Math 6730 : Asymptotic and Perturbation Methods

Hyunjoong Kim & Chee Han Tan

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Abstract: These notes are largely based on Math 6730: Asymptotic and Perturbation Methods course, taught by Paul Bressloff in Fall 2017, at the University of Utah. Additional examples or remarks or results from other sources are added as we see fit, mainly to facilitate our understanding. These notes are by no means accurate or applicable, and any mistakes here are of course our own. Please report any typographical errors or mathematical fallacy to us by email hkim@math.utah.edu or tan@math.utah.edu
Chapter 1

Introduction to Asymptotic Approximation

Our main goal is to construct approximate solutions of differential equations to gain insight of the problem, since they are nearly impossible to solve analytically in general due to the nonlinear nature of the problem. Among the most important machinery in approximating functions in some small neighbourhood is the Taylor’s theorem. It says that given \( f \in C^{(N+1)}(B_\delta(x_0)) \), we can write \( f(x) \) as

\[
f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_{N+1}(x),
\]

where \( R_{N+1}(x) \) is the remainder term

\[
R_{N+1}(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1},
\]

for some \( \xi \) between \( x \) and \( x_0 \), provided \( x \in B_\delta(x_0) \). Taylor’s theorem can be used to solve the following problem:

Given a fixed \( \varepsilon = |x - x_0| > 0 \), how many terms should we include in the Taylor polynomial to achieve a certain accuracy?

Asymptotic approximation concerns about a slightly different problem:

Given a fixed number of terms \( N \), how accurate is the asymptotic approximation as \( \varepsilon \to 0 \)?

We want to avoid from including as many terms as possible as \( \varepsilon \to 0 \) and in contrast to Taylor’s theorem, we do not care about convergence of the asymptotic approximation. In fact, most asymptotic approximations diverge as \( N \to \infty \) for a fixed \( \varepsilon \).

Remark 1.0.1. If the given function is sufficiently differentiable, then Taylor’s theorem offers a reasonable approximation and we can easily analyse the error as well.
1.1 Asymptotic Expansion

We begin the section with a motivating example. Suppose we want to evaluate the integral
\[ f(\varepsilon) = \int_0^{\infty} \frac{e^{-t}}{1 + \varepsilon t} dt, \quad \varepsilon > 0. \]

We can develop an approximation of \( f(\varepsilon) \) for sufficiently small \( \varepsilon > 0 \) by repeatedly integrating by parts. Indeed,
\[
  f(\varepsilon) = 1 - \varepsilon \int_0^{\infty} \frac{e^{-t}}{(1 + \varepsilon t)^2} dt = 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3 + \ldots + (-1)^N N! \varepsilon^N + R_N(\varepsilon)
\]
where
\[
  R_N(\varepsilon) = (-1)^{N+1} (N + 1)! \varepsilon^{N+1} \int_0^{\infty} \frac{e^{-t}}{(1 + \varepsilon t)^{N+2}} dt.
\]

Since
\[
  \int_0^{\infty} \frac{e^{-t}}{(1 + \varepsilon t)^{N+2}} dt \leq \int_0^{\infty} e^{-t} dt = 1,
\]
it follows that
\[
  |R_N(\varepsilon)| \leq |(N + 1)! \varepsilon^{N+1}|.
\]
Thus, for fixed \( N > 0 \) we have that
\[
  \lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon) - \sum_{k=0}^{N} a_k \varepsilon^k}{\varepsilon^N} \right| = 0,
\]
or
\[
  f(\varepsilon) = \sum_{k=0}^{N} a_k \varepsilon^k + o(\varepsilon^N) = \sum_{k=0}^{N} a_k \varepsilon^k + O(\varepsilon^{N+1}).
\]

The formal series \( \sum_{k=0}^{N} a_k \varepsilon^k \) is said to be an asymptotic expansion of \( f(\varepsilon) \) such that for fixed \( N \), it provides a good approximation to \( f(\varepsilon) \) as \( \varepsilon \to 0 \). However, this expansion is not convergent for any fixed \( \varepsilon > 0 \), since
\[
  (-1)^N N! \varepsilon^N \to \infty \quad \text{as} \quad \varepsilon \to 0,
\]
\( i.e. \) the correction term actually blows up!

Remark 1.1.1. Observe that for sufficiently small \( \varepsilon > 0 \),
\[
  |R_N(\varepsilon)| \ll |(-1)^N N! \varepsilon^N|,
\]
which means that the remainder \( R_N(\varepsilon) \) is dominated by the \( (N + 1) \)th term of the approximation, \( i.e. \) the error is of higher order of the approximating function. This property is something that we would want to impose on the asymptotic expansion, and this idea can be made precise using the Landau symbols.
Definition 1.1.2.

(a) \( f(\varepsilon) = O(g(\varepsilon)) \) as \( \varepsilon \to 0 \) means that there exists a finite \( M \) for which\
\[ |f(\varepsilon)| \leq M|g(\varepsilon)| \quad \text{as} \quad \varepsilon \to 0. \]

(b) \( f(\varepsilon) = o(g(\varepsilon)) \) as \( \varepsilon \to 0 \) means that\
\[ \lim_{\varepsilon \to 0} \frac{|f(\varepsilon)|}{|g(\varepsilon)|} = 0. \]

(c) The ordered sequence of functions \( \{\phi_k(\varepsilon)\}_{k=0}^{\infty} \) is called an asymptotic sequence as \( \varepsilon \to 0 \) if and only if \( \phi_{k+1}(\varepsilon) = o(\phi_k(\varepsilon)) \) as \( \varepsilon \to 0 \) for each \( k \).

(d) Let \( f(\varepsilon) \) be a continuous function of \( \varepsilon \) and \( \{\phi_k(\varepsilon)\}_{k=0}^{\infty} \) an asymptotic sequence. The formal series expansion\
\[ \sum_{k=0}^{N} a_k \phi_k(\varepsilon) \]

is called an asymptotic expansion valid to order \( \phi_N(\varepsilon) \) if for any \( N \geq 0 \),
\[ \lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon) - \sum_{k=0}^{N} a_k \phi_k(\varepsilon)}{\phi_N(\varepsilon)} \right| = 0. \]

We typically write \( f(\varepsilon) \sim \sum_{k=0}^{N} a_k \phi_k(\varepsilon) \) as \( \varepsilon \to 0 \).

Remark 1.1.3. Intuitively, an asymptotic expansion of a given function \( f \) is a finite sum which might diverges, yet it still provides an increasingly accurate description of the asymptotic behaviour of \( f \) as \( \varepsilon \to 0 \). There is a caveat here: for a divergent asymptotic expansion, for some \( \varepsilon \), there exists an optimal \( N_0 = N_0(\varepsilon) \) that gives best approximation to \( f \), i.e. adding more terms actually gives worse accuracy. However, for values of \( \varepsilon \) sufficiently close to the limiting value 0, the optimal number of terms required increases, i.e. for every \( \varepsilon_1 > 0 \), there exists an \( \delta \) and an optimal \( N_0 = N_0(\delta) \) such that
\[ \left| f(\varepsilon) - \sum_{k=0}^{N} a_k \phi_k(\varepsilon) \right| < \varepsilon_1 \quad \text{for every} \quad |z - z_0| < \delta \quad \text{and} \quad N > N_0. \]

In solutions to ODEs, we will need to consider time-dependent asymptotic expansions. Suppose \( \dot{x} = f(x, \varepsilon), x \in \mathbb{R}^n \). We seek a solution of the form
\[ x(t, \varepsilon) \sim \sum_{k=0}^{N} a_k(t) \phi_k(\varepsilon) \quad \text{as} \quad \varepsilon \to 0, \]
which will tend to be valid over some range of times $t$. It is often useful to characterise the
time interval over which the asymptotic expansion exists. We say that this estimate is valid
on a time-scale $\frac{1}{\delta(\varepsilon)}$ if

$$\lim_{\varepsilon \to 0} \left| \frac{x(t, \varepsilon) - \sum_{k=0}^{N} a_k(t) \phi_k(\varepsilon)}{\phi_N(\varepsilon)} \right| = 0 \quad \text{for } 0 \leq \delta(\varepsilon) t \leq C,$$

for some $C$ independent of $\varepsilon$.

**Accuracy vs Convergence**

In the case of a Taylor series expansion, one can increase the accuracy (for fixed $\varepsilon$) by including
more terms in the approximation, assuming we are expanding within the radius of convergence.
This is not usually the case for an asymptotic expansion because the asymptotic expansion
concerns the limit as $\varepsilon \to 0$ whereas increasing the number of terms concerns $N \to \infty$ for
fixed $\varepsilon$.

**Manipulating Asymptotic Expansions**

Two asymptotic expansions can be added together term by term, assuming both involve the
same basis functions $\{\phi_k(\varepsilon)\}$. Multiplication can also be carried out provided the asymptotic
sequence $\{\phi_k(\varepsilon)\}$ can be ordered in a particular way. What about differentiation? Suppose

$$f(x, \varepsilon) \sim \phi_1(x, \varepsilon) + \phi_2(x, \varepsilon) \quad \text{as } \varepsilon \to 0.$$

It is not necessarily the case that

$$\frac{d}{dx} f(x, \varepsilon) \sim \frac{d}{dx} \phi_1(x, \varepsilon) + \frac{d}{dx} \phi_2(x, \varepsilon) \quad \text{as } \varepsilon \to 0.$$

**Example 1.1.4.** Consider $f(x, \varepsilon) = e^{-x/\varepsilon} \sin(e^{x/\varepsilon})$. Observe that for $x > 0$ we have that

$$\lim_{\varepsilon \to 0} \left| \frac{f(x, \varepsilon)}{\varepsilon^n} \right| = 0 \quad \text{for all finite } n,$$

which means that

$$f(x, \varepsilon) \sim 0 + 0 \cdot \varepsilon + 0 \cdot \varepsilon^2 + \ldots \quad \text{as } \varepsilon \to 0.$$

However,

$$\frac{d}{dx} f(x, \varepsilon) = -\frac{1}{\varepsilon} e^{-x/\varepsilon} \sin(e^{x/\varepsilon}) + \frac{1}{\varepsilon} \cos(e^{x/\varepsilon}) \to \infty \quad \text{as } \varepsilon \to 0,$$

i.e. the derivative cannot be expanded using the asymptotic sequence $\{1, \varepsilon, \varepsilon^2, \ldots\}$.

**Example 1.1.5.** Even if $\{\phi_k(\varepsilon)\}$ is an ordered asymptotic sequence, its derivative $\{\phi'_k(\varepsilon)\}$
need not be. Consider $\phi_1(x) = 1 + x$, $\phi_2(x) = \varepsilon \sin(x/\varepsilon)$ for $x \in (0, 1)$. Then $\phi_2 = o(\phi_1)$ but

$$\phi'_1(x) = 1, \quad \phi'_2(x) = \cos(x/\varepsilon),$$

which are not ordered!
The good news is, if
\[ f(x, \varepsilon) \sim a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon) \quad \text{as } \varepsilon \to 0, \] (1.1.1)
and if
\[ \frac{d}{dx} f(x, \varepsilon) \sim b_1(x)\phi_1(\varepsilon) + b_2(x)\phi_2(\varepsilon) \quad \text{as } \varepsilon \to 0, \] (1.1.2)
then \( b_k = \frac{d a_k}{dx} \), i.e. the asymptotic expansion for \( \frac{df}{dx} \) can be obtained from term by term differentiation of (1.1.1). Throughout this course, we will assume that (1.1.2) holds whenever we are given (1.1.1) which is almost always true in practice. Integration, on the other hand, is less problematic. If
\[ f(x, \varepsilon) \sim a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon) \quad \text{as } \varepsilon \to 0 \quad \text{for } x \in [a, b], \]
and all the functions are integrable, then
\[ \int_a^b f(x, \varepsilon) \, dx \sim \left( \int_a^b a_1(x) \, dx \right) \phi_1(\varepsilon) + \left( \int_a^b a_2(x) \, dx \right) \phi_2(\varepsilon) \quad \text{as } \varepsilon \to 0. \]

1.2 Asymptotic Solution of Algebraic and Transcendental Equations

We study three examples where approximate solutions are found using asymptotic expansions, but each uses different method. They serve to illustrate the important point that instead of performing the routine procedure with standard asymptotic sequence, we should tailor our asymptotic expansion to extract the physical property or behavior of our problem.

1.2.1 Singular Quadratic Equation

Consider the quadratic equation
\[ \varepsilon x^2 + 2x - 1 = 0. \] (1.2.1)
This is known as a singular problem since the order of the polynomial (and thus the nature of the equation) changes when \( \varepsilon = 0 \); in this case the unique solution is \( x = 1/2 \). It is clear from Figure 1.1 that there are two real roots for sufficiently small \( \varepsilon \); one is located slightly to the left of \( x = 1/2 \) and one far left on the \( x \)-axis. This means that the asymptotic expansion should not start out as
\[ x(\varepsilon) \sim \varepsilon x_0 + \ldots, \]
because then \( x(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Therefore, we try the asymptotic expansion
\[ x(\varepsilon) \sim x_0 + \varepsilon^\alpha x_1 + \ldots \quad \text{as } \varepsilon \to 0, \] (1.2.2)
for some \( \alpha > 0 \). Substituting (1.2.2) into (1.2.1) leads to
\[
\varepsilon \left[ x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \ldots \right] + 2 \left[ x_0 + \varepsilon^\alpha x_1 + \ldots \right] - 1 = 0. \] (1.2.3)
Since we require (1.2.3) to hold as \( \varepsilon \to 0 \), this results in the \( O(1) \) equation
\[
2x_0 - 1 = 0 \implies x_0 = \frac{1}{2}.
\]
Next, the \( O(\varepsilon) \) in (1) must be balanced by the \( O(\varepsilon^\alpha) \) term in (2). This means we must choose \( \alpha = 1 \) and the \( O(\varepsilon) \) equation has the form
\[
x_0^2 + 2x_1 = 0 \implies x_1 = -\frac{1}{8}.
\]
Consequently, a two-term expansion of one of the roots is
\[
x^{(1)}(\varepsilon) \sim \frac{1}{2} - \frac{\varepsilon}{8} + \ldots \quad \text{as } \varepsilon \to 0.
\]
The chosen ansatz (1.2.2) produce an approximation for the root near \( x = \frac{1}{2} \) and we missed the other root because it approaches negative infinity as \( \varepsilon \to 0 \). One possible method to generate the other root is to consider solving
\[
\varepsilon(x - x_1)(x - x_2) = 0,
\]
but a more systematic method which is applicable to ODEs is to avoid the \( O(1) \) solution. Take
\[
x \sim \varepsilon^\gamma (x_0 + \varepsilon^\alpha x_1 + \ldots) \quad \text{as } \varepsilon \to 0, \tag{1.2.4}
\]
for some \( \alpha > 0 \). Substituting (1.2.4) into (1.2.1) gives
\[
\varepsilon^{1+2\gamma} \left[ x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \ldots \right] + 2\varepsilon^\gamma \left[ x_0 + \varepsilon^\alpha x_1 + \ldots \right] - \frac{1}{3} = 0. \tag{1.2.5}
\]
The terms on the LHS must balance to produce zero, and we need to determine the order of the problem that comes from this balancing. There are 3 possibilities on leading order:

1. Set \( \gamma = 0 \) and we recover the root \( x^{(1)}(\varepsilon) \) on balancing (2) and (3).
2. Balance 1 and 3 and 2 is higher order. The condition $1 \sim 3$ requires
\[
1 + 2\gamma = 0 \implies \gamma = -\frac{1}{2},
\]
so that the leading order term in 1, 3 are of $O(1)$, whilst 2 = $O(\varepsilon^{-1/2})$ which is lower order than 1.

3. Balance 1 and 2 and 3 is higher-order. The condition $1 \sim 2$ requires
\[
1 + 2\gamma = \gamma \implies \gamma = -1,
\]
so that the leading order term in 1, 2 are of $O(\varepsilon^{-1})$ and 3 = $O(1)$. This is consistent with the assumption!

Setting $\gamma = -1$ in (1.2.5) and multiplying by $\varepsilon$ result in
\[
(x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \ldots) + 2(x_0 + \varepsilon^\alpha x_1 + \ldots) - \varepsilon = 0. \tag{1.2.6}
\]
The $O(1)$ equation is
\[
x_0^2 + 2x_0 = 0 \implies x_0 = 0 \text{ or } x_0 = -2.
\]
The solution $x_0 = 0$ gives rise to the root $x^{(1)}(\varepsilon)$ by choosing $\alpha = 1$, so the new root is given by taking $x_0 = -2$. Balancing the equation as before means we must choose $\alpha = 1$ and the $O(\varepsilon)$ equation is
\[
2x_0 x_1 + 2x_1 - 1 = 0 \implies x_1 = -\frac{1}{2}.
\]
Hence, a two-term expansion of the second root is
\[
x(\varepsilon) \sim \frac{1}{\varepsilon} \left(-2 - \frac{\varepsilon}{2}\right) \text{ as } \varepsilon \to 0.
\]

Remark 1.2.1. We may choose $x_0 = 1/2$ in (1.2.2) since one of the root should be close to $x = 1/2$ as we “switch on” $\varepsilon$ in the term $\varepsilon x^2$.

1.2.2 Exponential Equation

Unlike algebraic equations, it is harder to determine the number of solutions of transcendental equations in most cases and we must resort to graphical method. Consider the equation
\[
x^2 + e^{\varepsilon x} = 5 \tag{1.2.7}
\]
From Figure 1.2, we see that there are two real solutions nearby $x = \pm 2$. We assume an asymptotic expansion of the form
\[
x(\varepsilon) \sim x_0 + \varepsilon^\alpha x_1 + \ldots \text{ as } \varepsilon \to 0, \tag{1.2.8}
\]
for some $\alpha > 0$. Substituting (1.2.8) into (1.2.7) and Taylor expanding the exponential term $e^{\varepsilon x}$ we obtain
\[
[x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \ldots] + [1 + \varepsilon x_0 + \ldots] = 5. \tag{1.2.9}
\]
1.2. Asymptotic Solution of Algebraic and Transcendental Equations

The \( O(1) \) equation is

\[ x_0^2 + 1 = 5 \implies x_0 = \pm 2. \]

Balancing (1.2.9) means \( \alpha = 1 \) and the \( O(\varepsilon) \) equation is

\[ 2x_0x_1 + x_0 = 0 \implies x_1 = -\frac{1}{2}. \]

Hence, a two-term asymptotic expansion of each solution is

\[ x(\varepsilon) \sim \pm 2 - \frac{\varepsilon}{2} \text{ as } \varepsilon \to 0. \]

1.2.3 Trigonometric Equation

Consider the equation

\[ x + 1 + \varepsilon \sech \left( \frac{x}{\varepsilon} \right) = 0. \]  \hspace{1cm} (1.2.10)

It appears from Figure 1.3 that there exists a real solution and it approaches \( x = -1 \) as \( \varepsilon \to 0. \)
If we naively try
\[ x \sim x_0 + \varepsilon^\alpha x_1 + \ldots \quad \text{as } \varepsilon \to 0, \]
we obtain
\[ [x_0 + \varepsilon^\alpha x_1 + \ldots] + 1 + \varepsilon \text{sech} \left( \frac{x_0 + \varepsilon^\alpha x_1 + \ldots}{\varepsilon} \right) = 0, \]
and see that \( x_0 = -1 \) since \( \text{sech}(x) \leq (0, 1] \) for any \( x \in \mathbb{R} \). However, we cannot balance terms since it is not possible to find \( \alpha \) due to the behavior of \( \text{sech}(x) \). Since any asymptotic sequences works, we assume an asymptotic expansion of the form
\[ x(\varepsilon) \sim x_0 + \mu(\varepsilon) \quad \text{as } \varepsilon \to 0, \] (1.2.11)
where we impose the condition \( \mu(\varepsilon) \ll 1 \) when \( \varepsilon \ll 1 \). Substituting (1.2.12) into (1.2.10) we obtain
\[ [x_0 + \mu(\varepsilon)] + 1 + \varepsilon \text{sech} \left[ \frac{x_0}{\varepsilon} + \frac{\mu(\varepsilon)}{\varepsilon} \right] = 0. \] (1.2.12)
The \( O(1) \) equation is \( x_0 = -1 \) and (1.2.12) reduces to
\[ \mu(\varepsilon) + \varepsilon \text{sech} \left[ \frac{x_0}{\varepsilon} + \frac{\mu(\varepsilon)}{\varepsilon} \right] = 0. \]
Since
\[ \text{sech} \left( \frac{x_0}{\varepsilon} + \frac{\mu(\varepsilon)}{\varepsilon} \right) \sim \text{sech} \left[ -\frac{1}{\varepsilon} \right] = \frac{2}{e^{1/\varepsilon} + e^{-1/\varepsilon}} \sim 2e^{-1/\varepsilon}, \]
we require
\[ \mu(\varepsilon) = -2e^{-1/\varepsilon} = o(1) \quad \text{as } \varepsilon \to 0. \]
To construct the third term in the expansion, we extend (1.2.12) into
\[ x \sim -1 - 2\varepsilon e^{-1/\varepsilon} + \nu(\varepsilon), \]
where we impose the condition \( \nu(\varepsilon) \ll \varepsilon e^{-1/\varepsilon} \).

### 1.3 Differential Equations: Regular Perturbation Theory

Roughly speaking, regular perturbation theory is a variant of Taylor’s theorem, in the sense that we look for power series solution in \( \varepsilon \). More precisely, we assume that the solution takes the form
\[ x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \quad \text{as } \varepsilon \to 0, \]
where \( x_0 \) is the zeroth-order solution, \( i.e. \) the solution for the case \( \varepsilon = 0 \).
1.3.1 Projectile Motion

Consider the motion of a gerbil projected radially upward from the surface of the Earth. Let \( x(t) \) be the height of the gerbil from the surface of the Earth. Newton’s law of motion asserts that

\[
\frac{d^2x}{dt^2} = - \frac{gR^2}{(x + R)^2},
\]

(1.3.1)

where \( R \) is the radius of the Earth and \( g \) is the gravitational constant. If \( x \ll R \), then to a first approximation we obtain the initial value problem

\[
\frac{d^2x}{dt^2} \approx - \frac{gR^2}{R^2} = -g, \quad x(0) = 0, \quad x'(0) = v_0,
\]

where \( v_0 \) is some initial velocity. The solution is

\[
x(t) = -\frac{gt^2}{2} + v_0 t.
\]

(1.3.2)

Unfortunately, this simplification does not determine a correction to the approximate solution (1.3.2). To this end, we nondimensionalise (1.3.1) with the dimensionless variables

\[
\tau = t / t_c, \quad y = x / x_c,
\]

where \( t_c = v_0 / g \) and \( x_c = v_0^2 / g \) are the chosen characteristic time and length scale respectively. This results in the dimensionless initial-value problem

\[
\frac{d^2y}{d\tau^2} = - \frac{1}{(1 + \varepsilon y)^2}, \quad y(0) = 0, \quad y'(0) = 1.
\]

(1.3.3)

Observe that the dimensionless parameter \( \varepsilon = \frac{x_c}{R} = \frac{v_0^2}{gR} \) measures how high the projectile gets in comparision to the radius of the Earth. Consider an asymptotic expansion

\[
y(\tau) \sim y_0(\tau) + \varepsilon^2 y_1(\tau) + \ldots \quad \text{as} \quad \varepsilon \to 0.
\]

(1.3.4)
where the exponent $\alpha > 0$ is included since a-priori there is no reason to assume $\alpha = 1$. Assume that we can differentiate (1.3.4) term by term, we obtain using generalised Binomial theorem

$$[y''_0 + \varepsilon^\alpha y'_1 + \ldots] = -\frac{1}{[1 + \varepsilon y_0 + \ldots]^2} \sim -1 + 2\varepsilon y_0 + \ldots,$$

with

$$y_0(0) + \varepsilon^\alpha y_1(0) + \ldots = 0, \quad y'_0(0) + \varepsilon^\alpha y'_1(0) = 1.$$ 

The $O(1)$ equation is

$$y''_0 = -1, \quad y_0(0) = 0, \quad y'_0(0) = 1 \implies y_0(\tau) = -\frac{\tau^2}{2} + \tau,$$

and we must choose $\alpha = 1$ to balance the term $2\varepsilon y_0$. Consequently, the $O(\varepsilon)$ equation is

$$y''_1 = 2y_0, \quad y_1(0) = 0, \quad y'_1(0) = 0 \implies y_1(\tau) = \frac{\tau^3}{3} - \frac{\tau^4}{12}.$$  

Hence, a two-term asymptotic expansion of the solution of (1.3.3) is

$$y(\tau) \sim \tau \left(1 - \frac{1}{2} \tau\right) + \frac{1}{3} \varepsilon \tau^3 \left(1 - \frac{\tau}{4}\right).$$

Note that the $O(1)$ term is the scaled solution of (1.3.1) in a uniform gravitational field and the $O(\varepsilon)$ term (first-order correction) contains the nonlinear effect of the problem.

### 1.3.2 Nonlinear Potential Problem

An interesting physical problem is the model of the diffusion of ions through a solution containing charged molecules. Assuming the solution occupies a domain $\Omega$, the electrostatic potential $\phi(x)$ in the solution satisfies the Poisson-Boltzmann equation

$$\nabla^2 \phi = -\sum_{i=1}^{k} \alpha_i z_i e^{-z_i \phi}, \quad x \in \Omega, \quad (1.3.5)$$

where $\alpha_i$’s are positive constants and $z_i$ is the valence of the $i$th ionic species. The whole system must be neutral and this gives the electroneutrality condition

$$\sum_{i=1}^{k} \alpha_i z_i = 0. \quad (1.3.6)$$

We impose the Neumann boundary condition in which we assume the charge is uniform on the boundary

$$\nabla \phi \cdot n = \partial_n \phi = \varepsilon \quad \text{on} \quad \partial \Omega, \quad (1.3.7)$$

where $n$ is the unit outward normal to $\partial \Omega$.

This nonlinear problem has no known solutions. To deal with this, we invoke the classical Debye-Hückle theory in electrochemistry which assumes that the potential is small enough so that the Poisson-Boltzmann equation can be linearised. Because of the boundary condition
(1.3.7), we may assume the zeroth order solution is 0 and try an asymptotic expansion of the form
\[
\phi \sim \varepsilon (\phi_0(x) + \varepsilon \phi_1(x) + \ldots) \quad \text{as } \varepsilon \rightarrow 0,
\]
(1.3.8)
where a small potential means \(\varepsilon\) is small. Substituting (1.3.8) into (1.3.5) and expanding the exponential function around the point 0 we obtain
\[
\varepsilon \left( \nabla^2 \phi_0 + \varepsilon \nabla^2 \phi + \ldots \right) = -\sum_{i=1}^{k} \alpha_i z_i e^{-\varepsilon z_i (\phi_0 + \varepsilon \phi_1 + \ldots)}
\]
\[
= -\sum_{i=1}^{k} \alpha_i z_i \left[ 1 - \varepsilon z_i (\phi_0 + \varepsilon \phi_1 + \ldots) + \frac{1}{2} \varepsilon^2 z_i^2 (\phi_0 + \varepsilon \phi_1 + \ldots)^2 + \ldots \right]
\]
\[
= -\sum_{i=1}^{k} \alpha_i z_i \left[ 1 - \varepsilon z_i \phi_0 + \varepsilon^2 \left( -z_i \phi_1 + \frac{1}{2} z_i^2 \phi_0^2 \right) + \ldots \right]
\]
\[
\sim \varepsilon \left( \sum_{i=1}^{k} \alpha_i z_i^2 \phi_0 \right) + \varepsilon^2 \left( \sum_{i=1}^{k} \alpha_i z_i^2 \left( \phi_1 - \frac{1}{2} z_i \phi_0^2 \right) \right).
\]
Setting \(\kappa^2 = \sum_{i=1}^{k} \alpha_i z_i^2\), the \(O(\varepsilon)\) equation is
\[
\nabla^2 \phi_0 = \kappa^2 \phi_0 \quad \text{in } \Omega,
\]
(1.3.9a)
\[
\partial_n \phi_0 = 1 \quad \text{on } \partial \Omega.
\]
(1.3.9b)
Setting \(\lambda = \frac{1}{2} \sum_{i=1}^{k} \alpha_i z_i^3\), the \(O(\varepsilon^2)\) equation is
\[
(\nabla^2 - \kappa^2) \phi_1 = -\lambda \phi_0^2 \quad \text{in } \Omega,
\]
(1.3.10a)
\[
\partial_n \phi_1 = 0 \quad \text{on } \partial \Omega.
\]
(1.3.10b)

Take \(\Omega\) to be the region outside the unit sphere, which is radially symmetric. Writing the Laplacian operator \(\nabla^2\) in terms of spherical coordinates, the solution must be independent of the angular variables since the boundary condition is independent of the angular variables. With \(\phi_0 = \phi_0(r)\), the \(O(\varepsilon)\) equation now has the form
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \phi_0}{dr} \right) - \kappa^2 \phi_0 = 0 \quad \text{for } 1 < r < \infty,
\]
(1.3.11a)
\[
\phi_0'(1) = -1,
\]
(1.3.11b)
where the negative sign is due to \(n = -\hat{r}\). The bounded solution of (1.3.11) is
\[
\phi_0(r) = \frac{1}{(1 + \kappa)r} e^{\kappa(1-r)},
\]
where the exponential term is the screening term. With \(\phi_1 = \phi_1(r)\), the \(O(\varepsilon^2)\) equation takes the form
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \phi_0}{dr} \right) - \kappa^2 \phi_1 = -\frac{\lambda}{(1 + \kappa)^2 r^2} e^{2\kappa(1-r)} \quad \text{for } 1 < r < \infty,
\]
(1.3.12a)
Using the method of variation of parameters, the solution of (1.3.12) is
\[ \phi_1(r) = \frac{\alpha}{r} e^{-\kappa r} + \frac{\gamma}{\kappa r} \left[ e^{\kappa r} E_1(3\kappa r) - e^{-\kappa r} E_1(\kappa r) \right] \]
\[ \gamma = \frac{\gamma}{2\kappa(1 + \kappa)^2} \]
\[ \alpha = \frac{\gamma}{\kappa(1 + \kappa)} \left[ (\kappa - 1)e^{2\kappa} E_1(3\kappa) + (\kappa + 1)E_1(\kappa) \right] \]
\[ E_1(z) = \int_{z}^{\infty} \frac{e^{-t}}{t} dt. \]

### 1.3.3 Fredholm Alternative

Let \( L_0 \) and \( L_1 \) be linear differential or integral operators on the Hilbert space \( L^2(\mathbb{R}) \) with inner product
\[ \langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x) \, dx. \]
Consider the perturbed eigenvalue problem
\[ (L_0 + \varepsilon L_1) \phi = \lambda \phi. \] (1.3.13)
Spectral problems are widely studied in the context of time-dependence PDEs when time-harmonic solutions are sought for instance, and we are interested in the behaviour of the spectrum of \( L_0 \) as we perturb \( L_0 \). Suppose further that for \( \varepsilon = 0 \), the unperturbed equation has a unique solution \((\lambda_0, \phi_0)\) with \( \lambda_0 \) non-degenerate. For simplicity, take \( L_0 \) to be self-adjoint, that is
\[ \langle f, L_0 g \rangle = \langle L_0 f, g \rangle. \]
Since \( L_0, L_1 \) are linear, we introduce the asymptotic expansions for both \( \phi \) and \( \lambda \) with asymptotic sequence \( \{ 1, \varepsilon, \varepsilon^2, \ldots \} \)
\[ \phi \sim \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \ldots \]
\[ \lambda \sim \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \ldots. \]

We obtain
\[ (L_0 + \varepsilon L_1) \left[ \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \ldots \right] = \left[ \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \ldots \right] \left[ \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \ldots \right]. \]
The \( O(1) \) equation is \( L_0 \phi_0 = \lambda_0 \phi_0 \) and the \( O(\varepsilon) \) equation is
\[ L_0 \phi_1 + L_1 \phi_0 = \lambda_0 \phi_1 + \lambda_1 \phi_0 \]
\[ (L_0 - \lambda_0 I) \phi_1 = \lambda_1 \phi_0 - L_1 \phi_0. \]
It follows from the Fredholm alternative that a necessary condition for the existence of \( \phi_1 \in L^2(\mathbb{R}) \) is that
\[ (\lambda_1 \phi_0 - L_1 \phi_0) \in \ker((L_0 - \lambda_0 I)^*)^\perp = \ker(L_0 - \lambda_0 I)^\perp, \]
and this in turn provides the solvability condition for $\lambda_1$. Since $\ker(L_0 - \lambda_0 I) = \text{span}(\phi_0)$ and $L_0$ is self-adjoint,

$$0 = \langle \phi_0, (L_0 - \lambda_0 I) \phi_1 \rangle = \lambda_1 \langle \phi_0, \phi_0 \rangle - \langle \phi_0, L_1 \phi_0 \rangle$$

$$\lambda_1 = \frac{\langle \phi_0, L_1 \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle}.$$

This expression of $\lambda_1$ represents the first-order correction to the eigenvalue of the operator $(L_0 + \varepsilon L_1)$. The $O(\varepsilon^n)$ equation can be analysed in a similar manner, where $\lambda_n$ can be found using the solvability condition from the Fredholm alternative, assuming \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\} are non-degenerate.

### 1.4 Problems

1. Consider the transcendental equation

$$1 + \sqrt{x^2 + \varepsilon} = e^x.$$

Explain why there is only one small root for small $\varepsilon$. Find the three term expansion of the root

$$x \sim x_0 + x_1 \varepsilon^\alpha + x_2 \varepsilon^\beta, \quad \beta > \alpha > 0.$$

**Solution:**

2. A classical eigenvalue problem is the transcendental equation

$$\lambda = \tan(\lambda).$$

(a) After sketching the two functions in the equation, establish that there is an infinite number of solutions, and for sufficiently large $\lambda$ takes the form

$$\lambda = \pi n + \pi \frac{\pi}{2} - x_n,$$

with $x_n$ small.

**Solution:**

(b) Find an asymptotic expansion of the large solutions of the form

$$\lambda \sim \varepsilon^{-\alpha} \left( \lambda_0 + \varepsilon^\beta \lambda_1 \right),$$

and determine $\varepsilon, \alpha, \beta, \lambda_0, \lambda_1$.

**Solution:**
3. In the study of porous media one is interested in determining the permeability \( k(s) = F'(c(s)) \), where
\[
\int_0^1 F^{-1}(c - \epsilon r) \, dr = s
\]
\[
F^{-1}(c) - F^{-1}(c - \epsilon) = \beta,
\]
and \( \beta \) is a given positive constant. The functions \( F(c) \) and \( c \) both depend on \( \epsilon \), whereas \( s \) and \( \beta \) are independent of \( \epsilon \). Find the first term in the expansion of the permeability for small \( \epsilon \). Hint: consider an asymptotic expansion of \( c \) and use the fact that \( s \) is independent of \( \epsilon \).

**Solution:**

4. Let \( A \) and \( D \) be real \( n \times n \) matrices.

(a) Suppose \( A \) is symmetric and has \( n \) distinct eigenvalues. Find a two-term expansion of the eigenvalues of the perturbed matrix \( A + \epsilon D \), where \( D \) is positive definite.

**Solution:** We assume the asymptotic expansions of the eigenpairs \((\lambda, x)\):
\[
\lambda \sim \lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \ldots
\]
\[
x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots.
\]
Substituting these into the eigenvalue equation \((A + \epsilon D)x = \lambda x\) yields
\[
(A + \epsilon D)(x_0 + \epsilon x_1 + \ldots) = (\lambda_0 + \epsilon \lambda_1 + \ldots)(x_0 + \epsilon x_1 + \ldots).
\]
The \( O(1) \) equation is \( Ax_0 = \lambda_0 x_0 \) which means that \((\lambda_0, x_0)\) is the eigenpair of the matrix \( A \). The \( O(\epsilon) \) equation is
\[
Ax_1 + Dx_0 = \lambda_0 x_1 + \lambda_1 x_0,
\]
or
\[
Lx_1 = (A - \lambda_0 I)x_1 = \lambda_1 x_0 - Dx_0.
\]
It follows from the Fredholm Alternative that the solvability condition for \( \lambda_1 \) is
\[
\lambda_1 \in \ker(L^T)^\perp = \ker(L)^\perp = \text{span}(x_0).
\]
Consequently,
\[
0 = x_0^T Lx_1 = x_0^T (\lambda_1 x_0 - Dx_0) \implies \lambda_1 = \frac{x_0^T Dx_0}{x_0^T x_0}.
\]

(b) Consider the matrices
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]
Use this example to show that the \( O(\epsilon) \) perturbation of a matrix need not result in a \( O(\epsilon) \) perturbation of the eigenvalues, nor that the perturbation is smooth (at \( \epsilon = 0 \)).
5. The eigenvalue problem for the vertical displacement $y(x)$ of an elastic string with variable density is

$$\begin{align*}
y'' + \lambda^2 \rho(x, \varepsilon)y &= 0, \quad 0 < x < 1, \\
y(0) = y(1) &= 0.
\end{align*}$$

where $y(0) = y(1) = 0$. For small $\varepsilon$, assume $\rho \sim 1 + \varepsilon \mu(x)$, where $\mu(x)$ is positive and continuous. Consider the asymptotic expansions

$$y \sim y_0(x) + \varepsilon y_1(x), \quad \lambda \sim \lambda_0 + \varepsilon \lambda_1.$$  

(a) Find $y_0, \lambda_0$ and $\lambda_1$. (The latter will involve an integral expression.)

**Solution:** Substituting the given asymptotic expansions together with the approximation $\rho \sim 1 + \varepsilon \mu(x)$ gives

$$\begin{align*}
y_0'' + \varepsilon y_1'' + \ldots + [\lambda_0 + \varepsilon \lambda_1 + \ldots]^2 [1 + \varepsilon \mu(x)] [y_0 + \varepsilon y_1 + \ldots] &= 0.
\end{align*}$$

The $\mathcal{O}(1)$ equation is

$$y_0'' + \lambda^2 y_0 = 0, \quad y_0(0) = y_0(1) = 0,$$

and this boundary value problem has solutions

$$y_{0,n}(x) = A \sin(\lambda_{0,n} x) = A \sin(n\pi x), \quad n \in \mathbb{Z}.$$  

The $\mathcal{O}(\varepsilon)$ equation is

$$y_1'' + \lambda_0^2 y_1 + \lambda_0^2 \mu(x)y_0 + 2\lambda_0 \lambda_1 y_0 = 0, \quad y_1(0) = y_1(1) = 0.$$  

Using integration by parts, one can show that the linear operator $L = \frac{d^2}{dx^2} + \lambda_0^2$ with domain

$$\mathcal{D}(L) = \{ f \in C^2[0, 1]: f(0) = f(1) = 0 \},$$

is self-adjoint with respect to the $L^2$ inner product over $[0, 1]$. Moreover, for a fixed $\lambda_0$ it has a one-dimensional kernel $\ker(L) = \text{span}(\sin(\lambda_0 x))$. We can now determine $\lambda_1$ using Fredholm alternative, this results in

$$0 = \langle y_0, \lambda^2 \mu(x)y_0 \rangle + \langle y_0, 2\lambda_0 \lambda_1 y_0 \rangle$$

$$\lambda_1 = -\frac{\lambda_0^2 \langle y_0, \mu(x)y_0 \rangle}{2\lambda_0 \langle y_0, y_0 \rangle}$$

$$= -\lambda_0 \int_0^1 \mu(x) \sin^2(n\pi x) \, dx.$$
since
\[
\langle y_0, y_0 \rangle = \int_0^1 A^2 \sin^2(n \pi x) \, dx = A^2 \int_0^1 \frac{1 - \cos(2n \pi x)}{2} \, dx = \frac{A^2}{2}.
\]

(b) Using the equation for \( y_1 \), explain why the asymptotic expansion can break down when \( \lambda_0 \) is large.

**Solution:**

6. Consider the following eigenvalue problem:
\[
\int_0^a K(x, s) y(s) \, ds = \lambda y(x), \quad 0 < x < a.
\]
This is a Fredholm integral equation, where the kernel \( K(x, d) \) is known and is assumed to be smooth and positive. The eigenfunction \( y(x) \) is taken to be positive and normalized so that
\[
\int_0^a y^2(s) \, ds = a.
\]
Both \( y(x) \) and \( \lambda \) depend on the parameter \( a \), which is assumed to be small.

(a) Find the first two terms in the expansion of \( \lambda \) and \( y(x) \) for small \( a \).

**Solution:**

(b) By changing variables, transform the integral equation into
\[
\int_0^1 K(a \xi, a r) \phi(r) \, dr = \frac{\lambda}{a} \phi(\xi), \quad 0 < \xi < 1.
\]
Write down the normalisation condition for \( \phi \).

**Solution:**

(c) From part (b) find the two-term expansion for \( \lambda \) and \( \phi(\xi) \) for small \( a \).

**Solution:**

(d) Explain why the expansions in parts (a) and (c) are the same for \( \lambda \) but not the eigenfunction.

**Solution:**

7. In quantum mechanics, the perturbation theory for bound states involves the time-independent Schrödinger equation
\[
\psi'' - [V_0(x) + \varepsilon V_1(x)] \psi = -E \psi, \quad -\infty < x < \infty,
\]
where \( \psi(-\infty) = \psi(\infty) = 0 \). In this problem, the eigenvalue \( E \) represents energy and \( V_1 \) is a perturbing potential. Assume that the unperturbed \( (\varepsilon = 0) \) eigenvalue is nonzero and nondegenerate.
(a) Assuming
\[ \psi(x) \sim \psi_0(x) + \varepsilon \psi_1(x) + \varepsilon^2 \psi_2(x), \quad E \sim E_0 + \varepsilon E_1 + \varepsilon^2 E_2, \]
write down the equation for \( \psi_0(x) \) and \( E_0 \). We will assume in the following that
\[ \int_{-\infty}^{\infty} \psi_0^2(x) \, dx = 1, \quad \int_{-\infty}^{\infty} |V_1(x)| \, dx < \infty. \]

Solution:

(b) Substituting \( \psi(x) = e^{\phi(x)} \) into the Schrodinger equation and derive the equation for \( \phi(x) \).

Solution:

(c) By expanding \( \phi(x) \) for small \( \varepsilon \), determine \( E_1 \) and \( E_2 \) in terms of \( \psi_0 \) and \( V_1 \).

Solution:
Chapter 2
Matched Asymptotic Expansions

CHT: Motivation

2.1 Introductory Example
Consider the boundary value problem
\[\varepsilon y'' + 2y' + 2y = 0, \quad 0 < x < 1\]  
\[y(0) = 0, \quad y(1) = 1.\]  
(2.1.1a)  
(2.1.1b)
with boundary condition \(y(0) = 0, y(1) = 1\). This is a singular perturbation problem since if \(\varepsilon = 0\) then we have a first-order ODE which only requires one boundary condition. As we will see below, for sufficiently small \(\varepsilon > 0\) there exists a thin transition layer with width of \(O(\varepsilon)\) near \(x = 0\), where the solution changes rapidly to match the boundary condition \(y(0) = 0\).

2.1.1 Outer Solution
Since the boundary conditions are of \(O(1)\), we expect the leading-order solution to be of \(O(1)\) as well. Let \(y(x)\) be the outer solution and assume it has an asymptotic expansion
\[y(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \ldots \quad \text{as } \varepsilon \to 0.\]  
(2.1.2)
The fact that we are using regular expansion for the outer solution means that there will be no rapid change of derivative in the region where the outer solution is valid. Substituting (2.1.2) into (2.1.1) and collecting \(O(1)\) terms we obtain
\[2y'_0 + 2y_0 = 0 \implies y_0 = Ae^{-x}.\]
An immediate problem arises: there is only one arbitrary constant but there are 2 boundary conditions to choose. This means that the outer solution (2.1.2) cannot describe the solution over the whole domain \([0, 1]\) and there must be a transition layer near the boundary where \(\varepsilon\) is no longer negligible compared to \(x\).

We make an educated guess about the location of the boundary layer by looking at the signs of derivatives. Suppose the boundary layer is at \(x = 1\), then \(y_0\) should satisfy the left boundary condition which results in \(y_0(x) \equiv 0\). As \(x\) approaches 1, the solution rapidly increases to match the right boundary condition. In this boundary layer, \(y'' \gg 1, y' \gg 1\) but \(y = O(1)\),
2.1. Introductory Example

Figure 2.1: We deduce the sign of $y''$ near the boundary layer by looking at the concavity of the function.

\[ i.e. \text{ there is no way to cancel two large terms. Hence, we guess that the boundary layer is at } x = 0. \]

2.1.2 Inner Solution

Assuming the boundary layer is at $x = 0$, we introduce the **stretched (boundary layer) coordinate**

\[ \tilde{x} = \frac{x}{\varepsilon^\alpha}, \quad \text{where } \alpha > 0. \]

What this transformation does is that it stretches the boundary layer to $\infty$ as $\varepsilon$ becomes small. After changing the variable from $x$ to $\tilde{x}$, we will treat $\tilde{x}$ as fixed when $\varepsilon$ is reduced. From chain rule, it follows that

\[ \frac{d}{dx} = \frac{1}{\varepsilon^\alpha} \frac{d}{d\tilde{x}}. \]

Setting $Y(\tilde{x}) = y(x)$, the ODE becomes

\[ \varepsilon^{1-2\alpha} \frac{d^2 Y}{d\tilde{x}^2} + 2\varepsilon^{-\alpha} \frac{dY}{d\tilde{x}} + 2Y = 0, \quad \text{with } Y(0) = 0. \quad (2.1.3) \]

We try an asymptotic expansion of the form

\[ Y(\tilde{x}) \sim Y_0(\tilde{x}) + \varepsilon^\gamma Y_1(\tilde{x}) + \ldots, \quad \gamma > 0. \quad (2.1.4) \]

A-priori there is no reason to assume that $\gamma = \alpha$. Substituting (2.1.4) into the inner equation (2.1.3) yields

\[ \underbrace{\varepsilon^{1-2\alpha} \frac{d^2}{d\tilde{x}^2}}_{\text{1}} \underbrace{(Y_0 + \varepsilon^\gamma Y_1 + \ldots)}_{\text{2}} + \underbrace{2\varepsilon^{-\alpha} \frac{d}{d\tilde{x}}}_{\text{2}} \underbrace{(Y_0 + \varepsilon^\gamma Y_1 + \ldots)}_{\text{2}} + \underbrace{2(Y_0 + \varepsilon^\gamma Y_1 + \ldots)}_{\text{3}} = 0 \quad (2.1.5) \]

It is now necessary to determine the correct balancing conditions:
1. 1 ∼ 3 and 2 is higher order. This requires
\[ 1 - 2\alpha = 0 \implies \alpha = \frac{1}{2}. \]
Thus 1, 3 = O(1) but 2 = O(ε^{-1/2}), which contradicts the assumption that 2 is higher order.

2. 1 ∼ 2 and 3 is higher order. This requires
\[ 1 - 2\alpha = -\alpha \implies \alpha = 1. \]
Thus 1, 2 = O(ε^{-1}) and 3 = O(1).

3. 2 ∼ 3 reduces to the regular perturbation of the outer solution.
Consequently, we have the O(ε^{-1}) equation
\[ Y''_0 + 2Y'_0 = 0 \quad \text{for } 0 < \tilde{x} < \infty, \text{ with } Y(0) = 0. \] (2.1.6)
The solution is
\[ Y_0(\tilde{x}) = B(1 - e^{-2\tilde{x}}), \text{ where } B \text{ is an unknown constant for now.} \]
We assume that there are no other transition layers, so that the outer solution y_0(x) = Ae^{-x} applies outside the boundary layer. Solving y_0(1) = 1 then gives \( y_0(x) = e^{1-x} \).

2.1.3 Matching
It remains in this problem to determine the constant A. The inner and outer solutions are both approximations of the same function. Hence they should agree in the transition zone between inner and outer layers. One way is that
\[ \lim_{\tilde{x} \to \infty} Y(\tilde{x}) = \lim_{x \to 0^+} y_0(x), \]
or in short hand, \( Y_0(+\infty) = y_0(0^+) \). This means that \( A = e \) and thus
\[ Y_0(\tilde{x}) = e - e^{1-2\tilde{x}}. \]

2.1.4 Composite Expansion
So far, we have a solution in two pieces - neither is uniformly valid for all \( x \in [0,1] \). We would like to construct a composite solution that holds everywhere to some approximation (asymptotically).

Remark 2.1.1.
1. The matching condition
\[ \lim_{\tilde{x} \to \infty} Y_0(\tilde{x}) = \lim_{x \to 0^+} y_0(x), \]
may not work. First, the limit might not exist! Second, complications may arise when constructing second order terms. To deal with this, a more general approach is to explicitly introduce an intermediate region between inner and outer domains. Introducing an intermediate variable \( x_\eta = \frac{x}{\eta(\varepsilon)} \) with \( \varepsilon \ll \eta \ll 1 \). The inner and outer solution should give the same result when expressed in terms of \( x_\eta \).
(a) Change from $x$ to $x_\eta$ in the outer expansion $y_{\text{outer}}(x_\eta)$. Assume there exists $\eta_1(\varepsilon)$ such that $y_{\text{outer}}$ is valid for $\eta_1(\varepsilon) \ll \varepsilon(\varepsilon) \leq 1$.

(b) Change variables $\tilde{x} \mapsto x_\eta$ in inner expansion to obtain $Y_{\text{inner}}$. Assume there exists $\eta_2(\varepsilon)$ such that $Y_{\text{inner}}$ is valid for $\varepsilon \leq \eta_2(\varepsilon) \ll \eta_2$.

(c) If $\eta_1 \ll \eta_2$, then the domains of validity overlap and we require $y_{\text{outer}} \sim y_{\text{inner}}$ in the overlap region.

Let us return to our particular example. Let $x_\eta = x / \varepsilon^\beta$ with $0 < \beta < 1$.

$$
Y_{\text{inner}} \sim A \left(1 - e^{-2x_\eta / \varepsilon^{1-\beta}}\right) \sim A
$$

$$
y_{\text{outer}} \sim e^{1-x_\eta \varepsilon^\beta} \sim e \left[1 - x_\eta \varepsilon^\beta\right] = e + \mathcal{O}(\varepsilon^\beta)
$$

2. Finding the second term. Going back to our problem, the $\mathcal{O}(\varepsilon)$ outer equation is

$$
y_1' + y_1 = -1/2y_0' \quad y_1(1) = 0,
$$

which has solution $y_1 = (1/2)(1 - x)e^{1-x}$. The $\mathcal{O}(1)$ inner equation is

$$
Y_1'' + 2Y_1' = -2Y_0, \quad Y_1(0) = 0,
$$

which has solution

$$
Y_1(\tilde{x}) = B \left(1 - e^{-2\tilde{x}}\right) - \tilde{x}e \left(1 + e^{-2\tilde{x}}\right).
$$

We determine $B$ using matching in the intermediate regime.

$$
y_{\text{outer}} \sim e^{1-x_\eta \varepsilon^\beta} + \frac{\varepsilon}{2} \left[1 - x_\eta \varepsilon^\beta\right] e^{1-x_\eta \varepsilon^\beta}
$$

$$
\sim e - \varepsilon^\beta x_\eta e + \frac{\varepsilon}{2} e + \frac{1}{2} \varepsilon^2 e x_\eta^2 + \ldots
$$

$$
Y_{\text{inner}} \sim e \left(1 - e^\xi\right) + \varepsilon \left[B \left(1 - e^\xi\right) - \frac{x_\eta}{\varepsilon^{1-\beta}} e(1 + e^\xi)\right], \quad \xi = -\frac{2x_\eta}{\varepsilon^{1-\beta}}
$$

$$
\sim e - \varepsilon^\beta x_\eta e + B \varepsilon + \ldots.
$$

This gives $B = \frac{\xi}{2}$. The composite solution is

$$
y \sim y_0 + \varepsilon y_1 + Y_0 + \varepsilon Y_1 - \left(e - x_\eta \varepsilon^\beta e + \frac{\varepsilon}{2} e\right)
$$

2.1.5 Things to look for in more general problem on $[0,1]$

1. The boundary layer could be at $x = 1$ or there could be a boundary layer at both ends. At $x = 1$, the stretched coordinate is $\tilde{x} = (x - 1) / \varepsilon^\alpha$.

2. There is an interior layer at some $x_0(\varepsilon)$, so

$$
\tilde{x} = \frac{x - x_0}{\varepsilon^\alpha}.
$$

3. $\varepsilon$-dependence could be funky, for example

$$
\nu = \frac{1}{\ln \varepsilon}.
$$

4. The solution does not have a layered structure, for example, this matching method is not able to catch the behavior if we have an asymptotic sequence $e^{-a/\varepsilon}$.
2.2 Extensions: Multiple Boundary Layers

2.2.1 Multiple Boundary Layers

Consider the boundary value problem

\[ \varepsilon^2 y'' + \varepsilon xy' - y = -e^x \]
\[ y(0) = 2, \quad y(1) = 1 \]

2.2.2 Interior Boundary Layers

It is also possible for a boundary layer to occur in the interior of the domain rather than at a physical boundary, matching now has to determine the location of the interior layers.

2.3 Partial Differential Equations

Consider Burger’s equation

\[ u_t + uu_x = \varepsilon u_{xx}, \quad -\infty < x < \infty, t > 0, \]

with initial condition \( u(x, 0) = \phi(x) \). This is a second-order quasilinear PDEs which can be solved using method of characteristics. Assume that \( \phi(x) \) is smooth and bounded, except for a jump discontinuity at \( x = 0 \) with \( \phi(0^-) > \phi(0^+) \), \( \phi' \geq 0 \). For concreteness, take

\[ u(x, 0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases} \]

This is an example of a Riemann problem that evolves into a travelling front that sharpens as \( \varepsilon \to 0 \). Observes that this is a singular problem, since the parabolic PDE becomes a first-order hyperbolic PDE when \( \varepsilon = 0 \).

2.3.1 Outer Solution

2.4 Eigenvalue Asymptotics in 3D

Let \( \Omega \) be a three-dimensional bounded domain with a hole of “radius” \( O(\varepsilon) \) removes from \( \Omega \). We are interested in solving

\[ \Delta u + \lambda u = 0 \quad \text{in } \Omega \setminus \Omega_\varepsilon \]  \hspace{1cm} (2.4.1a)
\[ u = 0 \quad \text{on } \partial \Omega \] \hspace{1cm} (2.4.1b)
\[ u = 0 \quad \text{on } \partial \Omega_\varepsilon \] \hspace{1cm} (2.4.1c)
\[ \int_\Omega u^2 \, dx = 1. \] \hspace{1cm} (2.4.1d)

We assume that \( \Omega_\varepsilon \) shrinks to a point \( x_0 \) as \( \varepsilon \to 0 \). For example, we could take \( \Omega_\varepsilon \) to be the sphere \( |x - x_0| \leq \varepsilon \). The unperturbed problem is

\[ \Delta \phi + \mu \phi = 0 \quad \text{in } \Omega \] \hspace{1cm} (2.4.2a)
\[ \phi = 0 \quad \text{on } \partial \Omega \quad (2.4.2b) \]
\[ \int_{\Omega} \phi^2 \, dx = 1 \quad (2.4.2c) \]

We assume that this has eigenpairs \((\phi_j(x), \mu_j)\), \(j = 0, 1, 2, \ldots\) with
\[ \int_{\Omega} \phi_j \phi_k \, dx = 0 \quad \text{for } j \neq k, \]
and \(\phi_0(x) > 0\) for \(x \in \Omega\). [Self-adjoint operators, discrete spectrum] We look for an eigenpair \(\varepsilon > 0\) near the \((\phi_0, \mu_0)\). Expand
\[ \lambda \sim \mu_0 + \nu(\varepsilon) \lambda_1 + \ldots, \]
where \(\nu(\varepsilon) \to 0\) as \(\varepsilon \to 0\). In the outer region away from the hole, we take
\[ u = \phi_0(x) + \nu(\varepsilon) u_1(x). \]

Since \(\Omega_\varepsilon \to \{x_0\}\) as \(\varepsilon \to 0\), the \(O(\nu(\varepsilon))\) outer equation is
\[ \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 \quad \text{in } \Omega \setminus \{x_0\} \quad (2.4.3a) \]
\[ u_1 = 0 \quad \text{on } \partial \Omega \quad (2.4.3b) \]
\[ \int_{\Omega} 2u_1 \phi_0 \, dx = 1 \quad (2.4.3c) \]

Construct the inner solution near the hole. Let
\[ y = \frac{|x - x_0|}{\varepsilon}. \]

Set \(V(x; \varepsilon) = u(x_0 + \varepsilon y)\). We find that \(v\) satisfies the following inner equation
\[ \Delta_y v + \lambda \varepsilon^2 v = 0 \quad \text{outside } \Omega_0 := \Omega_\varepsilon / \varepsilon \]

Let \(V = V_0 + \nu(\varepsilon)V_1 + \ldots\), then
\[ \Delta_y V_0 = 0 \quad \text{for } y \notin \Omega_0 \quad (2.4.4a) \]
\[ V_0 = 0 \quad \text{on } \partial \Omega_0 \quad (2.4.4b) \]
\[ V_0 \to \phi_0(x_0) \quad \text{as } |y| \to \infty \quad (2.4.4c) \]
Chapter 3

Method of Multiple Scales

3.1 Introductory Example

As in the previous chapter, we will introduce the ideas underlying the method by a simple example. Consider the initial value problem

\[ y'' + \varepsilon y' + y = 0 \quad \text{for } t > 0 \]  
\[ y(0) = 0, \quad y'(0) = 1 \]

which models a linear oscillator with weak damping. This reduces to the linear oscillator model when \( \varepsilon = 0 \) and we do not expect boundary layers since this is not a singular problem. This suggests that the solution might have a regular asymptotic expansion, i.e. we try a regular expansion

\[ y(t) \sim y_0(t) + \varepsilon y_1(t) + \ldots \quad \text{as } \varepsilon \to 0 \]

Substituting (3.1.2) into (3.1.1) and collecting terms in equal powers of \( \varepsilon \) gives

\[ y''_0 + y_0 = 0 \]
\[ y''_n + y_n = -y'_{n-1}, \quad n \geq 1, \]

with initial conditions

\[ y_0(0) = 0, \quad y'_0(0) = 1, \quad y_n(0) = y'_{n}(0) = 0, \quad n \geq 1. \]

Solving \( O(1) \) and \( O(\varepsilon) \) equations we obtain

\[ y(t) \sim \sin(t) - \frac{1}{2} \varepsilon t \sin(t), \]

but this is problematic since the correction term \( y_1(t) \) contains a secular term \( t \sin(t) \) which blows up as \( t \to \infty \). Consequently, the asymptotic expansion is valid for only small values of \( t \), since \( \varepsilon y_1(t) \sim y_0(t) \) when \( \varepsilon t \sim 1 \). The problem is that regular perturbation theory does not capture the correct behavior of the exact solution. Indeed, (3.1.1) is a constant-coefficient linear ODE, and it has the exact solution given by

\[ y(t) = \frac{1}{\sqrt{1 - \varepsilon^2/4}} e^{-\sqrt{1 - \varepsilon^2/4}} \sin\left(t \sqrt{1 - \varepsilon^2/4}\right) \]
3.1. Introductory Example

Figure 3.1: Comparison between the regular asymptotic approximation (3.1.3) and the exact solution (3.1.4) for $\varepsilon = 0.1$.

We immediately see that the exact solution decays but the first term in our regular asymptotic approximation (3.1.3) does not. Note also that we will pick up the secular terms if we naively expand the exponential function around $t = 0$, since

$$y(t) \approx \left(1 - \frac{\varepsilon t}{2} + \frac{\varepsilon^2 t^2}{8} + \ldots\right) \sin(t).$$

3.1.1 Multi-Scale Expansion

In fact, there are two time-scales in the exact solution:

1. The slowly decaying exponential component which varies on a time-scale of $O(1/\varepsilon)$;
2. The fast oscillating component which varies on a time-scale of $O(1)$.

To identify or separate these time-scales, we introduce the variables

$$t_1 = t, \quad t_2 = \varepsilon^\alpha t, \quad \alpha > 0,$$

where $t_2$ is called the slow time-scale because it does not affect the asymptotic expansion until $t\varepsilon^\alpha \sim 1$. We treat these two time-scales as independent variables and as a consequence, the original time derivative becomes

$$\frac{d}{dt} = \frac{dt_1}{dt} \frac{\partial}{\partial t_1} + \frac{dt_2}{dt} \frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \varepsilon^\alpha \frac{\partial}{\partial t_2}. \quad (3.1.5)$$

Substituting (3.1.5) into (3.1.1) yields the transformed problem

$$\left[\partial^2_{t_1} + 2\varepsilon^\alpha \partial_{t_1} \partial_{t_2} + \varepsilon^{2\alpha} \partial^2_{t_2}\right] y + \varepsilon (\partial_{t_1} + \varepsilon^\alpha \partial_{t_2}) y + y = 0, \quad (3.1.6a)$$
Unlike the original problem, additional constraints are needed for (3.1.6) to have a unique solution, and it is precisely this degree of freedom that allows us to eliminate the secular terms.

We now introduce an asymptotic expansion

\[ y \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \ldots \]  

Substituting (3.1.7) into (3.1.6) yields

\[
\frac{\partial^2}{\partial t_1^2} y_0 + 2\varepsilon^\alpha \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} y_0 + \varepsilon^2 \alpha^2 \frac{\partial^2}{\partial t_2^2} y_0 + \varepsilon \left( \frac{\partial}{\partial t_1} + \varepsilon^\alpha \frac{\partial}{\partial t_2} \right) (y_0 + \ldots) + (y_0 + \varepsilon y_1 + \ldots) = 0.
\]

- The \(\mathcal{O}(1)\) equation is

\[
\left( \frac{\partial^2}{\partial t_1^2} + 1 \right) y_0 = 0, \\
y_0(0, 0) = 0, \quad \partial_{t_1} y_0(0, 0) = 1,
\]

which has the general solution

\[ y_0(t_1, t_2) = a_0(t_2) \sin(t_1) + b_0(t_2) \cos(t_1), \]

where \(a_0(0) = 1, b_0(0) = 0\). Note that \(y_0(t_1, t_2)\) consists of purely harmonic components with slowly varying amplitude. We now need to determine \(\alpha\) in the slow time-scale \(t_2\).

Observe that the \(\mathcal{O}(\varepsilon)\) equation is

\[
\left( \frac{\partial^2}{\partial t_1^2} + 1 \right) y_1 = -\partial_{t_1} y_0,
\]

and the inhomogeneous term \(\partial_{t_1} y_0\) will generate secular terms, since it belongs to the kernel of homogeneous linear operator \(\left( \frac{\partial^2}{\partial t_1^2} + 1 \right)\). More importantly, there is no way to generate non-trivial solution that will cancel the secular term. This can be prevented by choosing \(\alpha = 1\).

- The \(\mathcal{O}(\varepsilon)\) equation is

\[
\left( \frac{\partial^2}{\partial t_1^2} + 1 \right) y_1 = -2\partial_{t_1} \partial_{t_2} y_0 - \partial_{t_1} y_0, \\
y_1(0, 0) = 0, \quad \partial_{t_1} y_1(0, 0) + \partial_{t_2} y_0(0, 0) = 0.
\]

Substituting \(y_0\) gives

\[
\left( \frac{\partial^2}{\partial t_1^2} + 1 \right) y_1 = -2(a'_0 \cos(t_1) - b'_0 \sin(t_1)) - (a_0 \cos(t_1) - b_0 \sin(t_1)) \\
= (2b'_0 + b_0) \sin(t_1) - (2a'_0 + a_0) \cos(t_1).
\]

The general solution of the \(\mathcal{O}(\varepsilon)\) problem is

\[ y_1(t_1, t_2) = a_1(t_2) \sin(t_1) + b_1(t_2) \cos(t_1) \\
- \frac{1}{2} \left( \frac{b_0}{2} \right) \frac{\partial}{\partial t_1} \cos(t_1) - \frac{1}{2} \left( \frac{a_0}{2} \right) \frac{\partial}{\partial t_1} \sin(t_1), \]
with \( a_1(0) = a'_0(0), b_1(0) = 0 \). CHT: I think this is wrong, it should be \( a_1(0) = -b'_0(0) \). We can choose the functions \( a_0, b_0 \) to remove the secular terms, which results in

\[
2b'_0 + b_0 = 0 \implies b_0(t_2) = \beta_0 e^{-t_2^2/2} = 0, \quad \text{since } b_0(0) = 0,
\]

and

\[
2a'_0 + a_0 = 0 \implies a_0(t_2) = \alpha_0 e^{-t_2^2/2} = e^{-t_2^2/2}, \quad \text{since } a_0(0) = 0.
\]

Hence, the first term approximation of the solution \( y(t) \) of (3.1.1) is

\[
y \sim e^{-\varepsilon t/2} \sin(t).
\]

One can prove that this asymptotic expansion is uniformly valid for \( 0 \leq t \leq \mathcal{O}(1/\varepsilon) \).

### Remark 3.1.1.

1. Many problems have the \( \mathcal{O}(1) \) equation as

\[
y''_0 + \omega^2 y_0 = 0.
\]

which has general solutions

\[
y_0(t) = a \cos(\omega t) + b \sin(\omega t).
\]

If the original problem is nonlinear and the \( \mathcal{O}(1) \) equation is as above, then it is usually more convenient to use a complex representation of \( y_0 \), i.e.

\[
y(t) = A e^{i \omega t} + \bar{A} e^{-i \omega t} = B \cos(\omega t + \theta).
\]

The reason is that these complex representations make identify the secular terms much easier.

2. Often, higher-order equations have the form

\[
y''_n + \omega^2 y_n = f(t).
\]

A secular term arises if \( f(t) \) contains a solution of the \( \mathcal{O}(1) \) problem, e.g. \( \cos(\omega t) \) or \( \sin(\omega t) \). We can avoid secular terms by requiring the \( t_2 \)-dependent coefficients of \( \cos(\omega t_1) \) and \( \sin(\omega t_1) \) to vanish. For example, there are no secular terms if

\[
f(t) = \sin(\omega t) \cos(\omega t) = \sin(2\omega t)/2,
\]

but there is a secular term if

\[
f(t) = \cos^3(\omega t) = \frac{1}{4} (3 \cos(\omega t) + \cos(3\omega t)).
\]

3. Variations on a theme.

(a) Several time-scales: e.g. \( t_1 = t/\varepsilon, t_2 = t, t_3 = \varepsilon t, \ldots \).
(b) More complex $\varepsilon$-dependency:

$$t_1 = \left(1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \ldots\right) t, \ t_2 = \varepsilon t.$$ 

This is sometimes called the **Lindstedt’s method** or the **method of strained coordinates**.

(c) Correct scaling may not be obvious:

$$t_1 = \varepsilon^\alpha t, \ t_2 = \varepsilon^\beta t, \ \alpha < \beta.$$ 

(d) Nonlinear time-dependence:

$$t_1 = f(t, \varepsilon), \ t_2 = \varepsilon t.$$ 

### 3.2 Forc ed Motion Near Resonance

Consider a damped nonlinear oscillator that is forced at a frequency near resonance. As an example, we will study the damped Duffing equation

$$y'' + \varepsilon \lambda y' + y + \varepsilon \kappa y^3 = \varepsilon \cos ((1 + \varepsilon \omega) t), \ t > 0,$$

with $y(0) = 0, y'(0) = 0$. Expect solution to be small. Consider the simpler equation

$$y'' + y = \varepsilon \cos(\Omega t), \ \Omega \neq \pm 1, \ y(t) = \frac{\varepsilon}{1 - \Omega^2} [\cos(\Omega t) - \cos t].$$

If $\Omega = 1 + \varepsilon w$, then the particular solution for $w \neq 0, -2/\varepsilon$

$$y(t) = -\frac{\cos(1 + \varepsilon w) t}{w(2 + \varepsilon w)}.$$

If $w = 0, -2/\varepsilon$, then

$$y(t) = \frac{1}{2} \varepsilon t \cdot \sin t,$$

which is secular. Take $t_1 = t$ and $t_2 = \varepsilon t$. Then we have

$$[\partial_1^2 + 2\varepsilon \partial_1 \partial_2 + \varepsilon^2 \partial^2_2]y + \varepsilon \lambda [\partial_1 + \varepsilon \partial_2]y + y + \varepsilon \kappa y^3 = \varepsilon \cos(t_1 + \varepsilon w t_1).$$

Assume that $y \sim \varepsilon^\beta y_0(t_1, t_2) + \varepsilon^\gamma y_1(t_1, t_2) + \cdots, \ \beta < \gamma$ and $\beta < 1$. Substituting into the equation gives

$$[\varepsilon^\beta \partial_1^2 y_0 + 2\varepsilon^{1+\beta} \partial_1 \partial_2 y_0 + \varepsilon^2 \partial^2_2 y_0 + \cdots] + [\varepsilon^{1+\beta} \lambda \partial_1 y_0 + \varepsilon^\beta y_0 + \varepsilon^\gamma y_1 + \cdots] + [\varepsilon^{1+3\beta} \kappa y_0^3 + \cdots] = \varepsilon \cos(t_1 + \varepsilon w t_1).$$

Balancing $O(\varepsilon^\beta)$ yields that

$$[\partial_1^2 + 1]y_0 = 0, \ y_0(0,0) = \partial_1 y(0,0) = 0.$$
Hence, \( y_0 = A(t_2) \cos(t_1 + \theta(t_2)) \), \( A(0) = 0 \). Terms involving \( y_0 \) or input term are all generating secular terms. They are all \( O(\epsilon) \) if \( \gamma = 1 \) and \( \beta = 0 \).

\[
\partial_t^2 y_0 - 2\partial_t y_0 - \lambda y_0 - \kappa y_0^3 + \cos(t_1 + wt_2) \\
= (2A' + \lambda A) \sin(t_1 + \theta) + 2\theta' A \cos(t_1 + \theta) \\
- \frac{\kappa}{4} A^3 [3 \cos(t_1 + \theta) + \cos 3(t_1 + \theta)] + \cos(t_1 + wt_2).
\]

Then we can remove \( \sin(t_1 + \theta) \) and \( \cos(t_1 + \theta) \) by requiring

\[
\begin{align*}
2A' + \lambda A &= -\sin(\theta - wt_2), \\
2\theta' A - 3\kappa A^3 / 4 &= -\cos(\theta - wt_2).
\end{align*}
\]

From \( A(0) = 0 \) and assuming \( A'(0) > 0 \), it follows that \( \theta(0) = -\pi/2 \). We now analyze the above amplitude equation with changing notation

\[
\begin{align*}
2r' &= -\kappa r - \frac{\gamma}{2} \sin \theta, \\
2\theta' &= \beta + 3r^2 - \frac{\gamma}{2r} \cos \theta. \tag{3.2.1}
\end{align*}
\]

The nullcline for \( r \) is \( r = -\gamma \sin \theta / 2\kappa \). Similarly, nullcline for \( \theta \) is given by \( \cos \theta = 2r(\beta + 3r^2)/\gamma \equiv F(r, \beta) \).

- If \( \beta > 0 \), there is unique \( r \) for each \( \theta \).
- If \( 0 > \beta > \beta_c \) where \( \min_r F(r, \beta_c) = -1 \), then there are two values of \( r \) for each \( \cos \theta \) in some interval \((-z, 0)\), \( 0 < z < 1 \). (It turns out that \( \beta_c^3 = -81\gamma^2 / 16 \).)
- If \( \beta < \beta_c \), then two values of \( r \) exist for all \( \cos \theta \) between \(-1\) and \( 0 \).

Then the nullcline growth with \( 1/\kappa \) is

- For small \( \kappa \), only one stable fixed point.
- If \( \kappa = \kappa_1 C \), there is a SN bifurcation, that is, saddle and a stable FP.
- At \( \kappa = \kappa_2 C \), there is a second SN bifurcation in which saddle and other stable FP (from \( A \)) annihilate leaning the stable FP (from B).

### 3.3 Periodically Forced Nonlinear Oscillators and Isochrones

Consider a general model of a nonlinear oscillator

\[
\frac{du}{dt} = f(u), \quad u = (u_1, \ldots, u_M), \quad \text{with } M \geq 2.
\]

Suppose there exists a stable periodic solution \( U(t) = U(t + \Delta_0) \), where the natural frequency is \( \omega_0 = \frac{2\pi}{\Delta_0} \). In state space, the solution is an isolated trajectory called a limit cycle. We can represent the dynamics in terms of a uniformly rotating phase, i.e.

\[
\frac{d\phi}{dt} = \omega_0 \quad \text{and} \quad U(t) = g(\phi(t)),
\]
where $g$ is $2\pi$-periodic.

Now, suppose a small external periodic input is applied to the oscillator:

$$\frac{du}{dt} = f(u) + \varepsilon P(u, t),$$

where $P(u, t) = P(u, t + \Delta)$ with $\omega = \frac{2\pi}{\Delta}$ the forcing frequency. If the perturbation is sufficiently small, then the resulting deviations transverse to the limit cycle will decay in time such that the main effect of the perturbation is a phase-shift along the limit cycle.

### 3.3.1 Isochrones

Suppose we observe the unperturbed system stroboscopically at time intervals of length $\Delta_0$. This leads to a Poincaré mapping

$$u(t) \rightarrow u(t + \Delta_0) \equiv \Phi(u(t)).$$

The map $\Phi$ has all points on the limit cycle as fixed points. Choose a point $U$ on the limit cycle and consider all points in a neighbourhood of $U$ in $\mathbb{R}^M$ that are attracted to it under the action of $\Phi$. These points form an $(M-1)$-dimensional hypersurface $I$, called an isochrone that crosses the limit cycle at $U$.

**Example 3.3.1.** Consider

$$\frac{dA}{dt} = (1 + i\eta) A - (1 + i\alpha)|A|^2, \quad A = Re^{i\theta}.$$ 

In polar coordinates, we have

$$\frac{dR}{dt} = R(1 - R^2)$$
$$\frac{d\theta}{dt} = \eta - \alpha R^2.$$ 

Observe that the origin is unstable and the unit circle is a limit cycle. The explicit solution is

$$R(t) = \left[1 + \frac{1 - R_0^2}{R_0} e^{-2t}\right]^{-1/2},$$

$$\theta(t) = \theta_0 + \omega_0 t - \frac{\alpha}{2} \ln \left[R_0^2 + (1 - R_0^2)e^{-2t}\right] \pmod{2\pi},$$

$$\omega_0 = \eta - \alpha.$$ 

Strobing the solution at time $t = n\Delta_0$, we see that

$$\lim_{n \to \infty} \theta(n\Delta_0) = \theta_0 - \alpha \ln R_0.$$ 

With this in mind, we define on the whole plane

$$\Phi(R, \theta) = \theta - \alpha \ln R.$$
### 3.3.2 Phase Equation

For an unperturbed oscillator in the vicinity of the limit cycle,

\[ \omega_0 = \frac{d\Phi(u)}{dt} = \sum_{k=1}^{M} \frac{\partial \Phi}{\partial u_k} \frac{du_k}{dt} = \sum_{k=1}^{n} \frac{\partial \Phi}{\partial u_k} f_k(u). \]

To a first approximation, we consider the perturbed system with the unperturbed isochrones:

\[ \frac{d\phi(u)}{dt} = \sum_{k=1}^{M} \frac{\partial \phi}{\partial u_k} \left[ f_k(u) + \varepsilon P_k(u, t) \right] = \omega_0 + \sum_{k=1}^{m} \frac{\partial \phi}{\partial u_k} P_k(u, t). \]

The next step is to neglect deviations of \( u \) from the limit cycle solution \( U \). Consequently,

\[ \frac{d\phi(u)}{dt} = \omega_0 + \sum_{k=1}^{m} \frac{\partial \phi(U)}{\partial u_k} P_k(U, t). \]

Finally, since points on the limit cycle are in one-to-one correspondence with the phase \( \phi \), we have a closed phase equation

\[ \frac{d\Phi}{dt} = \omega_0 + \varepsilon Q(\phi, t), \]

where

\[ Q(\phi, t) = \sum_{k=1}^{M} \frac{\partial \phi(U(\phi))}{\partial u_k} P_k(U(\phi), t). \]

**Example 3.3.2.** Consider the previous example in Cartesian coordinate:

\[
\begin{align*}
\frac{dx}{dt} &= x - \eta y - (x^2 + y^2) (x - \eta y) + \varepsilon \cos(\omega t) \\
\frac{dy}{dt} &= y + \eta y - (x^2 + y^2) (y + \alpha x)
\end{align*}
\]

We have established that the isochrone is given by

\[ \phi = \tan^{-1} \left( \frac{y}{x} \right) - \frac{\alpha}{2} \ln \left( x^2 + y^2 \right). \]

Differentiating with respect to \( x \) gives

\[ \frac{d\phi}{dx} = -\frac{y}{x^2 + y^2} - \frac{\alpha x}{x^2 + y^2}. \]

On the limit cycle \( u_0 = (x_0, y_0) = (\cos \phi, \sin \phi) \),

\[ \frac{d\phi}{dx}(u_0(\phi)) = -\sin \phi - \alpha \cos \phi, \]

and the phase equation is

\[ \frac{d\phi}{dt} = \omega_0 - \varepsilon (\alpha \cos \phi + \sin \phi) \cos (\omega t). \]
3.3.3 Phase Resetting Curves

$Q(\phi, t)$ can be related to a measurable property known as a phase resetting curve (PRC). Let us denote this by a $2\pi$-periodic function $R(\theta)$. The basic idea is that we perturb the oscillator (perhaps a particular degree of freedom) by an impulse at different times around the limit cycle and measure the resulting phase-shift relative to the unperturbed oscillator. Suppose we perturb $u_1$, then

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon \frac{\partial \phi}{\partial u_1} (U(\phi)) \delta(t - t_0).$$

Integrating over a small time interval around $t_0$, we obtain

$$\Delta \phi = \varepsilon R(\phi), \quad R(\phi) = \frac{\partial \phi}{\partial u_1} (U(\phi)).$$

Given $R(\phi)$, a general perturbation is

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon R(\phi) P(t).$$

3.3.4 Averaging Theory

Recall the phase equation

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon Q(\phi, t).$$

Expand $Q(\phi, t)$ as a double Fourier series

$$Q(\phi, t) = \sum_{i,k} a_{i,k} e^{ik\phi + il\omega t}.$$

where $\phi \sim \phi_0 + \omega_0 t$ to a leading order approximation. It follows that $Q$ contains fast oscillating terms (compared to $1/\varepsilon$) together with slowly varying terms that satisfies the resonance condition

$$k\omega_0 + l\omega \approx 0;$$

only the latter will lead to significant variations in the phase (about the unperturbed case). The simplest possibility is that $\omega \approx \omega_0$, which implies that $l = -k$. Consequently,

$$Q(\phi, t) = \sum_k a_{-k,k} e^{ik(\phi - \omega t)} = q(\phi - \omega t),$$

and

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon q(\phi - \omega t).$$

Here, $\phi - \omega t$ is the phase difference between the rotating phase and the phase of the forcing term. Introducing the difference between oscillator phase and external input $\omega t \psi = \phi - \omega t$, we obtain

$$\frac{d\psi}{dt} = -\Delta \omega + \varepsilon q(\psi),$$

where $\Delta \omega = \omega - \omega_0$ is the degree of frequency detuning. Similarly, if $\omega \approx \frac{m\omega_0}{n}$, then

$$Q(\phi, t) \approx \sum_k a_{mk,k} e^{ik(m\phi - n\omega t)} = \hat{q}(m\phi - n\omega t),$$
3.3. Periodically Forced Nonlinear Oscillators and Isochrones

and

\[ \frac{d\psi}{dt} = m\omega_0 - nw + \varepsilon m\hat{q}(\psi), \quad \psi = m\phi - n\omega t. \]

The above analysis involves an application of the averaging theorem. Assuming \( \Delta\omega = \omega - \omega_0 = O(\varepsilon) \) and setting \( \psi = \phi - \omega t \), we have

\[ \frac{d\psi}{dt} = -\Delta\omega + \varepsilon Q(\psi + \omega t, t) = O(\varepsilon). \]

Define

\[ q(\psi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T Q(\psi + \omega t, t) \, dt, \]

and consider the averaged equation

\[ \frac{d\bar{\psi}}{dt} = -\Delta\omega + \varepsilon q(\bar{\psi}). \]

The averaging theorem states that there exists a change of variable that maps solutions of the full equation to those of the averaged equation. In general, one can only establish that a solution of the full equation is \( \varepsilon \)-close to the corresponding solution of the average equation for times up to \( O(1/\varepsilon) \), i.e.

\[ \sup_{t \in I} |\psi(t) - \bar{\psi}(t)| \leq a \varepsilon. \]

3.3.5 Phase-Locking and Synchronisation

Suppose that the \( 2\pi \)-periodic function \( q(\psi) \) has a unique maximum at \( q_{\text{max}} \) and a unique minimum at \( q_{\text{min}} \).

**Synchronisation Regime**

If the degree of detuning is sufficiently small, in the sense that

\[ \varepsilon q_{\text{min}} < \Delta\omega < \varepsilon q_{\text{max}}, \]

then there exists at least one pair of stable/unstable fixed points \( \psi_s \) and \( \psi_u \). The system then evolves to the solution \( \phi(t) = \omega t + \psi_s \) and this is the phase-locked synchronise state.

**Drift Regime**

Increasing \( |\Delta\omega| \) means that \( \psi_s, \psi_u \) coalesce at a saddle point, beyond which there are no fixed points. In this case, the phase rotates through \( 2\pi \) with period

\[ T_\psi = \int_0^{2\pi} \frac{d\psi}{\varepsilon q(\psi) - \Delta\omega}. \]

Close to the bifurcation point, the period of the drifting oscillation is

\[ \Omega_\psi = \frac{2\pi}{T_\psi} \approx \sqrt{\varepsilon |q''(\psi_{\text{max}})| |\Delta\omega - \Delta\omega_{\text{max}}|} = O(\sqrt{\varepsilon}). \]

where \( \Delta\omega_{\text{max}} = \varepsilon q_{\text{max}} \).
3.4 Phase Reduction for Networks of Coupled Oscillators

Let us consider a network of $N$ oscillators. Denote the state of $i$th oscillator by $\mathbf{u}_i \in \mathbb{R}^M$, $i = 1, \ldots, N$ and

$$
\frac{d\mathbf{u}_i}{dt} = f(\mathbf{u}_i) + \varepsilon \sum_{j=1}^{N} a_{ij} \mathbf{H}(\mathbf{u}_j).
$$

The first and second term represent the local dynamics and the interaction between oscillators respectively.

3.4.1 Method of Isochrones

Recall that

$$
\frac{d\phi_i}{dt} = \omega_0, \quad \phi_i \rightarrow \mathbf{U}(\phi_i).
$$

In a similar fashion to a single forced oscillator,

$$
\frac{d\phi_i(\mathbf{u}_i)}{dt} = \omega_0 + \varepsilon \frac{\partial \phi_i}{\partial \mathbf{u}_i} \cdot \left( \sum_{j=1}^{N} a_{ij} \mathbf{H}(\mathbf{u}_j) \right).
$$

Since the limit cycle is uniquely defined by phase,

$$
\frac{d\phi_i}{dt} = \omega_0 + \varepsilon \sum_{j=1}^{N} a_{ij} Q(\phi_i, \phi_j),
$$

where

$$
Q(\phi_i, \phi_j) = \frac{\partial \phi_i}{\partial \mathbf{u}_i} \cdot \mathbf{U}(\phi_i) \cdot \mathbf{H}(\mathbf{U}(\phi_j)).
$$

The final step is to use the method of averaging to obtain phase-difference equation. Introducing $\psi_i = \phi_i - \omega t$, then

$$
\frac{d\psi_i}{dt} = \varepsilon \sum_{i,j} a_{ij} Q(\psi_i + \omega_0 t, \psi_j + \omega t).
$$

Upon averaging over one period, we obtain

$$
\frac{d\psi_i}{dt} = \varepsilon \sum_{j=1}^{N} a_{ij} h(\psi_j - \psi_i),
$$

where

$$
h(\psi_j - \psi_i) = \frac{1}{\Delta_0} \int_{\mathbf{R}} R(\psi_i + \omega t) H(\mathbf{U}(\psi_j + \omega t)) \, dt
$$

$$
= \frac{1}{2\pi} \int_{\mathbf{R}} R(\phi + \psi_i - \psi_j) H(\mathbf{U}(\phi)) \, d\phi.
$$

Here, $R$ is the phase-resetting curve.
### 3.4.2 Phase-Locked Solutions

We define a one-to-one phase-locked solutions to be

\[ \psi_i(t) = t \Delta w + \bar{\psi}_i. \]

Then

\[ \Delta \omega = \varepsilon \sum_{j=1}^{N} \omega_{ij} h (\bar{\psi}_j - \bar{\psi}_i), \quad i = 1, \ldots, N. \]

We have \( N \) equations in \( N \) unknowns \( \Delta \omega \) and \( N - 1 \) phases \( \bar{\psi}_j - \bar{\psi}_1 \), for example. (We only care about phase difference.) In order to determine local stability, we set

\[ \bar{\psi}_i(t) = \bar{\psi}_i(t) + t \Delta \omega + \Delta \psi_i(t), \]

and linearise:

\[
\frac{d\Delta \psi_i}{dt} = \varepsilon \sum_{j=1}^{N} \hat{H}_{ij}(\Phi) \Delta \psi_j \\
\hat{H}_{ij}(\Phi) = \omega_{ij} h' (\bar{\psi}_j - \bar{\psi}_i) - \delta_{ij} \sum_k \omega_{ik} h' (\bar{\psi}_k - \bar{\psi}_i) \\
\Phi = (\bar{\psi}_1, \ldots, \bar{\psi}_N).
\]

### 3.4.3 Pair of Identical Oscillators

In this case, \( N = 2 \) and we assume symmetric coupling, i.e. \( \omega_{12} = \omega_{21} \) and \( \omega_{11} = \omega_{22} = 0 \) (no self-interaction). Let \( \psi = \psi_2 - \psi_1 \). Then

\[ \frac{d\psi}{dt} = \varepsilon H^{-1}(\psi), \quad H^{-1}(\psi) = h(-\psi) - h(\psi). \]

Phase-locked states are given by zeros of the odd function \( H^{-1}(\psi) = 0 \) and is stable if

\[ \varepsilon \frac{dH^{-1}(\psi)}{d\psi} < 0. \]

By symmetry, the in-phase solution \( \psi = 0 \) and anti-phase solution \( \psi = \pi \) are guaranteed to exist.

### 3.5 PDEs

#### 3.5.1 Wave Equation with Weak Damping

We add a damping term to the classical weak equation

\[
\begin{align*}
\partial_x^2 u &= \partial_t^2 u + \varepsilon \partial_t u, \quad 0 < x < 1, \quad t > 0 \\
u &= 0 \quad \text{at} \ x = 0 \ \text{and} \ x = 1 \\
u(x, 0) &= g(x), \quad \partial_t u(x, 0) = 0.
\end{align*}
\]
Introduce two separate time scales $t_1 = t, t_2 = \varepsilon t$, we obtain

$$\partial_x^2 u = \left[ \partial_{t_1}^2 + 2\varepsilon \partial_{t_1} \partial_{t_2} + \varepsilon^2 \partial_{t_2}^2 \right] u + \varepsilon \left( \partial_{t_1} + \varepsilon \partial_{t_2} \right) u,$$

with $u = 0$ at $x = 0$ and $x = 1$, $u(x, 0) = g(x)$ and $(\partial_{t_1} + \varepsilon \partial_{t_2}) u = 0$ at $t_1 = t_2 = 0$. Try the asymptotic expansion

$$u \sim u_0(x, t_1, t_2) + \varepsilon u_1(x, t_1, t_2) + \ldots.$$

The $O(1)$ equation is

$$\partial_x^2 u_0 = \partial_{t_1}^2 u_0,$$

with $u_0(x, 0, 0) = g(x)$ and $\partial_{t_1} u_0(x, 0, 0) = 0$. The method of separation of variables yields

$$u_0(x, t_1, t_2) = \sum_{n=1}^{\infty} \left[ a_n(t_2) \sin(\lambda_n t_1) + b_n(t_2) \cos(\lambda_n t_1) \right] \sin(\lambda_n x),$$

with $\lambda_n = n\pi$. We impose initial condition once we determine $a_n(t_2), b_n(t_2)$. The $O(\varepsilon)$ equation is

$$\partial_x^2 u_1 = \partial_{t_1}^2 u_1 + 2\partial_{t_1} \partial_{t_2} u_0 + \partial_{t_1} u_0 = \partial_{t_1}^2 u_1 + \sum_{n=1}^{\infty} A_n(t_1, t_2) \sin(\lambda_n x),$$

where

$$A_n = (2a_n' + a_n) \lambda_n \cos(\lambda_n t_1) - (2b_n' + b_n) \lambda_n \sin(\lambda_n t_1).$$

Introducing the Fourier expansion, we must have that

$$u_1 = \sum_{n=1}^{\infty} V_n(t_1, t_2) \sin(\lambda_n x)$$

$$\partial_{t_1}^2 V_n + \lambda_n^2 V_n = -(2a_n' + a_n) \lambda_n \cos(\lambda_n t_1) + (2b_n' + b_n) \lambda_n \sin(\lambda_n t_1).$$

We can eliminate secular terms if

$$2a_n' + a_n = 0, \quad 2b_n' + b_n = 0,$$

which implies

$$a_n(t_2) = a_n(0) e^{-t_2/2}, \quad b_n(t_2) = b_n(0) e^{-t_2/2}. $$

Hence,

$$u(x, t) \sim \sum_{n=1}^{\infty} \left[ a_n(0) e^{-t/2} \sin(\lambda_n t) + b_n(0) e^{-t/2} \cos(\lambda_n t) \right] \sin(\lambda_n x), \quad \lambda_n = n\pi.$$

Applying the initial condition, $a_n(0) = 0$ and

$$b_n(0) = 2 \int_0^1 g(x) \sin(\lambda_n x) \, dx.$$
3.6 Pattern Formation and Amplitude Equations

Neural field equations on a ring. Take $\theta \in [0, \pi]$,

$$\frac{\partial a}{\partial t} = -a(\theta, t) + \frac{1}{\pi} \int_0^\pi w(\theta - \theta') f(a(\theta', t)) \, d\theta'$$

(3.6.1)

$$f(a) = \frac{1}{1 + e^{-\eta(a - \kappa)}}.$$  

(3.6.2)

From periodicity of $w(\theta)$, we have

$$w(\theta) = W_0 + 2 \sum_{n \geq 0} W_n \cos(2\pi \theta).$$

Suppose there exists a uniform equilibrium solution $\bar{a}$, it must satisfy

$$\bar{a} = f(\bar{a}) \int_0^\pi \frac{w(\theta)}{\pi} \, d\theta.$$  

The stability of the equilibrium solution is determined by setting $a(\theta, t) = \bar{a} + a(\theta) e^{\lambda t}$. Substituting these into (3.6.2) and expanding $f$ around $\bar{a}$ yields:

$$f(\bar{a} + a(\theta) e^{\lambda t}) \approx f(\bar{a}) + f'(\bar{a}) a(\theta) e^{\lambda t},$$

and

$$\lambda a(\theta) = -a(\theta) + \frac{f'(\bar{a})}{\pi} \int_0^\pi w(\theta - \theta') a(\theta') \, d\theta' = \hat{L}a.$$  

The linear operator $\hat{L}$ is compact on $L^2(S^1)$ with eigenvalues $\lambda_n, n \in \mathbb{Z}$ (Take Fourier transform) satisfying

$$\lambda_n = -1 + f'(\bar{a}) W_n,$$

and eigenfunctions

$$a(\theta) = z_n e^{2in\theta} + z_n^* e^{-2in\theta}.$$  

The eigenvalue expression reveals the bifurcation parameter $\mu = f'(\bar{a})$. For sufficiently small $\mu$ corresponding to a low activity state, $\lambda_n < 0$ for all $n$ and the fixed point is stable. As $\mu$ increases beyond a critical value $\mu_c$, the fixed point becomes unstable due to excitation of the eigenfunctions associated with the largest Fourier component of $w(\theta)$. Suppose that

$$W_1 = \max_m W_m, \quad \text{with eigenfunction } \cos(2\theta).$$

Then $\mu_c = \frac{1}{W_1}$. We predict that for $\mu > \mu_c$, the excited mode will be

$$a(\theta) = ze^{2i\theta} + z^*e^{-2i\theta} = |z| \cos(2(\theta - \theta_0)).$$

We expect this mode to grow and stop at a maximum amplitude as $\mu$ approaches $\mu_c$, mainly because of the saturation of $f$.
3.6.1 Derivation of Amplitude Equation Using the Fredholm Alternative

Suppose that the system is just above the bifurcation point, that is, \( \mu - \mu_c = \varepsilon \Delta \mu, 0 < \varepsilon \ll 1; \) here \( \Delta \mu = O(1) \). We are going to carry out a perturbation expansion in \( \varepsilon \) and Taylor expand the nonlinearity:

\[
f(a) - f(\bar{a}) = \mu(a - \bar{a}) + \frac{f''(\bar{a})}{2}(a - \bar{a})^2 + \frac{f'''(\bar{a})}{6}(a - \bar{a})^3 + \ldots.
\]

Next we take \( a = \bar{a} + \sqrt{\varepsilon} a_1 + \varepsilon^{3/2} a_3 + \ldots \). Finally, the dominant temporal behaviour just beyond the bifurcation point is \( e^{\varepsilon \Delta \mu t} = e^{\tau \Delta \mu} \). Substituting these into the full nonlinear equation (3.6.2) and expanding in powers of \( \varepsilon \), we find the following equations. The \( O(1) \) equation is

\[
\bar{a} = f(\bar{a}) \int_0^{\pi} \frac{w(\theta)}{\pi} d\theta,
\]

and the \( O(\sqrt{\varepsilon}) \) equation is

\[
\hat{L} a_1 = -a_1 + \frac{\mu_c}{\pi} \int_0^{\pi} w(\theta - \theta') a_1(\theta') d\theta' = 0.
\]

The \( O(\varepsilon) \) equation is

\[
\hat{L} a_2 = V_2 := -\frac{f''(\bar{a})}{2} \int_0^{\pi} w(\theta - \theta') a_1^2(\theta') d\theta',
\]

and the \( O(\varepsilon^{3/2}) \) equation is

\[
\hat{L} a_3 = V_3 := -\frac{f'''(\bar{a})}{6} \int_0^{\pi} w(\theta - \theta') a_1^3(\theta') d\theta' - 2 \frac{f''(\bar{a})}{2} \int_0^{\pi} w(\theta - \theta') a_1(\theta') a_2(\theta') d\theta' + \left[ \frac{\partial a_1}{\partial \tau} - \Delta \mu \frac{1}{\pi} \int_0^{\pi} w(\theta - \theta') a_1(\theta') d\theta' \right].
\]

The \( O(\sqrt{\varepsilon}) \) equation has solutions of the form

\[
a_1 = z(\tau)e^{2i\theta} + z^*(\tau)e^{-2i\theta}.
\]

A dynamical equation for \( z(\tau) \) can be obtained by deriving solvability conditions for the higher order equations using Fredholm alternative. For any two periodic functions, define the following inner product

\[
\langle U, V \rangle = \frac{1}{\pi} \int_0^{\pi} U^*(\theta)V(\theta) d\theta.
\]

The linear operator \( \hat{L} \) is self-adjoint with respect to this inner product. We can write the higher order equations in the general form

\[
\hat{L} a_n = V_n(a_0, a_1, \ldots, a_{n-1}).
\]

Note that \( \hat{L} \bar{a} = 0 \) for \( \bar{a} = e^{\pm 2i\theta} \). Since \( \hat{L} a_n = V_n \), the Fredholm alternative requires that \( V_n \perp \bar{a} \), i.e.

\[
\langle \bar{a}, V_n \rangle = 0, \quad n \geq 2.
\]
Chapter 4

The Wentzel-Kramers-Brillouin (WKB) Method

The WKB method, named after Wentzel, Kramers and Brillouin, is a method for finding approximate solutions to linear differential equations with spatially varying coefficients. The origin of WKB theory dates back to 1920s where it was developed by Wentzel, Kramers and Brillouin to study time-independent Schrodinger equation.

This often arises from the following problem: Let \( x = \varepsilon \hat{x}, \) then

\[
\frac{d^2y}{d\hat{x}^2} - q(\varepsilon \hat{x})y = 0.
\]

4.1 Introductory Example

Consider the differential equation

\[
\varepsilon^2 y'' - q(x)y = 0, \quad x \in [0, 1],
\]

where \( q \) is a smooth function. For constant \( q \in \mathbb{R} \), the general solution of (4.1.1) is

\[
y(x) = a_0 e^{-x\sqrt{q}/\varepsilon} + b_0 e^{x\sqrt{q}/\varepsilon},
\]

and the solution oscillates rapidly on a scale of \( O(\varepsilon) \). The hypothesis of the WKB method is that the exponential solutions (4.1.5) can be generalised to obtain an approximate solution of the full problem (4.1.1).

Because we expect that the solutions of (4.1.1) to have rapid oscillations on a scale of \( O(\varepsilon) \) with slowly-varying amplitude and phase, we start with the following general WKB ansatz:

\[
y(x) \sim e^{\theta(x)/\varepsilon^\alpha} \left[ y_0(x) + \varepsilon^\alpha y_1(x) + \ldots \right] \quad \text{as} \; \varepsilon \to 0,
\]

for some \( \alpha > 0 \). Here, we assume that the solution varies exponentially with respect to the fast variation. From (4.1.3) we obtain:

\[
y' \sim \left\{ \varepsilon^{-\alpha} \theta_x y_0 + y_0' + \theta_x y_1 + \ldots \right\} e^{\theta/\varepsilon^\alpha}
\]

\[
y'' \sim \left\{ \varepsilon^{-2\alpha} \theta_x^2 y_0 + \varepsilon^{-\alpha} \left( \theta_{xx} y_0 + 2\theta_x y_0' + \theta_x^2 y_1 \right) + \ldots \right\} e^{\theta/\varepsilon^\alpha}
\]
Substituting both (4.1.3) and (4.1.4) into (4.1.1) and cancelling the exponential term yields
\[ \varepsilon^2 \left[ \frac{\theta^2 y_0}{\varepsilon^{2\alpha}} + \frac{1}{\varepsilon^\alpha} (\theta_{xx} y_0 + 2\theta_x y'_0 + \theta^2 y_1) + \ldots \right] - q(x) [y_0 + \varepsilon^\alpha y_1 + \ldots] = 0. \] (4.1.5)

Such cancellation is possible due to the linearity of the equation! Balancing leading-order terms in (4.1.5) we see that \( \alpha = 1. \) The \( O(1) \) equation is the well-known eikonal equation:
\[ \theta_x^2 = q(x), \] (4.1.6)
and its solutions (in one-dimensional) are
\[ \theta(x) = \pm \int x \sqrt{q(s)} \, ds. \] (4.1.7)

To determine \( y_0(x) \), we need to solve the \( O(\varepsilon) \) equation which is the transport equation:
\[ \theta_{xx} y_0 + 2\theta_x y'_0 + \theta^2 y_1 = q(x) y_1, \] (4.1.8)
The \( y_1 \) terms cancel out due to the eikonal equation (4.1.6), so (4.1.8) reduces to
\[ \theta_{xx} y_0 + 2\theta_x y'_0 = 0. \] (4.1.9)
The equation (4.1.9) can be easily solved since it is separable:
\[ \frac{y'_0}{y_0} = -\frac{\theta_{xx}}{2\theta_x}, \]
\[ \ln |y_0| = -\frac{1}{2} \ln |\theta_x| + C \]
\[ \ln |y_0| = -\ln \sqrt{|\theta_x|} + C \]
\[ y_0(x) = \frac{C}{\sqrt{\theta_x}} = \pm Cq(x)^{-1/4}, \]
where \( C \) is an arbitrary nonzero constant and the last line follows from (4.1.7). Hence, a first-term asymptotic approximation of the general solution of (4.1.1) is
\[ y(x) \sim q(x)^{-1/4} \left[ a_0 \exp \left( -\frac{1}{\varepsilon} \int x \sqrt{q(s)} \, ds \right) + b_0 \exp \left( \frac{1}{\varepsilon} \int x \sqrt{q(s)} \, ds \right) \right], \] (4.1.10)
where \( a_0, b_0 \) are arbitrary constants, possibly complex. It is evident that (4.1.10) is valid if \( q(x) \neq 0 \) on \([0,1]\). The \( x \)-values where \( q(x) = 0 \) are called turning points and we will see how to handle this in the Section 4.2.

**Example 4.1.1.** Choose \( q(x) = -e^{2x} \). Then the WKB approximation (4.1.10) is
\[ y(x) \sim e^{-x/2} \left[ a_0 e^{-i\lambda x}/\varepsilon + b_0 e^{i\lambda x}/\varepsilon \right] = e^{-x/2} \left[ a_0 \cos(\lambda e^x) + b_0 \sin(\lambda e^x) \right], \]
where \( \lambda = 1/\varepsilon \). With boundary conditions \( y(0) = a, y(1) = b \), we obtain
\[ y(x) \sim e^{-x/2} \left( \frac{b\sqrt e \sin (\lambda (e^x - 1)) - a \sin (\lambda (e^x - e))}{\sin (\lambda (e - 1))} \right), \]
The exact solution of (4.1.1) with the given $q(x)$ can be solved as follows. Making a change of variable $\tilde{x} = e^x/\varepsilon = \lambda e^x$, we obtain

$$x = \ln(\varepsilon) + \ln(\tilde{x}) \implies \frac{dx}{d\tilde{x}} = \frac{1}{\tilde{x}}.$$ 

Setting $Y(\tilde{x}) = y(x)$ and using Chain Rule give

$$\frac{dY}{d\tilde{x}} = \frac{dy}{dx} \frac{dx}{d\tilde{x}} = \frac{y'}{\tilde{x}}$$

$$\frac{d^2Y}{d\tilde{x}^2} = -\frac{y'}{\tilde{x}^2} + \frac{y''}{\tilde{x}^2} = -\frac{1}{\tilde{x}} \frac{dy}{d\tilde{x}} + \frac{y''}{\tilde{x}^2}.$$ 

Consequently, the equation of $Y(\tilde{x})$ is the zeroth-order Bessel’s differential equation

$$\tilde{x}^2 \frac{d^2Y}{d\tilde{x}^2} + \tilde{x} \frac{dY}{d\tilde{x}} + \tilde{x}^2 Y = 0,$$

which has solutions

$$Y(\tilde{x}) = c_0 J_0(\tilde{x}) + d_0 Y_0(\tilde{x}) = c_0 J_0(\lambda e^x) + d_0 Y_0(\lambda e^x) = y(x),$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are the zeroth-order Bessel functions of the first and second kinds respectively. Finally, solving for $c_0$ and $d_0$ using the boundary conditions yields

$$c_0 = \frac{1}{D} [bY_0(\lambda) - aY_0(\lambda e)]$$

$$d_0 = \frac{1}{D} [aJ_0(\lambda e) - bJ_0(\lambda)]$$

$$D = J_0(\lambda e)Y_0(\lambda) - Y_0(\lambda e)J_0(\lambda).$$

One can plot the exact solution and the WKB approximation and see that their difference is almost zero!

To measure the error of the WKB approximation (4.1.10), we look at the $O(\varepsilon^2)$ equation which has the form

$$\theta_{xx} y_1 + 2\theta_x y'_1 + \theta_x^2 y_2 + y''_0 = q(x)y_2. \quad (4.1.11)$$

The $y_2$ terms vanish due to the eikonal equation (4.1.6), so (4.1.11) reduces to

$$\theta_{xx} y_1 + 2\theta_x y'_1 + y''_0 = 0. \quad (4.1.12)$$

Because the first two terms of (4.1.12) are similar to the transport equation (4.1.9), we make an ansatz $y_1(x) = y_0(x)w(x)$ and so (4.1.12) reduces to

$$2\theta_x y_0 w' + y''_0 = 0. \quad (4.1.13)$$

Suppose $q(x) > 0$ so that $\theta_x$ is a real-valued function. Rearranging (4.1.13) in terms of $w'$ and integrating by parts with respect to $x$ we obtain

$$\frac{2C \theta_x w'}{\sqrt{\theta_x}} = -\frac{d^2}{dx^2} \left( \frac{C}{\sqrt{\theta_x}} \right) = \frac{d}{dx} \left( \frac{C \theta_x}{2 \theta_x^{3/2}} \right)$$
\[
\begin{align*}
\frac{d}{dx} w' &= \frac{1}{4} d \left( \frac{\theta_{xx}}{\theta_x^{3/2}} \right) \left( \frac{1}{\sqrt{\theta_x}} \right) \\
\frac{d}{dx} w(x) &= \frac{1}{4} \int x \left( \frac{\theta_{xx}}{\theta_x^{3/2}} \right) \left( \frac{1}{\sqrt{\theta_x}} \right) ds \\
&= d + \frac{1}{4} \left( \frac{\theta_{xx}}{\theta_x^2} \right) - \frac{1}{4} \int x \left( \frac{\theta_{xx}}{\theta_x^{3/2}} \right) d \left( \frac{1}{\sqrt{\theta_x}} \right) ds \\
&= d + \frac{1}{4} \left( \frac{\theta_{xx}}{\theta_x^2} \right) + \frac{1}{8} \int x \left( \frac{\theta_{xx}}{\theta_x^2} \right) ds,
\end{align*}
\]

where \( d \) is an arbitrary constant. On the other hand, \( \theta_x \) is a complex-valued function if \( q(x) < 0 \). Suppose \( \theta_x = \pm i \sqrt{-q} \), then

\[
\begin{align*}
\theta_{xx} &= \pm \frac{i}{2} \left( \frac{-q_x}{\sqrt{-q}} \right) = \mp \frac{i q_x}{2 \sqrt{-q}} \\
\frac{\theta_{xx}}{\theta_x^2} &= \mp \frac{i q_x}{2 q \sqrt{-q}} = \pm \frac{i q_x}{2 (-q)^{3/2}} \\
\theta_{xx}^2 &= \frac{-q_x^2}{4 (-q)} = \frac{q_x}{4q} \\
\theta_x^3 &= (\pm i)^3 (\sqrt{-q})^3 = \mp i (-q)^{3/2} \\
\frac{\theta_{xx}}{\theta_x^3} &= \mp 4 i q (-q)^{3/2} = \mp \frac{i q_x^2}{4 (-q)^{5/2}}.
\end{align*}
\]

Consequently,

\[
w(x) = \begin{cases} 
& d + \frac{1}{8} \frac{q_x}{q^{3/2}} + \frac{1}{32} \int x \left( \frac{q_x^2}{q^{5/2}} \right) ds & \text{if } \theta_x(x) = \sqrt{q(x)}, \\
& d - \frac{1}{8} \frac{q_x}{q^{3/2}} - \frac{1}{32} \int x \left( \frac{q_x^2}{q^{5/2}} \right) ds & \text{if } \theta_x(x) = -\sqrt{q(x)}, \\
& d + \frac{1}{8} \frac{i q_x}{(q)^{3/2}} - \frac{1}{32} \int x \left( \frac{i q_x^2}{(q)^{5/2}} \right) ds & \text{if } \theta_x(x) = i \sqrt{-q(x)}, \\
& d - \frac{1}{8} \frac{i q_x}{(q)^{3/2}} + \frac{1}{32} \int x \left( \frac{i q_x^2}{(q)^{5/2}} \right) ds & \text{if } \theta_x(x) = -i \sqrt{-q(x)}.
\end{cases}
\]

Finally, for small \( \varepsilon \) the WKB ansatz (4.1.3) is well-ordered provided

\[
|\varepsilon y_1(x)| \ll |y_0(x)|, \quad \text{or} \quad |\varepsilon w(x)| \ll 1.
\]

In terms of the function \( q(x) \) and its first derivatives, for \( x \in [x_0, x_1] \) we will have an accurate approximation if

\[
\varepsilon \left[ |d| + \frac{1}{32} \left| \frac{q_x}{q^{3/2}} \right| \left( 4 + \int_{x_0}^{x_1} \left| \frac{q_x}{q} \right| dx \right) \right] \ll 1,
\]

where \( | \cdot | := \| \cdot \|_\infty \) over the interval \([x_0, x_1]\). We stress that this condition holds if the interval \([x_0, x_1]\) does not contain a turning point.

**Remark 4.1.2.** The constants \( a_0, b_0 \) in (4.1.10) and \( d \) in \( w(x) \) are determined from boundary conditions. However, it is very possible that these constants depend on \( \varepsilon \). It is therefore necessary to make sure this dependence does not interfere with the ordering assumed in the WKB ansatz (4.1.3).
4.2 Turning Points

This section is devoted to the analysis of turning points of \( q(x) \). Assume \( q(x) \) is smooth and has a simple zero at \( x_t \), i.e. \( q(x_t) = 0 \) and \( q'(x_t) \neq 0 \). For concreteness, we take \( q'(x_t) > 0 \) and so we expect solutions of (4.1.1) to be oscillatory for \( x < x_t \) and exponential for \( x > x_t \). We can apply the WKB method on the regions \( \{ x < x_t \} \) and \( \{ x > x_t \} \). More precisely, from (4.1.10) we have

\[
y \sim \begin{cases} 
  y_L(x, x_t) & \text{if } x < x_t, \\
  y_R(x, x_t) & \text{if } x > x_t,
\end{cases}
\]  

where

\[
y_L(x, x_t) = \frac{1}{q(x_t)^{1/4}} \left[ a_L \exp \left( -\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} \, ds \right) + b_L \exp \left( \frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} \, ds \right) \right], \quad (4.2.2a) \\
y_R(x, x_t) = \frac{1}{q(x_t)^{1/4}} \left[ a_R \exp \left( -\frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} \, ds \right) + b_R \exp \left( \frac{1}{\varepsilon} \int_{x_t}^{x} \sqrt{q(s)} \, ds \right) \right]. \quad (4.2.2b)
\]

An important realization is that these coefficients \( a_L, b_L, a_R, b_R \) are not all independent. In addition to the two boundary conditions at \( x = 0 \) and \( x = 1 \), we also have matching conditions in a transition layer centered at \( x = x_t \).

4.2.1 Transition layer

Following the boundary layer analysis, we introduce the stretched coordinate

\[
\tilde{x} = \frac{x - x_t}{\varepsilon^\beta}, \quad \text{or equivalently } x = x_t + \varepsilon^\beta \tilde{x}.
\]

We can reduce (4.1.1) by expanding the function \( q(x) \) around the turning point \( x_t \):

\[
q(x) = q(x_t + \varepsilon^\beta \tilde{x}) = q(x_t) + q'(x_t)\varepsilon^\beta \tilde{x} + \ldots \\
\approx \varepsilon^\beta \tilde{x} q'(x_t),
\]

since we assume \( x_t \) is a simple zero. Denote the inner solution by \( Y(\tilde{x}) \). Transforming (4.1.1) using

\[
\frac{d}{dx} = \frac{1}{\varepsilon^\beta} \frac{d}{d\tilde{x}}
\]

gives the inner equation

\[
\varepsilon^{2-2\beta} Y''_{\varepsilon} - (\varepsilon^\beta \tilde{x} q'_{\varepsilon} + \ldots) Y = 0, \quad (4.2.3)
\]

where \( q_{\varepsilon}' := q'(x_t) \). Balancing leading-order terms in (4.2.3) means we require

\[
2 - 2\beta = \beta \implies \beta = \frac{2}{3}.
\]

Since it is not clear what the asymptotic sequence should be, we take the asymptotic expansion to be

\[
Y \sim \varepsilon^\gamma Y_0(\tilde{x}) + \ldots. \quad (4.2.4)
\]

The \( \mathcal{O}(\varepsilon^{2/3}) \) equation is

\[
Y_0'' - \tilde{x} q_{\varepsilon}' Y_0 = 0, \quad -\infty < \tilde{x} < \infty. \quad (4.2.5)
\]
Performing a coordinate transformation \( s = (q_{\ell}^{1/3}) \tilde{x} \), (4.2.5) becomes Airy’s equation:

\[
\frac{d^2 Y_0}{ds^2} - s Y_0 = 0, \quad -\infty < s < \infty,
\]

and this can be solved either using power series expansion or Laplace transform. The general solution of (4.2.6) is

\[
Y_0(s) = a \text{Ai}(s) + b \text{Bi}(s),
\]

where \( \text{Ai}(\cdot) \) and \( \text{Bi}(\cdot) \) are Airy functions of the first and the second kinds respectively. It is well-known that

\[
\text{Ai}(x) = \frac{1}{3^{2/3}\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma \left( \frac{k+1}{3} \right) \sin \left( \frac{2\pi}{3} (k+1) \right) \left( 3^{1/3} x \right)^k
\]

and

\[
\text{Bi}(x) = e^{i\pi/6} \text{Ai}(xe^{2\pi i/3}) + e^{-i\pi/6} \text{Ai}(xe^{-2\pi i/3})
\]

where \( \Gamma(\cdot) \) is the gamma function. Setting \( \xi = \frac{2}{3} |x|^{3/2} \), we also have that

\[
\text{Ai}(x) \sim \begin{cases} 
\frac{1}{\sqrt{\pi|x|^{1/4}}} \left[ \cos \left( \xi - \frac{\pi}{4} \right) + \frac{5}{72} \sin \left( \xi - \frac{\pi}{4} \right) \right] & \text{if } x \to -\infty, \\
\frac{1}{2\sqrt{\pi|x|^{1/4}}} e^{-\xi} \left[ 1 - \frac{5}{72} \xi \right] & \text{if } x \to +\infty,
\end{cases}
\]

\[
\text{Bi}(x) \sim \begin{cases} 
\frac{1}{\sqrt{\pi|x|^{1/4}}} \left[ \cos \left( \xi + \frac{\pi}{4} \right) + \frac{5}{72} \sin \left( \xi + \frac{\pi}{4} \right) \right] & \text{if } x \to -\infty, \\
\frac{1}{\sqrt{\pi|x|^{1/4}}} e^{\xi} \left[ 1 + \frac{5}{72} \xi \right] & \text{if } x \to +\infty.
\end{cases}
\]

### 4.2.2 Matching

From (4.2.7), the general solution of (4.1.1) in the transition layer is

\[
Y_0(\tilde{x}) = a \text{Ai} \left( (q_{\ell})^{1/3} \tilde{x} \right) + b \text{Bi} \left( (q_{\ell})^{1/3} \tilde{x} \right).
\]

We now have 6 undetermined constants from (4.2.2) and (4.2.9), but these are all connected since the inner solution (4.2.9) must match the outer solutions (4.2.2) and this will result in two arbitrary constants in the general solution. Since the inner solution is unbounded, we introduce an intermediate variable

\[
x_\eta = x - \frac{x_t}{\varepsilon^\eta}, \quad 0 < \eta < \frac{2}{3},
\]

where the interval for \( \eta \) comes from the requirement that the scaling for the intermediate variable must lie between the outer scale, \( \mathcal{O}(1) \) and the inner scale, \( \mathcal{O}(\varepsilon^{2/3}) \).
4.2.3 Matching for $x > x_t$

We first change the stretched variable $\tilde{x}$ to the intermediate variable $x$:

$$\tilde{x} = \frac{x - x_t}{\varepsilon^\beta} = \frac{x - x_t}{\varepsilon^{\eta - \beta}} = \varepsilon^{\eta - \beta} x_\eta = \varepsilon^{\eta - 2/3} x_\eta.$$

Note that $x_\eta > 0$ since $x > x_t$. From (4.2.4) and (4.2.9), the inner solution $Y(\tilde{x})$ now becomes

$$Y \sim \varepsilon^\gamma Y_0 \left( \varepsilon^{\eta - 2/3} x_\eta \right) + \ldots$$

$$= \varepsilon^\gamma \left[ a \text{Ai} \left( (q'_t)^{1/3} \varepsilon^{\eta - 2/3} x_\eta \right) + b \text{Bi} \left( (q'_t)^{1/3} \varepsilon^{\eta - 2/3} x_\eta \right) \right] + \ldots$$

$$= \varepsilon^\gamma \left[ a \text{Ai}(r) + b \text{Bi}(r) \right] + \ldots$$

$$\sim \varepsilon^\gamma \left[ \frac{a}{2\sqrt{\pi} r^{1/4}} \exp \left( \frac{2}{3} r^{3/2} \right) + \frac{b}{\sqrt{\pi} r^{1/4}} \exp \left( \frac{2}{3} r^{3/2} \right) \right], \quad (4.2.10)$$

where $r = (q'_t)^{1/3} \varepsilon^{\eta - 2/3} x_\eta > 0$ and the last line follows from (4.2.8). On the other hand, since

$$\int_{x_t}^x \sqrt{q(s)} \, ds \sim \int_{x_t}^{x_t + \varepsilon^\eta x_\eta} \sqrt{(s - x_t) q'_t} \, ds$$

$$= \sqrt{q'_t} \left[ \frac{2}{3} (s - x_t)^{3/2} \right]_{x_t}^{x_t + \varepsilon^\eta x_\eta}$$

$$= \frac{2}{3} \sqrt{q'_t} (\varepsilon^\eta x_\eta)^{3/2}$$

$$= \frac{2}{3} \varepsilon^{r^{3/2}},$$

and

$$q(x)^{-1/4} \sim [q(x_t) + (x - x_t) q'_t]^{-1/4} = [\varepsilon^\eta x_\eta q'_t]^{-1/4} = \varepsilon^{-1/6} (q'_t)^{-1/6} r^{-1/4},$$

the right outer solution becomes

$$y_R \sim \varepsilon^{-1/6} \left[ a R \exp \left( -\frac{2}{3} r^{3/2} \right) + b R \exp \left( \frac{2}{3} r^{3/2} \right) \right]. \quad (4.2.11)$$

Consequently, matching (4.2.10) the right outer solution $y_R$ with (4.2.11) the inner solution $Y$ yields the following:

$$\gamma = -\frac{1}{6}, \quad a_R = \frac{a}{2\sqrt{\pi}} (q'_t)^{1/6}, \quad b_R = \frac{b}{\sqrt{\pi}} (q'_t)^{1/6}. \quad (4.2.12)$$

4.2.4 Matching for $x < x_t$

Because $x < x_t$, we have $x_\eta < 0$ which introduces complex numbers into the outer solution $y_L$. Using the asymptotic properties of Airy functions as $r \to -\infty$ (see (4.2.8)), the inner solution becomes

$$Y \sim \varepsilon^\gamma \left[ a \text{Ai}(r) + b \text{Bi}(r) \right] + \ldots$$
\[
\sim \varepsilon^7 \left[ \frac{a}{\sqrt{\pi |r|^{1/4}}} \cos \left( \frac{2}{3} |r|^{3/2} - \frac{\pi}{4} \right) + \frac{b}{\sqrt{\pi |r|^{1/4}}} \cos \left( \frac{2}{3} |r|^{3/2} + \frac{\pi}{4} \right) \right]
\]

Using the identity \( \cos \theta = (e^{i\theta} + e^{-i\theta})/2 \), a more useful form of the inner expansion \( Y \) as \( r \to -\infty \) is

\[
Y \sim \frac{\varepsilon^7}{2\sqrt{\pi |r|^{1/4}}} \left[ (ae^{-i\pi/4} + be^{i\pi/4}) e^{i\zeta} + (ae^{i\pi/4} + be^{-i\pi/4}) e^{-i\zeta} \right], \tag{4.2.13}
\]

where \( \zeta = \frac{2}{3} |r|^{3/2} \). On the other hand, since

\[
\int_x^{x_t} \sqrt{q(s)} \, ds \sim \int_{x_t + \varepsilon x_\eta}^{x_t} \sqrt{(s - x_t)q_t} \, ds
\]

\[
= \sqrt{q_t} \left[ \frac{2}{3} (s - x_t)^{3/2} \right]_{x_t + \varepsilon x_\eta}^{x_t}
\]

\[
= \frac{2}{3} \sqrt{q_t} (\varepsilon x_\eta)^{3/2}
\]

\[
= \frac{2}{3} \varepsilon |r|^{3/2} (-1)^{3/2}
\]

\[
= \frac{2}{3} i \varepsilon |r|^{3/2},
\]

and

\[
q(x)^{-1/4} \sim [\varepsilon x_\eta q_t]^{-1/4}
\]

\[
= \varepsilon^{-1/6} (q_t)^{-1/6} |r|^{-1/4} (-1)^{-1/4}
\]

\[
= \varepsilon^{-1/6} (q_t)^{-1/6} |r|^{-1/4} e^{-i\pi/4},
\]

the left outer solution becomes

\[
y_L \sim \varepsilon^{-1/6} e^{-i\pi/4} \left[ a_L e^{-i\zeta} + b_L e^{i\zeta} \right]. \tag{4.2.14}
\]

Consequently, matching (4.2.14) the left outer solution \( y_L \) with (4.2.13) the inner solution \( Y \) yields the following:

\[
a_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}} (ia + b), \quad b_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}} (a + ib) = i\bar{a}_L. \tag{4.2.15}
\]

From (4.2.12), it follows that

\[
a_L = ia_R + \frac{b_R}{2}, \quad b_L = a_R + \frac{i}{2} b_R, \tag{4.2.16}
\]

or in matrix form,

\[
\begin{bmatrix} a_L \\ b_L \end{bmatrix} = \begin{bmatrix} i & 1/2 \\ 1 & i/2 \end{bmatrix} \begin{bmatrix} a_R \\ b_R \end{bmatrix}. \tag{4.2.17}
\]
4.2.5 Conclusion

Because we assume $q(t) < 0$ for $x < x_t$, this introduces complex numbers on $y_L$:

$$q(x)^{-1/4} = e^{-ix/4}|q(x)|^{-1/4}$$

$$\int_x^{x_t} \sqrt{q(s)} \, ds = i \int_x^{x_t} \sqrt{|q(s)|} \, ds$$

In conclusion, we have

$$y(x) = \begin{cases} y_L(x, x_t) & \text{if } x < x_t, \\ y_R(x, x_t) & \text{if } x > x_t, \end{cases}$$

where

$$y_L(x, x_t) = \frac{1}{|q(x)|^{1/4}} \left[ i a_R + \frac{b_R}{2} \right] e^{-i\theta(x)/\varepsilon} e^{-i\pi/4} + \left[ a_R + \frac{ib_R}{2} \right] e^{i\theta(x)/\varepsilon} e^{-i\pi/4}$$

$$y_R(x, x_t) = \frac{1}{|q(x)|^{1/4}} \left[ a_R e^{-\kappa(x)/\varepsilon} + b_R e^{\kappa(x)/\varepsilon} \right]$$

$$\theta(x) = \int_x^{x_t} \sqrt{|q(s)|} \, ds$$

$$\kappa(x) = \int_x^{x_t} \sqrt{|q(s)|} \, ds.$$

Example 4.2.1. Consider $q(x) = x(2 - x)$, where $-1 < x < 1$. The simple turning point is at $x_t = 0$, with $q'(0) = 2 > 0$. One can compute and show that:

$$\theta(x) = \frac{1}{2} \left( 1 - x \right) \sqrt{x(x - 2)} - \frac{1}{2} \ln \left[ 1 - x + \sqrt{x(x - 2)} \right], \quad x < 0$$

$$\kappa(x) = \frac{1}{2} (x - 1) \sqrt{x(2 - x)} - \frac{1}{2} \ln \left( 1 - x + \sqrt{x(x - 2)} \right) + \frac{\pi}{4}, \quad x > 0.$$

4.2.6 The opposite case: $q'_t < 0$

CHT: Don’t really know how to do the change of variable? Note that if $q'(x_t) < 0$, then we instead have

$$\begin{bmatrix} a_L \\ b_L \end{bmatrix} = \begin{bmatrix} i/2 & 1 \\ 1/2 & i \end{bmatrix} \begin{bmatrix} a_R \\ b_R \end{bmatrix}.$$  

Consequently,

$$y_L(x) = \frac{1}{|q(x)|^{1/4}} \left[ a_L e^{\theta(x)/\varepsilon} + b_L e^{-\theta(x)/\varepsilon} \right]$$

$$y_R(x) = \frac{1}{|q(x)|^{1/4}} \left[ 2b_L \cos \left( \frac{1}{\varepsilon} \kappa(x) - \frac{\pi}{4} \right) + a_L \cos \left( \frac{1}{\varepsilon} \kappa(x) + \frac{\pi}{4} \right) \right]$$
4.3 Wave Propagation and Energy Methods

In this section, we study how to obtain an asymptotic approximation of a travelling-wave solution of the following PDE which models the string displacement

\[ u_{xx} = \mu^2(x)u_{tt} + \alpha(x)u_t + \beta(x)u, \quad 0 < x < \infty, t > 0 \]  
\[ u(0, t) = \cos(\omega t) \]  

(4.3.1a)  

(4.3.1b)

The terms \( \alpha(x)u_t \) and \( \beta u \) correspond to damping and elastic support respectively. From the initial condition, we see that the string is periodically forced at the left end and so the solution will develop into a wave that propagates to the right.

Observe that there is no obvious small parameter \( \varepsilon \), but we will extract one from the following observation. In the special case where \( \alpha = \beta = 0 \) and \( \mu \) equals some constant, (4.3.1) reduces to the classical wave equation and we obtain the right-moving plane waves

\[ u(x, t) = e^{i(w t - k x)}, \quad \text{where the wavenumber } k \text{ satisfies } k = \pm \omega \mu. \]  

(4.3.2)

For higher temporal frequencies \( \omega \gg 1 \), these waves have short wavelength, i.e. \( \lambda = \frac{2\pi}{|k|} \ll 1 \).

Motivated by this, we choose \( \varepsilon = 1/\omega \) and construct an asymptotic approximation of the travelling-wave solution of (4.3.1) in the case of a high frequency. The WKB ansatz is assumed to be

\[ u(x, t) \sim \exp \left[ i \left( wt - w^\gamma \theta(x) \right) \right] \left\{ u_0(x) + \frac{1}{w^\gamma} u_1(x) + \ldots \right\}, \]  

(4.3.2)

Substituting (4.3.2) into (4.3.1) we obtain

\[ -\omega^2 \gamma^2 x (u_0 + w^{-\gamma} u_1 + \ldots) + i w^\gamma \theta_x (\theta_x u_0 + \ldots) + \frac{d}{dx} (i \omega^\gamma \theta_x u_0 + \ldots) \]

\[ = -\mu^2 \omega^2 (u_0 + \omega^{-\gamma} u_1 + \ldots) - i \omega \alpha (u_0 + \ldots) + \beta (u_0 + \ldots). \]

Balancing the first terms on each side of this equation gives \( \gamma = 1 \). The \( O(\omega^2) = O(1/\varepsilon^2) \) equation is the eikonal equation:

\[ \theta_x^2 = \mu^2(x), \]  

(4.3.3)

and its solutions are

\[ \theta(x) = \pm \int_0^x \mu(s) \, ds. \]  

(4.3.4)

We choose the positive solution as we are considering the right-moving waves. The \( O(\omega) = O(1/\varepsilon) \) equation is the transport equation:

\[ -\theta_x^2 u_1 + i \theta_x \partial_x u_0 + i (\theta_x \partial_x u_0 + \theta_{xx} u_0) = -\mu^2 u_1 - i \alpha u_0. \]  

(4.3.5)

The \( u_1 \) terms cancel out due to the eikonal equation (4.3.3), so (4.3.5) reduces to

\[ \theta_{xx} u_0 + 2 \theta_x \partial_x u_0 = -\alpha u_0. \]  

(4.3.6)

With \( \theta_x = \mu(x) \), we can rearrange (4.3.6) and obtain a first order ODE in \( u_0 \):

\[ \partial_x u_0 + \left( \frac{\mu_x + \alpha}{2\mu} \right) u_0 = 0, \]  

(4.3.7)
which can be solved using the method of integrating factor. The integrating factor is given by
\[ I(x) = \exp \left( \int_{0}^{x} \left( \frac{\mu(s) + \alpha(s)}{2\mu(s)} \right) \, ds \right) = \sqrt{\mu(x)} \exp \left( \frac{1}{2} \int_{0}^{x} \frac{\alpha(s)}{\mu(s)} \, ds \right), \]
and so (4.3.7) can be written as
\[ \frac{d}{dx} (I(x)u_0) = 0, \quad u_0 = \frac{a_0}{I(x)} = \frac{a_0}{\sqrt{\mu(x)}} \exp \left( -\frac{1}{2} \int_{0}^{x} \frac{\alpha(s)}{\mu(s)} \, ds \right). \quad (4.3.8) \]

Finally, imposing the boundary condition at \( x = 0 \) we obtain a first-term asymptotic expansion of the travelling-wave solution of (4.3.1)
\[ u(x, t) \sim \sqrt{\frac{\mu(0)}{\mu(x)}} \exp \left[ -\frac{1}{2} \int_{0}^{x} \frac{\alpha(s)}{\mu(s)} \, ds \right] \cos \left( \omega t - \omega \int_{0}^{x} \mu(s) \, ds \right). \quad (4.3.9) \]
Observe that in (4.3.9) the amplitude and phase of the travelling wave depend on the spatial position \( x \). Interestingly, (4.3.9) is independent of \( \beta(x) \)!

### 4.3.1 Connection to energy methods

Energy methods are extremely powerful in the study of wave-related problems. To determine the energy equation in this case, we multiply (4.3.1) by \( u_t \):
\[ u_t u_{xx} = \mu^2(x) u_t u_{tt} + \alpha(x) u_t^2 + \beta(x) u u_t \]
\[ \partial_x (u_t u_x) - \frac{1}{2} \partial_t (u_x^2) = \frac{1}{2} \mu^2(x) \partial_t (u_t^2) + \alpha(x) u_t^2 + \beta(x) \partial_t (u^2) \]
\[ \partial_t \left[ \frac{1}{2} \mu^2(x) (u_t^2) + \frac{1}{2} \beta(x) u^2 + \frac{1}{2} (u_x^2) \right] - \partial_x (u_t u_x) = -\alpha(x) u_t^2 \]
\[ \partial_t E(x, t) + \partial_x S(x, t) = -\Phi(x, t), \]
where
\[ E(x, t) = \text{energy density} \quad := \frac{1}{2} \mu^2(x) (\partial_t u)^2 + \frac{1}{2} (\partial_x u)^2 + \frac{1}{2} \beta(x) u^2 \]
\[ S(x, t) = \text{energy flux} \quad := -\partial_t u \partial_x u \]
\[ \Phi(x, t) = \text{dissipation function} \quad := \alpha(x) (\partial_t u)^2. \]

We are interested in the energy over some spatial interval of the form \([x_1(t), x_2(t)]\). It follows from Leibniz’s rule,
\[ \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} E(x, t) \, dx = E(x_2(t), t) \dot{x}_2 - E(x_1(t), t) \dot{x}_1 + \int_{x_1(t)}^{x_2(t)} \partial_x E(x, t) \, dx \]
\[ = E(x_2(t), t) \dot{x}_2 - E(x_1(t), t) \dot{x}_1 - S(x_2(t), t) + S(x_1(t), t) - \int_{x_1(t)}^{x_2(t)} \Phi(x, t) \, dx. \quad (4.3.10a) \]
The term $E(x_j(t), t)\dot{x}_j$ is the change of energy due to the motion of the endpoint, $S(x_j(t), t)$ is the flux of energy across the endpoint due to wave motion and $-\int_{[x_1(t), x_2(t)]} \Phi(x, t) \, dx$ is the energy loss over the interval due to dissipation.

The WKB solution can be written in the more general form:

$$u(x, t) \sim \underbrace{A(x)}_{\text{slowly changing amplitude}} \cos \left( wt - \underbrace{\varphi(x)}_{\text{rapidly changing phase}} \right), \quad \varphi(x) = \omega \theta(x).$$

(4.3.11)

It follows that

$$E(x, t) \sim \frac{1}{2} A^2 \left( \mu^2 \omega^2 + \varphi_x^2 \right) \sin^2 \left[ \omega t - \varphi(x) \right]$$

(4.3.12a)

$$S(x, t) \sim \omega \varphi_x A^2 \sin^2 \left[ \omega t - \varphi(x) \right]$$

(4.3.12b)

$$\Phi(x, t) \sim \alpha \omega A^2 \sin^2 \left[ \omega t - \varphi(x) \right].$$

(4.3.12c)

Suppose we choose $x_i(t)$ satisfying

$$\dot{x}_i = \frac{\omega}{\varphi_x(x_i)} = \text{phase velocity}.$$  

Such curves in the $x - t$ plane are called phase lines. Then

$$E \dot{x} - S \sim \frac{1}{2} \omega A^2 \left[ \mu^2 \omega^2 + \varphi_x^2 \right] \sin^2 \left[ \omega t - \varphi(x) \right] - \omega \varphi_x A^2 \sin^2 \left[ \omega t - \varphi(x) \right]$$

$$= \frac{1}{2} \omega A^2 \left[ \mu^2 \omega^2 - \varphi_x^2 \right] \sin^2 \left[ \omega t - \varphi(x) \right] = 0,$$

since $\theta(x) = \varphi(x)/\omega$ satisfies the eikonal equation (4.3.3). Hence, if $x_2 - x_1 = O(1/\omega)$ then it follows from (4.3.10) that $\frac{dE}{dt} \approx 0$, i.e. the total energy remains constant (to the first term) between any two phase lines $x_1(t), x_2(t)$ that are $O(1/\omega)$ apart.

Recall the energy equation that

$$\partial_t E + \partial_x S = -\Phi.$$  

Averaging the energy equation over one period in time results in

$$\partial_x \left( \int_0^{2\pi/\omega} S(x, t) \, dt \right) = -\int_0^{2\pi/\omega} \Phi(x, t) \, dt,$$

where the average of $\partial_t E$ over one period vanishes using (4.3.12) for $E$. Substituting (4.3.12) for $S$ and $\Phi$, we obtain

$$\partial_x (\varphi_x A^2) = -\alpha \omega A^2$$

$$\partial_x (\theta_x A^2) = -\alpha A^2$$

$$\theta_{xx} A^2 + 2 \theta_x A A_x = -\alpha A^2$$

$$\theta_{xx} A + 2 \theta_x A A_x = -\alpha A,$$

which implies that $A = u_0$ since the last equation is precisely the transport equation (4.3.6). Physically, this means that the transport equation corresponds to the balance of energy over one period in time.
4.4 Higher-Dimensional Waves - Ray Methods

The extension of the WKB method to higher dimensions is relatively straightforward, but the equations could be difficult to solve explicitly. Consider the \( n \)-dimensional wave equation

\[
\nabla^2 u = \mu^2(x) \partial_t^2 u, \quad x \in \mathbb{R}^n, \quad n = 2, 3.
\]

(4.4.1)

We look for time-harmonic solutions \( u(x,t) = e^{-i\omega t} V(x) \) and (4.4.1) reduces to the Helmholtz equation

\[
\nabla^2 V + \omega^2 \mu^2(x) V = 0.
\]

(4.4.2)

It is more instructive to have some understanding of what properties the solution has and how the WKB approximation takes advantage of them. Suppose \( \mu \) is some constant and we want to solve (4.4.2) in the region exterior to the circle \( \|x\| = a \) in \( \mathbb{R}^2 \). Exploiting the geometry leads to the choice of polar coordinates

\[
x = \rho \cos(\varphi), \quad y = \rho \sin(\varphi).
\]

We impose the Dirichlet boundary condition \( V = f(\varphi) \) at \( \rho = a \) and the Sommerfeld radiation condition which ensures that waves only propagate outward from the circle:

\[
\sqrt{\rho} \left[ \partial_\rho V - i\omega \mu V \right] = 0 \quad \text{for} \quad \rho \to \infty.
\]

Using separation of variables, the general solution of (4.4.2) is given by

\[
V(\rho, \varphi) = \sum_{n=-\infty}^{\infty} \alpha_n \left( \frac{H_n^{(1)}(\omega \mu \rho)}{H_n^{(1)}(\omega \mu a)} \right) e^{-in\varphi},
\]

(4.4.3)

where \( H_n^{(1)} \) is the Hankel function of first kind and the \( \alpha_n \) are determined from the boundary condition at \( \rho = a \). It is known that for large values of \( z \)

\[
H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp \left( i \left( z - \frac{n\pi}{2} - \frac{\pi}{4} \right) \right).
\]

Consequently, in the regime of higher frequency \( \omega \gg 1 \) (4.4.3) reduces to

\[
V(\rho, \varphi) \sim f(\varphi) \sqrt{\frac{a}{\rho}} e^{i\omega \mu (\rho - a)}.
\]

(4.4.4)

Thus we have a WKB-like solution for constant \( \mu \). Radial lines in this example correspond to rays and from (4.4.4) we see that along a ray (so that \( \varphi \) is fixed), the solution has a highly oscillatory component that is multiplied by a slowly varying amplitude \( V_0 = f(\varphi) \sqrt{a/\rho} \) that decays as \( \rho \) increases.

4.4.1 WKB Expansion

We first specify the domain and boundary conditions. (4.4.2) is to be solved in a region exterior to a smooth surface \( S \), where \( S \) encloses a bounded convex domain. This means that there
is a well-defined unit outward normal at every point on the surface. We impose the Dirichlet boundary condition
\[
V(x_0) = f(x_0) \quad \text{for } x_0 \in S,
\]
and focus only on outward propagating waves.

For higher frequency waves, we take a WKB ansatz of the form
\[
V(x) \sim e^{i \omega \theta(x)} \left[ V_0(x) + \frac{1}{\omega} V_1(x) + \ldots \right]. \quad (4.4.5)
\]
Then
\[
\nabla V \sim \left\{ i \omega \nabla \theta V_0 + i \nabla \theta V_1 + \nabla V_1 + \ldots \right\} e^{i \omega \theta} \quad (4.4.6a)
\]
\[
\nabla^2 V \sim \left\{ -\omega^2 \nabla \theta \cdot \nabla \theta V_0 + \omega \left( -\nabla \theta \cdot \nabla \theta V_1 + 2i \nabla \theta \cdot \nabla V_0 + \nabla^2 \theta V_0 \right) + \ldots \right\} e^{i \omega \theta}. \quad (4.4.6b)
\]
Substituting (4.4.6) into (4.4.2) and rearranging we find that
\[
\omega^2 \left( -\nabla \theta \cdot \nabla \theta V_0 + \mu^2 V_0 \right) + \omega \left[ -\nabla \theta \cdot \nabla \theta V_1 + 2i \nabla \theta \cdot \nabla V_0 + \mu^2 V_1 \right] + O(1) = 0
\]
\[
(\nabla \theta \cdot \nabla \theta - \mu^2) V_0 + \frac{1}{\omega} \left[ (\nabla \theta \cdot \nabla \theta - \mu^2) V_1 - i \nabla^2 \theta V_0 - 2i \nabla \theta \cdot \nabla V_0 \right] + O \left( \frac{1}{\omega^2} \right) = 0.
\]
The $O(1)$ equation is the eikonal equation which is now nontrivial to solve:
\[
\nabla \theta \cdot \nabla \theta = \mu^2. \quad (4.4.7)
\]
After cancelling the $V_1$ term using the eikonal equation (4.4.7), the $O(1/\omega)$ equation is the transport equation:
\[
2 \nabla \theta \cdot \nabla V_0 + (\nabla^2 \theta) V_0 = 0. \quad (4.4.8)
\]
Both $\pm \theta$ are solutions to the eikonal equation and we choose positive $\theta$ since this corresponds to the outward propagating waves.

### 4.4.2 Surfaces and wave fronts

The usual method method for solving the nonlinear eikonal equation (4.4.7) is to introduce **characteristic coordinates**. More precisely, we use curves that are orthogonal to the level surfaces of $\theta(x)$ which are also known as **wave fronts** or **phase fronts**.

First, note that the WKB approximation of (4.4.1) has the form
\[
u(x,t) \sim e^{i(\omega \theta(x) - \omega t)} V_0(x).
\]
We introduce the phase function
\[
\Theta(x,t) = \omega \theta(x) - \omega t.
\]
Suppose we start at $t = 0$ with the surface $S_c = \{ \theta(x) = c \}$, so that
\[
\Theta(x,0) = \omega c.
\]
As $t$ increases, the points where $\Theta = \omega c$ change, and therefore points forming $S_c$ move and form a new surface $S_{c+t} = \{\theta(x) = c + t\}$. We still have

$$\Theta(x, t) = \omega c.$$ 

The path each point takes to get from $S_c$ to $S_{c+t}$ is obtained from the solution of the eikonal equation and in the WKB method these paths are called rays.

The evolution of the wave front generates a natural coordinate system $(\alpha, \beta, s)$ where $\alpha, \beta$ comes from parameterising the wave front and $s$ from parameterising the rays. Note that these coordinates are not unique as there are no unique parameterisation for the surfaces and rays. It turns out that determining these coordinates is crucial in the derivation of the WKB approximation.

**Example 4.4.1.** Suppose we know a-priori that $\theta(x) = x \cdot x$. In this case, the surface $S_{c+t}$ is described by the equation $|x|^2 = c + t$, which is just the sphere with radius $c + t$. The rays are now radial lines and so the points forming $S_c$ move along radial lines to form the surface $S_{c+t}$.

To this end, we use a modified version of spherical coordinates:

$$(x, y, z) = \rho(s) (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha),$$

with

$$0 \leq \alpha < \pi, \ 0 \leq \beta \leq 2\pi, \ 0 \leq s.$$ 

The function $\rho(s)$ is required to be smooth and strictly increasing. Examples are $\rho = s$, $\rho = e^{s} - 1$ or $\rho = \ln(1 + s)$.

An important property of the preceding modified spherical coordinates is that $(s, \alpha, \beta)$ forms an orthogonal coordinate system. That is, under the change of variables $x = X(s, \alpha, \beta)$, the vector $\partial_s X$ tangent to the ray is orthogonal to the wave front $S_{c+t}$. We now in the opposite case: we need to find $\theta(x)$ given conditions on the map $X(s, \alpha, \beta)$. Observe the degree of freedom on specify $X$!

### 4.4.3 Solution of the eikonal equation

In what follows, we will assume that $(s, \alpha, \beta)$ forms an orthogonal coordinate system. This means that a ray’s tangent vector $\partial_s X$ points in the same direction as $\nabla \theta$ when $x = X(s, \alpha, \beta)$, or equivalently

$$\frac{\partial X}{\partial s} = \lambda \nabla \theta,$$ (4.4.9)

where $\lambda$ is a smooth positive function, to be specified later. WLOG, we assume that the rays are parameterised so that $s \geq 0$. One should not confuse $s$ with the arclength parameterisation.

Along a ray,

$$\partial_s \theta(x) = \partial_s \theta(X) = \nabla \theta \cdot \partial_s X = \lambda \nabla \theta \cdot \nabla \theta.$$ 

Therefore we can rewrite the eikonal equation as

$$\partial_s \theta = \lambda \mu^2,$$ (4.4.10)
4.4. Higher-Dimensional Waves - Ray Methods

which can be integrated directly to yield

\[
\theta(s, \alpha, \beta) = \theta(0, \alpha, \beta) + \int_0^s \lambda \mu^2 \, d\sigma,
\]

(4.4.11)

assuming we can find such a coordinate system \((s, \alpha, \beta)\). This amounts to solving (4.4.9) which is generally nonlinear and requires the assistance of numerical method. Nonetheless, we still have the freedom of choosing the function \(\lambda\).

### 4.4.4 Solution of the transport equation

It remains to find the first term \(V_0\) of the WKB approximation (4.4.5). Using (4.4.9) we have

\[
\partial_s V_0 = \nabla V_0 \cdot \partial_s \mathbf{X} = \lambda \nabla V_0 \cdot \nabla \theta.
\]

Consequently we can also rewrite the transport equation (4.4.8) as

\[
2 \partial_s V_0 + \lambda (\nabla^2 \theta) V_0 = 0.
\]

(4.4.12)

Using the identity

\[
\partial_s \left( \frac{J}{\lambda} \right) = J \nabla^2 \theta,
\]

(4.4.13)

where \(J = \left| \frac{\partial(x, y, z)}{\partial(s, \alpha, \beta)} \right|\) is the Jacobian of the transformation \(x = \mathbf{X}(s, \alpha, \beta)\), we can rewrite (4.4.12) as

\[
2 J \partial_s V_0 + \lambda \partial_s \left( \frac{J}{\lambda} \right) V_0 = 0
\]

\[
J \partial_s \left( V_0^2 \right) + \lambda V_0^2 \partial_s \left( \frac{J}{\lambda} \right) = 0
\]

\[
\left( \frac{J}{\lambda} \right) \partial_s \left( V_0^2 \right) + V_0^2 \partial_s \left( \frac{J}{\lambda} \right) = 0
\]

\[
\partial_s \left( \frac{1}{\lambda} J V_0^2 \right) = 0,
\]

and its general solution is

\[
V_0(x) = a_0 \sqrt{\frac{\lambda(x)}{J(x)}}.
\]

(4.4.14)

Imposing the boundary condition \(V_0(x_0) = f(x_0)\), we obtain

\[
V_0(x) = f(x_0) \sqrt{\frac{\lambda(x) J(x_0)}{\lambda(x_0) J(x)}},
\]

(4.4.15)

and this is true provided \(\theta(0, \alpha, \beta) = 0\) in (4.4.11).