplanes to \( C \), which are supporting hyperplanes to \( C \) that do not contain \( C \) itself.

**Theorem 11.6**

Let \( C \) be a convex set, \& let \( D \) be a non-empty, convex subset of \( C \) (e.g. a single point). In order that there exists a non-trivial supporting hyperplane to \( C \) containing \( D \), it is necessary and sufficient that \( D \) be disjoint from \( \text{ri} C \).

**Proof**

\[ D \subseteq C \Rightarrow \text{the non-trivial supporting hyperplanes (H) to } C \ s.t. D \cap H \text{ are hyperplanes separating } C \setminus D \text{ properly. Such } H \text{ exist iff } \text{ri} C \cap \text{ri} D = \emptyset \text{ (Thm 11.5)}. \]

Assume \( \text{ri} D \subseteq \text{ri} C \), then Cor 6.5.2 implies that \( \text{ri} D \subseteq \text{ri} C \) which is a contradiction. Therefore we have \( \text{ri} D \cap \text{ri} C = \emptyset \) is equivalent to \( D \cap \text{ri} C = \emptyset \) \( \square \)

**Corollary 11.6.1**

A convex set has a non-zero normal at each of its boundary points.

**Corollary 11.6.2**

Let \( C \) be a convex set. \( x \in C \) is a relative boundary pt iff there exists a linear funct \( h \) not constant on \( C \) s.t. \( h \) achieves its maximum over \( C \) at \( x \).