Invariants of Non-Isolated Singularities of Hypersurfaces

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Abstract

In this paper we generalize some results by Siersma, Pellikaan, and de Jong regarding morsifications of singular hypersurfaces whose singular locus is a smooth curve, and present some applications to the study of Yomdin-type isolated singularities. In order to prove these results, we discuss the transversal discriminant of such singularities and how it relates to other algebraic and topological invariants.

Contents

1 Introduction

Understanding the singular points of a singular hypersurface allows us to better understand the geometry and the topology of the hypersurface. An important tool used to understand such singular points is to deform the hypersurface and see how such singular points behave under "special" deformations.

The goal of this paper is to study singular hypersurfaces $(V(f), 0) \subset (\mathbb{C}^n, 0)$ such that $\mathrm{Sing}(V(f)) = V(I)$ is a smooth curve germ, $f \in I^p \setminus I^{p+1}$ for some $p \geq 2$, and its generic transversal type is an ordinary multiple point. Specifically, how such singular hypersurfaces behave under a special kind of deformation, and what bounds we can conclude on some topological and algebraic invariant of $V(f)$. We summarize the results we prove in the following theorem, which is a combination of Theorem 14, Theorem 45, Theorem 54, and Corollary 55.

Theorem. *Let $f \in I^p \setminus I^{p+1}$ be a germ of analytic function (where $p \geq 2$) with $\mathrm{Sing}(V(f)) = V(I)$ such that its generic transversal type is an ordinary multiple point. Then f has a relative morsification f_t (see Definition 10) and for every large enough k and small enough t_0 we have that:*

1. *If $n \neq 4$ then*
$$j(f) \geq \#A_1(f_{t_0}) + \deg(\Delta^{\perp}(f)),$$
where $\#A_1(f_{t_0})$ denoted the number of Morse points f_{t_0} has outside $V(I)$ (see Notation 15), $j(f)$ is defined in Definition 35, and $\deg(\Delta^{\perp}(f))$ is the degree of the transversal discriminant of f as a Cartier divisor (see Definition 5).

2. *$\delta(f) \geq \deg(\Delta^{\perp}(f))$, where $\delta(f)$ is defined in Section 5.*

3. *The Milnor number of $f + x_n^k$ is finite and bounded below by*
$$\#A_1(f_{t_0}) + (k-1)(p-1)^{n-1} + 2\deg(\Delta^{\perp}(f)).$$

We now describe some historical background and some motivation for this paper. The study of deformations of singular hypersurfaces started with the study of singular hypersurface whose singular locus is a single point (which we may assume is at the origin). Such singular hypersurfaces are said to have isolated singularities. Much is known about isolated singularities and about their topology and geometry, as discussed and reviewed in [1, 7, 12].

An important result regarding isolated singularities is the relationship between the Milnor number of an analytic germ $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ (which equals to $\mu(f) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1,\ldots,x_n\}/Jac(f)))$ and the number of Morse points in a morsification of f, which is a special kind of deformation of f (for more details see Section 3.8 in [9]). Therefore, a natural question to ask is how to generalize this relationship to hypersurfaces $V(f)$ whose singular locus is not isolated.

Siersma [23] studied the Milnor fiber of f where $\mathrm{Sing}(V(f)) = V(I)$ such that the germ of f at $(0, \ldots, 0, x_n)$ is equivalent to an A_∞ singularity for every small $x_n \neq 0$. Siersma proved that the Milnor fiber of f is homotopy equivalent to a bouquet of spheres S^{n-1}. In order to compute the number of spheres in this bouquet, Siersma studied deformations of such hypersurfaces into hypersurfaces with only A_1, A_∞, and D_∞ singularities. Siersma conjectured that the number of A_1 points plus the number of D_∞ points in such a deformation, that is, $\#A_1(f) + \#D_\infty(f)$, does not depend on the deformation and can be expressed as an algebraic invariant of f.

Pellikaan [18, 22], generalized Siersma's ideas regarding isolated line singularities to the case where the singular locus of $V(f)$ is a reduced curve defined by some ideal I. Using techniques from commutative and homological algebra developed in [20], together with analytic and topological tools, Pellikaan was able to prove Siersma's conjecture: $\#A_1(f)+\#D_\infty(f) = \dim_{\mathbb{C}}(I/Jac(f))$. In addition, Pellikaan [21] was able apply this result in order to compute the Milnor number of a related isolated singularities, inspired by the work of Yomdin [26] and later Lê [17]. Pellikaan's work on Siersma's conjecture was later expanded upon by de Jong [6] to the case where the transversal type of the singular locus is determined by some fixed simple (ADE) isolated singularity.

Therefore, inspired by Siersma, de Jong, and Pellikaan, we wish to generalize the formula $\#A_1(f) + \#D_\infty(f) = \dim_{\mathbb{C}}(I/Jac(f))$ (which we call the Siersma-Pellikaan-de Jong formula) and other related results to the non reduced case, i.e., where the singular locus is a non-reduced scheme. Yet, an obstacle that we face in this case is that the transversal type of $V(f)$ is not a simple singularity, but rather a multiple point. So, in order to understand the transversal type of our singular locus, we need to look at a special Cartier Divisor of $\mathrm{Sing}(V(f))$ called the tranversal discriminant of $V(f)$, denoted $\Delta^\perp(f)$, as described by Kerner and Némethi in [16, 15], which we discuss in Section 2.

In this paper, by studying a special kind of deformations of f, which we call a relative morsification, as constructed in Section 3, we are able to obtain a generalization of the the Siersma-Pellikaan-de Jong formula, as presented in the theorem above. In addition, as in Pellikann [21], we show how we can use the tools used to prove this generalizatiom in order to compute and bound the Milnor number of isolated singularities of the form $f + x_n^k$. In Section 6 we see why our results are indeed generalizations and discuss what are the challenges in the non reduced case.

Throughout the text, unless stated otherwise, we assume that $f \in I^p \setminus I^{p+1}$ for some $p \geq 2$ with $\mathrm{Sing}(V(f)) = V(I)$ such that its generic transversal type is an ordinary multiple point. Note that since $V(I)$ is a smooth curve germ, we can assume that $I = \langle x_1, \ldots, x_{n-1}\rangle$. We denote by $\mathbb{C}\{\underline{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ the ring of germs of analytic functions in $x_1, \ldots, x_n$ over $\mathbb{C}$. In addition, for an ideal $\mathfrak{a} \subset \mathbb{C}\{\underline{x}\}$, we view $V(\mathfrak{a})$ both as a subset of $\mathrm{Spec}(\mathbb{C}\{\underline{x}\})$ and as a germ in $(\mathbb{C}^n, \underline{0})$, as in the Rückert Nullstellensatz (See Theorem 1.72 in [12] for more details).

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2 Transversal Discriminant

In this section we summarize and review some basic properties of the transversal discriminant of a hypersurface, based upon [16, 15]. We start with a motivational example, the discriminant of a polynomial in one variable (note that we in fact use this example in the proof of Theorem 14).

Definition 1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j \in \mathbb{C}[z]$ be a complex polynomial (of degree $n \geq 2$) whose roots are $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then the* ***discriminant of*** *p is defined to be*

$$\Delta_z(p) = a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2$$

Proposition 2. *Let $p(z) = \sum_{j=0}^{n} a_j z^j \in \mathbb{C}[z]$ be a polynomial of degree n. Then $\Delta_z(p) = 0$ if and only if p is non reduced, and we have that*

$$\Delta_z(p) = \frac{(-1)^{\frac{n(n-1)}{2}}}{a_n} Res_z(p, \frac{dp}{dz}),$$

where $Res_z(p, \frac{dp}{dz})$ is the resultant of the polynomials p and $\frac{dp}{dz}$.

Proof. See Section 1.B of Chapter 12 of [11]. □

Remark 3. *1.* If we look at every polynomial of degree n as its tuple of coefficients, then by Proposition 2, the discriminant defines a polynomial map $\mathbb{C}^{n+1} \to \mathbb{C}^1$

2. One can generalize the concept of the discriminant of a polynomial to polynomials in a number of variables. For proofs and more details regarding the discriminant and the resultant, see [11].

One might ask if we can capture properties other than reducability in a similar way. Specifically, the property we are interested in is how the transversality type of our hypersurface along its singular locus changes and degenerates:

Let $(X, 0) = (V(f), 0) \subset (\mathbb{C}^n, 0)$ be a hypersurface with $Z = \text{Sing}(V(f))$, where we assume that Z is smooth. For every $o \in Z$ we consider a smooth germ $(L, o) \subset (\mathbb{C}^n, o)$ such that $(L, o) \cap (Z, o) = \{o\}$ and (L, o) is transversal to (Z, o). By looking at $(X \cap L, o)$ and changing o, we can see that its type is generically constant in some sense, but there are still points in which this transversal type degenerates.

Example 4. Let $X = V(f) \subset \mathbb{C}^3$ where $f(x,y,z) = x^p - y^p z$. Note that $\mathrm{Sing}(V(f)) = V(x,y)$. For every $z_0 \in \mathbb{C}$, we can view $f_{z_0}(x,y) = f(x,y,z_0)$ as a polynomial in two variables which defines a plane curve singularity. For every $z_0 \neq 0$, we have that f_{z_0} defines an ordinary multiple point with p tangents, but for $z_0 = 0$ we have that $f_0(x,y) = x^p$ which is a unique line of multiplicity p. So $z_0 = 0$ is a point in $V(x,y)$ where the transversal type degenerates.

This intuition leads us to the definition of the transversal discriminant of f.

Definition 5 (2.2.5 in [15])**.** *Given $f \in \mathbb{C}\{\underline{x}\}$ with Sing$(V(f)) = V(I)$, the **transversal discriminant** of f, is defined to be*

$$\Delta^{\perp}(f) = V(\mathrm{Fitt}_0(\pi_* \mathcal{O}_{\mathrm{Crit}(\pi)})),$$

where $\pi\colon \widetilde{V(f)} \to V(f)$ is the restriction of the blow up map $Bl_{V(I)}(\mathbb{C}^n) \to \mathbb{C}^n$ to the strict transform of $V(f)$, and Fitt_0 *is the zeroth fitting ideal.*

Example 6. Let $f(\underline{x}) = x_n x_{n-1}^p + \sum_{i=1}^{n-1} x_i^p \in I^p$ for some $p \geq 2$, where $I = \langle x_1, \ldots, x_{n-1} \rangle$. Then $\mathrm{Sing}(V(f)) = V(I)$, and thus

$$Bl_{V(I)}(\mathbb{C}^n) = \{(x_1, \ldots, x_n, [\sigma_1 \colon \ \ldots : \sigma_{n-1}]) \colon \forall i, j, x_i \sigma_j = x_j \sigma_i\}.$$

So, if we denote by E the exceptional divisor of $Bl_{V(I)}(\mathbb{C}^n)$, then

$$E = \{(0, \ldots, 0, x_n, [\sigma_1 \colon \ \cdots : \sigma_{n-1}])\},$$

and we can compute that

$$\widetilde{V(f)} \cap E = \{x_n \sigma_{n-1}^p + \sum_{i=1}^{n-2} \sigma_i^p = 0, x_1 = \cdots = x_{n-1} = 0\}.$$

Now, by restricting the blow up map to $\pi\colon \widetilde{V(f)} \cap E \to V(f)$ and looking at π with respect to each chart of the blow up, we can compute that $\mathrm{Crit}(\pi) = V(x_n, \sigma_1^{p-1}, \ldots, \sigma_{n-2}^{p-1})$ and so $\mathcal{O}_{\mathrm{Crit}(\pi)} = {}^{\mathbb{C}\{x_n, \sigma_1, \ldots, \sigma_{n-2}\}}/_{\langle x_n, \sigma_1^{p-1}, \ldots, \sigma_{n-2}^{p-1}\rangle}$. Yet we have an isomorphism of $\mathbb{C}[x_n]-$modules ${}^{\mathbb{C}[x_n, \sigma_1, \ldots, \sigma_{n-2}]}/_{\langle x_n, \sigma_1^{p-1}, \ldots, \sigma_{n-2}^{p-1}\rangle} \cong \bigoplus_{0 \leq i_1, \ldots, i_{n-2} \leq p-1} \mathbb{C} \cdot \sigma_1^{i_1} \cdots \sigma_{n-2}^{i_{n-2}} \cong ({}^{\mathbb{C}\{x_n\}}/_{\langle x_n \rangle})^{\oplus (p-1)^{n-2}}$, and so we have that $\pi_* \mathcal{O}_{\mathrm{Crit}(\pi)} = ({}^{\mathbb{C}\{x_n\}}/_{\langle x_n \rangle})^{\oplus (p-1)^{n-2}}$. Therefore, we can conclude that

$$\mathrm{Fitt}_0(\pi_* \mathcal{O}_{\mathrm{Crit}(\pi)}) = \mathrm{Fitt}_0({}^{\mathbb{C}[x_n]}/_{\langle x_n \rangle})^{(p-1)^{n-2}} = \langle x_n^{(p-1)^{n-2}} \rangle.$$

We end this section with a few remarks which play a crucial role in the proofs later on.

Remark 7. *1.* Proposition 4.4 in [16] tells us that $\Delta^{\perp}(f) \subset \mathrm{Sing}(V(f))$ is a Cartier Divisor. In fact, an explicit generator for $I(\Delta^{\perp}(f))$ (in the case we are interested in) is given in section 4.1.1 in [16], and in [15], the equivalence class of $\Delta^{\perp}(f)$ as a Cartier divisor in the Picard group $Pic(\mathrm{Sing}(V(f)))$ has been studied and computed using the Thom-Porteus Formula.

2. A useful conclusion (from item 1 of this remark) is that given a polynomial $f \in \mathbb{C}[\underline{x}] = \mathbb{C}[x_1, \ldots, x_n]$, then this explicit generator of $\Delta^{\perp}(f)$ is a polynomial in the coefficients of f, and if d is the total degree of f, then the degree of this polynomial is bounded by $\tilde{n} = (n-1)(d-1)^{n-2}$ (see Section 5.1 in [16]). Therefore, this induces an algebraic map $\mathbb{C}^N \to \mathbb{C}^{\tilde{n}}$, where we associate (by looking at the coefficient) $\mathbb{C}[\underline{x}]_{\leq d} = \{f \in \mathbb{C}[\underline{x}] \colon \deg_{\text{total}}(f) \leq d\}$ with $\mathbb{C}^N$ (for $N = \binom{d+n}{n}$) and $\mathbb{C}[x]_{\leq \tilde{n}} = \{f \in \mathbb{C}[x] \colon \deg(f) \leq \tilde{n}\}$ with $\mathbb{C}^{\tilde{n}}$.

Remark 8. Definition 5 gives us an additional topological understanding of the transversal discriminant (assuming that the generic transversal type of $V(f)$ along its singular locus is an ordinary multiple point): $\Delta^{\perp}(f)$ is empty if and only if the transversal sections of $V(f)$ along its singular locus are topologically equisingular, i.e., for every p_1 and p_2 in $\text{Sing}(V(f))$ (which we assume is smooth), if we consider $(L_1, p_1) \subset (\mathbb{C}^n, p_1)$ and $(L_2, p_2) \subset (\mathbb{C}^n, p_2)$ such that $(L_i, p_i) \cap (\text{Sing}(V(f)), p_i) = \{p_i\}$ and (L_i, p_i) is transversal to $(\text{Sing}(V(f)), p_i)$, for $i = 1, 2$, then the germs $(V(f) \cap L_1, p_1)$ and $(V(f) \cap L_2, p_2)$ are homeomorphic. For a deeper discussion on this result, see Section 2.2.3 in [15] and Section 2.7 in [16].

Remark 9. An important property of the transversal discriminant is that the transversal discriminant behaves nicely under flat deformations. Proposition 5.1 in [16] tells us that given $f \in \mathbb{C}\{\underline{x}\}$ and $f_t \in \mathbb{C}\{\underline{x}, t\}$ a flat deformation of f which deforms the singular locus of $V(f)$ flatly and the generic multiplicity of f along its singular locus remains constant, then the family $\{\Delta^{\perp}(f_t)\}_t$ is flat and $\deg(\Delta^{\perp}(f_{t_0})) = \deg(\Delta^{\perp}(f))$ for every t_0, where the degree of $\Delta^{\perp}(f)$ is the degree of the transversal discriminant of f as a Cartier divisor. For example, let $f(x, y, z) = x^p + y^p z$ and let $f_t(x, y, z) = x^p + y^p z + txy^{p-1}$. Then, one can compute that for every small t we have that

$$\Delta^{\perp}(f_t) = \{z^{p-1} - \left(\frac{t}{p}\right)^p (1-p)^{p-1} = 0\}$$

while $\Delta^{\perp}(f) = \{z^{p-1} = 0\}$.

3 Relative Morsification

In this section we discuss how we can deform f in a specific way (as we define in Definition 10), which plays a crucial role in the proof of Theorem 18, and later Theorem 45 and Theorem 54. The concept of a relative morsification generalizes the concept of a morsification of a hypersurface $V(f)$ with a reduced line singularity, as presented in [23, 22, 6], as we discuss in Section 6.

Definition 10. *1. Given a point $\underline{y} \in \text{Crit}(f)$, we say that $\underline{y}$ is a **Morse critical point of** f (or an A_1 point of f) if $\text{Hess}(f)(\underline{y})$, the Hessian matrix of f at the point $\underline{y}$, is invertible.*

*2. We say that $f \in \mathbb{C}\{\underline{x}\}$ is **Morse outside of** $V(I)$ if $Sing(V(f)) = V(I)$ and f has only Morse critical points in a small neighborhood outside its singular locus.*

3. *Let $f \in I^p \setminus I^{p+1}$ such that $\mathrm{Sing}(V(f)) = V(I)$. We say that a flat deformation $f_t \in \mathbb{C}\{\underline{x}, t\}$ of f is a* ***relative morsification*** *of f if $f_0 = f$ and there exists some neighborhood $0 \in U \subset \mathbb{C}$ such for every $0 \neq t_0 \in U$:*

   (a) *$\mathrm{Sing}(V(f_{t_0})) = V(I)$ and $f_{t_0} \in I^p \setminus I^{p+1}$.*

   (b) *$f_{t_0} \in I^p$ is Morse outside of $V(I)$.*

   (c) *The transversal discriminant $\Delta^{\perp}(f_{t_0})$ is a reduced subscheme of $V(I)$.*

Remark 11. An analogue concept of a relative morsification has been studied from a topological point of view in Bobadillia [5] which also generalizes the concept of a morsification presented in [23, 22, 6].

Example 12. *1.* The following table contains a few examples of relative morsifications (reducability of the transversal discriminant follows from similar computations to the ones preformed in Example 6):

f	A relative morsification of f
$x^p + y^p z$	$t(x^2y^{p-2} + y^2x^{p-2}) + y^p z + x^p$
$x^p + y^p z^q + y^{p+1}$	$t(x^2y^{p-2} + y^2x^{p-2}) + y^p(z^q - tz) + y^{p+1} + x^p$
$x^p z^{q_1} + y^p z^{q_2} + y^{p+1} + x^{p+1}$	$x^p(z^{q_1} - tz) + y^p(z^{q_2} - tz) + y^{p+1} + x^{p+1} + t(x^2y^{p-2} + y^2x^{p-2})$
$\prod_{i=1}^{n}(x^{p_i} + y^{p_i} z)$	$\prod_{i=1}^{n}(t(x^2y^{p_i-2} + y^2x^{p_i-2}) + y^{p_i} z + x^{p_i})$
$\prod_{i=1}^{n}(x^{p_i} + y^{p_i} z^{q_i} + y^{p_i+1})$	$\prod_{i=1}^{n}(t(x^2y^{p_i-2} + y^2x^{p_i-2}) + y^{p_i}(z^{q_i} - tz) + y^{p_i+1} + x^{p_i})$
$\sum_{i=1}^{n-2} x_i^p + x_{n-1}^p x_n^q + x_{n-1}^{p+1}$	$\sum_{i=1}^{n-2} x_i^p + x_{n-1}^p(x_n^q - tx_n) + x_{n-1}^{p+1} + t(\sum_{i\neq j<n} x_i^2 x_j^{p-2})$
$\sum_{i=1}^{n-2}(x_i^p x_n^{q_i} + x_i^{p+1})$	$\sum_{i=1}^{n-2}(x_i^p(x_n^{q_i} - tx_n) + x_i^{p+1}) + t(\sum_{i\neq j<n} x_i^2 x_j^{p-2})$

2. Let $f(\underline{x}) = \sum_{i=1}^{n} x_i^p$ where $p > 1$. Then f is Morse outside $V(I)$ as it has no non-singular critical points and $\Delta^{\perp}(f) = \emptyset$ since its transversal type does not change along its singular locus.

3. The deformation $f_t(x, y, z) = x^{p+1} + y^p + tx^p$ is not a relative morsification since for every $t_0 \neq 0$ we have that $(\frac{-tp}{p+1}, 0, z_0)$ is a non-Morse critical point for every z_0 which does not belong to $V(I)$.

Remark 13. If $\Delta^{\perp}(f)$ is empty then any morsification of f has no Morse critical points outside $V(I)$. This is true since its blow up is smooth and as a deformation of a smooth hypersurface is smooth, and the Milnor number stays zero (see Section 2.6 in [12] for more details). But the Milnor number of an isolates singularity is the number of Morse points in any morsification of f (see Proposition 3.19. in [9]). In fact, we can conclude that if $\Delta^{\perp}(f) = \emptyset$ then for every deformation f_t (as in Remark 9) and for every small ennough t_0 we have that $\Delta^{\perp}(f_{t_0}) = \emptyset$.

Theorem 14. *For every $f \in I^p \setminus I^{p+1}$ with $\mathrm{Sing}(V(f)) = V(I)$ such that its generic transversal type is an ordinary multiple point, there exists a relative morsification. Moreover, it can be taken in the form $f_t = f + tg$ for some $g \in I^p$.*

Proof. We prove this proposition in two steps. In the first step we assume that f is a polynomial in $\mathbb{C}[\underline{x}]$ and we find a polynomial relative morsification for f. In the second step we prove the result (for analytic functions), based upon the first step.

- Step 1: Assume that $f \in \mathbb{C}[\underline{x}]$.

Denote the (total) degree of f with respect to $x_1, \dots, x_n$ by d.

By viewing every polynomial in $\mathbb{C}$ as its tuple of coefficients, we identify $\mathbb{C}[x_1, \dots, x_n]_{\leq d}$ (the set of all polynomials $g \in \mathbb{C}[\underline{x}] = \mathbb{C}[x_1, \dots, x_n]$ whose degree with respect to $x_1, \dots, x_n$ does not surpass d), with the affine space $\mathbb{C}^N$ where $N = \binom{d+n}{n}$. Now, viewed as subsets of $\mathbb{C}^N$,

1. Denote by Σ_1 the set of polynomials $g \in I^p \cap \mathbb{C}[x_1, \dots, x_n]_{\leq d}$ such that g is not Morse outside $V(I)$, as in Definition 10.

2. Denote by Σ_2 the set of polynomials $g \in I^p \cap \mathbb{C}[x_1, \dots, x_n]_{\leq d}$ such that $\Delta^{\perp}(g) = \langle h \rangle$ where $h \in \mathbb{C}[x_n]$ is not reduced or $h = 0$.

We divide this step into three parts: In the first two parts (1.1 and 1.2) we show that $\overline{\Sigma_1}$ (the closure of Σ_1 in $\mathbb{C}^N$ with respect to the Euclidean topology) and Σ_2 are algebraic subsets which are proper subsets of I^p (where we consider I^p as a subset of $\mathbb{C}^N$). In the third part (1.3) we construct the desired relative morsification of f, using the fact that f_t is a relative morsification if and only if for every small enough t_0, $f_{t_0} \in I^p \setminus (\Sigma_1 \cup \Sigma_2)$ and $f_0 = f$.

We are interested in $\overline{\Sigma_1}$ since Σ_1 is not closed in the classical topology. This is true because if g_t has a unique non-Morse point at $(t, 0, .., 0)$, then as $t \to 0$ we have that $g_0 \notin \Sigma_1$.

Note that for every ideal $\mathfrak{a} \subset \mathbb{C}[\underline{x}]$ we have that the (set theoretical) intersection $\mathfrak{a} \cap \mathbb{C}[\underline{x}]_{\leq d}$ is a complex vector space, and thus it is an algebraic subset of $\mathbb{C}^N$ (as it is the intersection of linear hypersurfaces).

By Example 12, for every t, the function $\sum_{i=1}^{n-2} x_i^p + x_{n-1}^p(x_n^q - tx_n) + x_{n-1}^{p+1} + t(\sum_{i \neq j < n} x_i^2 x_j^{p-2})$ has a reduced non trivial transversal discriminant and no non-Morse critical points outside of $V(I)$, which tells us that $\Sigma_1 \cup \Sigma_2 \neq I^p$.

Part 1.1: Note that since Σ_1 is a cone in $\mathbb{C}^N$ (that is, closed under multiplication by a complex scalar), $\overline{\Sigma_1}$ is a cone as well. Therefore it is enough to prove that $\mathbb{P}(\overline{\Sigma_1}) \subset \mathbb{P}^{N-1}(\mathbb{C})$ is a projective algebraic subset. Now, look at the set

$$\Sigma_{1,pt} = \{([g], a) \in \mathbb{P}^{N-1}(\mathbb{C}) \times \mathbb{C}^n \colon a \notin V(I) \text{ is not a Morse point of } g \in I^p\}.$$

Note that $\Sigma_{1,pt}$ is well defined since Σ_1 is a cone. Take the compactification $\iota\colon \mathbb{P}^{N-1}(\mathbb{C}) \times \mathbb{C}^n \to \mathbb{P}^{N-1}(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ defined by

$$\iota([\sigma_1\colon \ldots : \sigma_N], z_1, \ldots, z_n) = ([\sigma_1\colon \ldots : \sigma_N], [z_1\colon \ldots : z_n\colon 1]),$$

and let $\overline{\iota(\Sigma_{1,pt})}$ be the Zariski closure of the image of $\Sigma_{1,pt}$ under this embedding. Note that

$$\Sigma_{1,pt} \subsetneq Y = \{([g], a)\colon \nabla(g)(a) = 0 = \det(\mathrm{Hess}(g)(a))\} \subsetneq \mathbb{P}^{N-1}(\mathbb{C}) \times \mathbb{C}^n.$$

Thus we have that $\overline{\iota(\Sigma_{1,pt})} \neq \mathbb{P}^{N-1}(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$, as it must be contained inside $\iota(Y) \subsetneq \mathbb{P}^{N-1}(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$.

Denote by $\pi\colon \mathbb{P}^{N-1}(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^{N-1}(\mathbb{C})$ the projection map. Since $\overline{\iota(\Sigma_{1,pt})}$ is an algebraic subset, by Theorem 3.12. in [13] we have that $\pi(\overline{\iota(\Sigma_{1,pt})})$ is an algebraic subset as well. But since $\pi(\iota(\Sigma_{1,pt})) = \mathbb{P}(\Sigma_1)$, we have that $\pi(\overline{\iota(\Sigma_{1,pt})}) = \mathbb{P}(\overline{\Sigma_1})$, and we can conclude that $\mathbb{P}(\overline{\Sigma_1})$ is an algebraic set.

Part 1.2: By Proposition 7 we know that $\Delta^{\perp}(g) \subset \mathrm{Sing}(V(g))$ is an algebraic subset for every $g \in I^p$. Denote by h_g the defining generator of $I(\Delta^{\perp}(g))$, as mentioned in Remark 7, and note that $h_g \in \mathbb{C}[x_n]$.

Let $\Phi\colon I^p \to \mathbb{C}^1$ be the map $\Phi(f) = \Delta_{x_n}(h_f)$, where Δ_{x_n} is defined in Definition 1 and, as before, we view I^p as a subset of $\mathbb{C}^N$. Φ is a polynomial map since by Remark 3, Δ_{x_n} defines a polynomial map, and, by Remark 7, the map $f \mapsto h_f$ is a polynomial map as well.

Thus, since we have that $\Phi^{-1}(\{0\}) = \Sigma_2$ and since $\{0\} \subset \mathbb{C}^1$ is an algebraic subset, then we get that $\Sigma_2 \subset I^p$ is an algebraic subset.

Part 1.3: By the previous parts, $\overline{\Sigma_1} \subsetneq I^p \cap \mathbb{C}[\underline{x}]_{\leq d}$ and $\Sigma_2 \subsetneq I^p \cap \mathbb{C}[\underline{x}]_{\leq d}$ are algebraic subsets. Therefore there exists some $g \in I^p$ such that $g \notin \Sigma_1 \cup \Sigma_2$, because I^p is a vector space which is irreducible as an algebraic subset. Denote $f_t = f + tg$. Then f_t is a relative morsification of f since $\{f_t\colon t \in \mathbb{C}\} \subset \mathbb{C}^N$ is a line and thus must intersect $\overline{\Sigma_1} \cup \Sigma_2$ in only finitely many points.

- Step 2 ($f \in \mathbb{C}\{\underline{x}\}$):

First, note that since $V(I)$ is a smooth curve and $f \in I^p \backslash I^{p+1}$, then $ht(Jac(f)) = 1$, and thus f is $\mathcal{R}-$equivalent to an element of $\mathbb{C}[x_1, \ldots, x_{n-1}]\{x_n\}$ (See Proposition 5.19 in [3]). Moreover, if f_t is a relative morsification and if the automorphism $\varphi\colon \mathbb{C}\{\underline{x}\} \to \mathbb{C}\{\underline{x}\}$ preserves the singular locus of f_t, then $\varphi^*(f_t) = f_t \circ \varphi$ is a relative morsification as well. Thus, it is enough to prove the existence of a relative morsification for analytic germs of the form $f \in \mathbb{C}\{x_n\}[x_1, \ldots, x_{n-1}]$.

We view f as an analytic function $f\colon U \to \mathbb{C}$, where $0 \in U \subset \mathbb{C}^n$ is an open cylinder $U = \mathbb{C}^{n-1} \times U_s$ with $U_s \subset \mathbb{C}^1_{x_1}$. For every $s_0 \in U_s$, denote by f_{s_0} the tuple corresponding with the polynomial in $(n-1)$ variables $f(x_1, \ldots, x_{n-1}, s_0)$.

(Here we identify $\mathbb{C}[x_1,\ldots,x_{n-1}]_{\leq d_{n-1}}$ with the affine space $\mathbb{C}^K$ where $K = \binom{d_{n-1}+n-1}{n-1}$ and d_{n-1} is the degree of f with respect to $x_1,\ldots,x_{n-1}$). Denote by Γ the set $\{f_s \colon s \in U_s\} \subset \mathbb{C}^K$. Since $f \in \mathbb{C}\{x_n\}[x_1,\ldots,x_{n-1}]$ then $(\Gamma,0)$ is an irreducible analytic curve-germ in $\mathbb{C}^K$.

Now, note that f is Morse outside $V(I)$ (as in Definition 10) if and only if for every small enough s, the polynomial $f_s \in \mathbb{C}[x_1,\ldots,x_{n-1}]$ has no non-Morse points which are not the origin. Denote

$$\Sigma_0 = \{g \in \langle x_1,\ldots x_{n-1}\rangle^p \subset \mathbb{C}[x_1,\ldots,x_{n-1}] \colon g \text{ is not Morse outside the origin}\}.$$

As in Part 1.1, we have that $\overline{\Sigma_0} \subset \mathbb{C}^K$ is an algebraic subset. Since $(\Gamma,0)$ is an irreducible analytic curve, then either $\Gamma \cap \overline{\Sigma_0}$ is a discrete set or $\Gamma \subset \overline{\Sigma_0}$.

If $\Gamma \cap \overline{\Sigma_0}$ is a discrete set, then we can choose a neighborhood $\tilde{U}_s \subset U_s$ such that $\{f_s \colon s \in \tilde{U}_s\} \cap \overline{\Sigma_0}$ is a single point (where $s = 0$) and thus f is Morse outside $V(I)$ in $\mathbb{C}^{n-1} \times U_s$.

Otherwise, $\Gamma \subset \overline{\Sigma_0}$, and since $\overline{\Sigma_0}$ is an algebraic subset then there exists some $g \in \langle x_1,\ldots,x_{n-1}\rangle^p \subset \mathbb{C}[x_1,\ldots,x_{n-1}]$ such that, for every s, $tg+(1-t)f_s$ intersects $\overline{\Sigma_0}$ only when $t = 0$, and therefore $tg+(1-t)f$ is Morse outside $V(I)$.

By Part 1.2, since $\Sigma_2 \subset \mathbb{C}[\underline{x}] \subset \mathbb{C}\{x_n\}[x_1,\ldots,x_{n-1}]$ is an algebraic subset, we can choose g such that $tg+(1-t)f$ is a relative morsification of f.

□

We end this section by presenting a sketch of a more analytic proof of Theorem 14. This proof is similar to the proof of Theorem 1.15 in [6] and to the proof of Proposition 7.18 in [18], and heavily relies on tools from differential geometry such as Sard's theorem and Fubini's theorem. In addition, it gives us a more explicit form for the relative morsification (similar to the relative morsifications in Example 12).

Proof. Consider f as analytic function $f \colon U \to \mathbb{C}$ where $U \subset \mathbb{C}^n$. Set $N = (n-1)^2 + n(n-1)$ and define $F \colon U \times \mathbb{C}^N \to \mathbb{C}$ by

$$F(\underline{x},\underline{a},\underline{b}) = f(x) - \Big(\sum_{i,j\leq n-1} a_{i,j}x_j^{p-2}x_i^2 + \sum_{k\neq l} b_{k,l}x_kx_l^p\Big).$$

We construct from $F(\underline{x},\underline{a},\underline{b})$ a relative morsification of f. Note that for every small enough $(\underline{a},\underline{b})$ we have that the singular locus of $F(\underline{x},\underline{a},\underline{b})$ (with respect to $\underline{x}$) is exactly $V(I)$.

First, denote $F_{\underline{a},\underline{b}}(\underline{x}) = F(\underline{x},\underline{a},\underline{b})$ and denote by $h_{\underline{a},\underline{b}} \in \mathbb{C}\{x_n\}$ the generating element of $I(\Delta^{\perp}(F_{\underline{a},\underline{b}}))$. Let $H = \{(x_n,\underline{a},\underline{b}) \in (V(I) \cap U) \times \mathbb{C}^N \colon h_{\underline{a},\underline{b}}(x_n) = 0\}$ where we identify $V(I)$ with a copy of $\mathbb{C}$. By directly computing the transversal discriminant of $F_{\underline{a},\underline{b}}$ one can verify that $h_{\underline{a},\underline{b}}$ is not the zero function, for every

$(\underline{a},\underline{b}) \in \mathbb{C}^N$. Thus the projection map $p\colon H \to \mathbb{C}^N$ is a finite map since $h_{\underline{a},\underline{b}}(x_n)$ has finitely many zeros in $V(I)\cap U$. By Sard's Theorem, the set $S_1 = p(\mathrm{Crit}(p))$ is a set of measure zero, and outside $p^{-1}(S_1)$ we have that p is a finite covering map. Thus for every $\underline{s} \notin S_1$ we have that $\Delta^{\perp}(F_{\underline{s}})(x_n)$ is reduced.

Second, denote $\Phi_i = \frac{\partial F}{\partial x_i}$ and consider the function $\Phi = (\Phi_1, \dots, \Phi_n)\colon U \times \mathbb{C}^N \to \mathbb{C}^n$. Note that in order to prove that $F(x, \underline{a}, \underline{b})$ is a Morse function outside $V(I)$, it is enough to show that $\Phi(x, \underline{a}, \underline{b})$ is a submersion outside $(V(I)\cap U)\times\mathbb{C}^N$. Because this would imply that $\det(\mathrm{Hess}(F)) \neq 0$. Accordingly, we study the critical locus of Φ via the differential $d\Phi \in \mathrm{M}_{n\times(n+N)}(\mathbb{C})$. One calculates that for every i, j we have that

$$\frac{\partial \Phi_i}{\partial a_{j,j}} = p x_j^{p-1}\delta_{i,j}$$

and for every $j \neq k$ and for every i we have that

$$\frac{\partial \Phi_i}{\partial b_{j,k}} = x_k^{p-1}(p x_i \delta_{k,i} + \delta_{i,j} x_k).$$

This gives us that

$$\det\left[\frac{\partial \Phi}{\partial b_{1,k}} - x_1\frac{\partial \Phi}{\partial a_{k,k}}, \dots, \frac{\partial \Phi}{\partial b_{n,k}} - x_n\frac{\partial \Phi}{\partial a_{k,k}}\right] = x_i^{p(n+1)}.$$

Thus, the ideal $\mathfrak{a} = \langle x_1^{p(n+1)}, \dots, x_{n-1}^{p(n+1)}\rangle$ is contained in the ideal generated by the $n\times n$ minors of $d\Phi$. So $V(\mathfrak{a}) = V(I)$ contains the critical locus of Φ, and thus Φ is a submersion outside $(V(I)\cap U)\times\mathbb{C}^N$.

Note that $\Phi^{-1}(0)\setminus((V(I)\cap U)\times\mathbb{C}^N)$ is smooth of dimension N. therefore if we look at the projection $\pi\colon \Phi^{-1}(0)\setminus((V(I)\cap U)\times\mathbb{C}^N) \to \mathbb{C}^N$, then this projection is non degenerate.

Now, define $\Psi\colon (U\setminus V(I))\times\mathbb{C}^N \to Gr(\mathbb{C}^n\times\mathbb{C}^N, N)$ by $\Psi(x) = \ker(d\Phi(x))$, where $Gr(\mathbb{C}^n\times\mathbb{C}^N, N)$ is the $N-$th Grassmannian of $\mathbb{C}^n\times\mathbb{C}^N$. Note that Ψ is C^∞ and its critical local is of measure zero. Observe that the set $\mathcal{H} = \{V \in Gr(\mathbb{C}^n\times\mathbb{C}^N, N)\colon V\cap(\mathbb{C}^n\times\{0\}) \neq \{0\}\}$ is a compact space of measure zero, and thus $\Psi^{-1}(\mathcal{H}) \subset (U\setminus V(I))\times\mathbb{C}^N$ is closed and of measure zero as well. Thus, by Fubini's theorem, almost every section of $\Psi^{-1}(\mathcal{H})$ with respect to $\mathbb{C}^N$ is closed and of measue zero as well, and so its complement is open and of full measure.

Therefore, for every $\underline{s} \notin S_2$ we have that $\Phi(\cdot, \underline{s})\colon U \to \mathbb{C}^n$ is a submersion, i.e., $F(\underline{x}, \underline{s})$ has only Morse critical points in U_1 and outside $V(I)$, where S_2 is the set of elements in $\mathbb{C}^N$ with respect to which the sections of $\Psi^{-1}(\mathcal{H})$ are not of measure zero (up to shrinking U).

Since S_1 and S_2 are sets of measure zero, then there exists some $s = (\underline{a}, \underline{b}) \in \mathbb{C}^N$ such that for every $0 \neq t \in \mathbb{C}$ small enough, we have that $t \cdot s \notin S_1 \cup S_2$. Therefore we can conclude that $f_t(\underline{x}) = F(\underline{x}, t \cdot s)$ is a relative morsification of f.

□

4 Siersma-Pellikaan-de Jong Formula

In this section we discuss how, inspired by Siersma, Pellikaan, and de Jong, we can bound the number of Morse points in any relative morsification and the degree of the transversal discriminant of f (as a Cartier divisor) by an algebraic invariant. We start with a general construction (Theorem 18) from which we give a more concrete and computational invariant (Theorem 45).

Recall that we assume that $f \in I^p \setminus I^{p+1}$ for some $p \geq 2$ with $\mathrm{Sing}(V(f) = V(I)$ such that its generic transversal type is an ordinary multiple point. In addition, throughout this section we assume that $n \neq 4$, as it is an important assumption in Proposition 27 and in its application in the proof of Lemma 29 (see Remark 26 for more details).

Notation 15. Let $f \in I^p \setminus I^{p+1}$ with $\mathrm{Sing}(V(f)) = V(I)$ and let f_t be a deformation of f. Consider the following objects:

1. $R = \mathbb{C}\{\underline{x}, t\}$ is a local ring with a unique maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n, t \rangle$.
2. The ideal $\mathfrak{p} = \langle x_1, \ldots, x_{n-1} \rangle \subset R$.
3. $\#A_1(f_{t_0})$ is the number of Morse points of a f_t outside $V(I)$, for a small and fixed t_0.
4. The ideal $Jac(f_t) = \langle \partial_1(f_t), \ldots, \partial_n(f_t) \rangle \subset \mathbb{C}\{\underline{x}, t\}$, where $\partial_i(f_t)$ is the partial derivative of f_t with respect to x_i.
5. The module $M(\mathfrak{a}) = \mathfrak{a}/Jac(f_t)$, and $j_0(\mathfrak{a}) = \dim_{\mathbb{C}}(M(\mathfrak{a}) \otimes_{\mathbb{C}\{\underline{x},t\}} \mathbb{C}\{\underline{x}, t\}/\langle t \rangle)$, for every ideal $Jac(f_t) \subset \mathfrak{a}$.

Remark 16. Note that we have an isomorphism

$$(M(\mathfrak{a}) \otimes_{\mathbb{C}\{\underline{x},t\}} \mathbb{C}\{\underline{x}, t\}/\langle t \rangle) = \mathfrak{a}/Jac(f_t) + \langle t \rangle \cdot \mathfrak{a}.$$

Definition 17. *Let f_t be a deformation of f. An ideal $Jac(f_t) \subset \mathfrak{a}$ is called a* ***generification of*** $Jac(f_t)$ ***over*** $\mathfrak{p}$ *if*

1. *The ring $R/\mathfrak{a}$ is a Cohen-Macaulay ring of dimension 2 such that t is not a zero-divisor of $R/\mathfrak{a}$.*
2. *$\sqrt{\mathfrak{a}} = \mathfrak{p}$ such that $Jac(f_t)_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}$.*

Theorem 18. *Let f_t be a relative morsification of f, and let $\mathfrak{a}$ be a generification of $Jac(f_t)$ over $\mathfrak{p}$. Then for every small t_0,*

$$j_0(\mathfrak{a}) \geq \#A_1(f_{t_0}) + \deg(\Delta^{\perp}(f)).$$

Before proving Theorem 18, we start with a few lemmata and a few additional remarks and notations, some of which are quotes of known results from the literature.

Notation 19. Let f_t be a relative morsification of f and let $\mathfrak{a}$ be a generification of $Jac(f_t)$ over $\mathfrak{p}$. Consider $F(\underline{x}, t) = f_t(\underline{x})$ as an analytic function $F\colon U \to \mathbb{C}$ where $U \subset \mathbb{C}^{n+1}$ is a bounded small neighborhood of 0 of the form $U = U_n \times U_1$, where $U_n \subset \mathbb{C}^n$ and $U_1 \subset \mathbb{C}^1$. Then for every $(p, t_0) \in U$ we define

$$j(\mathfrak{a}, p, t_0) = \dim_{\mathbb{C}}(\pi_*(\mathcal{M}_{(p,t_0)})),$$

where $\mathcal{M} = {}^{\mathcal{I}_F}/_{\mathcal{J}_F}$, in which $\mathcal{J}_F$ is the ideal sheaf generated by $\frac{\partial F}{\partial x_1} \dots, \frac{\partial F}{\partial x_n}$, $\mathcal{I}_F = \mathfrak{a} \cdot \mathcal{O}_U$, and $\pi\colon (U \cap \mathrm{Supp}(\mathcal{M})) \to U_1$ is the projection map.

Remark 20. As with $j_0(\mathfrak{a})$, the invariant $j(\mathfrak{a}, p, t_0)$ might be infinite. We address the finiteness of $j_0(\mathfrak{a})$ in Lemma 31, and of $j(\mathfrak{a}, p, t_0)$ in Lemma 33. Yet in general, since we are looking at $\pi_*(\mathcal{M}_{(p,t_0)})$, we are looking at the push-forward of a one-dimensional stalk onto a one dimensional ring, $\mathbb{C}\{t\}$.

The following proposition, Proposition 21, is a restatement of a result which was originally presented as Proposition 1.17 in Part 2 of [18]. This proposition plays a crucial role in the proof of Theorem 18, specifically as part of the proof of Lemma 23. Originally it was presented in more generality by Artin and Nagata as Theorem 2.1 in [2], yet a counterexample to their general case and a correction has been presented by Huneke [14] as Theorem 3.1, and the proof we present below was inspired by their results.

For more information on Cohen-Macaulay modules, regular sequences, and depth, see Section 17 of [10]. (Note that since our ring is local, by depth we mean depth of a module with respect to the corresponding maximal ideal.)

Proposition 21. *Let $(S, \mathfrak{n})$ be a regular local ring and let $\mathfrak{a}, \mathfrak{b} \subset S$ be ideals such that*

1. ${}^{S}/_{\mathfrak{a}}$ *is a Cohen-Macaulay ring with* $\dim({}^{S}/_{\mathfrak{a}}) = \dim(S) - r$,
2. $\mathfrak{b} = \langle b_1, \dots, b_{r+1} \rangle \subsetneq \mathfrak{a}$,
3. $\mathfrak{a}_{\mathfrak{q}} = \mathfrak{b}_{\mathfrak{q}}$ *for every prime ideal* $\mathfrak{q}$ *such that* $\dim({}^{S}/_{\mathfrak{q}}) > \dim(S) - (r+1)$.

Then ${}^{\mathfrak{a}}/_{\mathfrak{b}}$ is a Cohen-Macaulay module over S, with

$$\dim(Supp({}^{\mathfrak{a}}/_{\mathfrak{b}})) = \dim(S) - (r+1).$$

Proof. First, since $\mathfrak{a}_\mathfrak{q} = \mathfrak{b}_\mathfrak{q}$ for every prime ideal $\mathfrak{q}$ such that $\dim(S/\mathfrak{q}) > \dim(S) - (r+1)$, then for every such prime ideal $\mathfrak{q}$ we have that $(\mathfrak{a}/\mathfrak{b})_\mathfrak{q} = 0$. Therefore $\dim(\mathrm{Supp}(\mathfrak{a}/\mathfrak{b})) \leq \dim(S) - (r+1)$, and so we would like the prove that the depth of $\mathfrak{a}/\mathfrak{b}$ is at least $\dim(S) - (r+1)$. But, by looking at the exact sequence $0 \to \mathfrak{a}/\mathfrak{b} \to S/\mathfrak{b} \to S/\mathfrak{a} \to 0$, we can conclude that it is enough to show that the depth of $S/\mathfrak{b}$ is at least $\dim(S) - (r+1)$.

Now, if $\mathfrak{b}$ is a complete intersection (i.e. can be generated by r elements in S), then we would get that $S/\mathfrak{b}$ is a Cohen-Macaulay ring, and so $\mathrm{depth}(S/\mathfrak{b}) = \dim(S/\mathfrak{b}) = \dim(S) - r$. Otherwise, since $\mathfrak{b}$ is generated by $r+1$ elements, namely $b_1, \ldots, b_{r+1}$, we can assume without loss of generality that $b_1, \ldots, b_r$ is an $S-$regular sequence. This is true since S is regular and so the grade of S over $\mathfrak{b}$ is exactly $ht(\mathfrak{b}) = r$ (see Section 1.2 in [4] or the introduction to Chapter 2 of [18]). Denote $\mathfrak{c} = \langle b_1, \ldots, b_r \rangle$. Since $\dim(S/\mathfrak{a}) = \dim(S) - r$ and $\mathfrak{c} \subset \mathfrak{a}$, then $b_1, \ldots, b_r$ is a maximal $S-$regular sequence in $\mathfrak{a}$. So, from Propositions 1.6 and 1.7 in [14] we have that $S/(\mathfrak{c} \colon \mathfrak{a})$ is a Cohen-Macaulay ring of dimension $\dim(S) - r$ and b_{r+1} is a non zero-divisor of $S/(\mathfrak{c} \colon \mathfrak{a})$.

Thus, if we look at the exact sequence of $S-$modules $0 \to \mathfrak{c} \to S \to S/\mathfrak{c} \to 0$, since $\mathfrak{c}$ is generated by an $S-$regular of length r, then we get that $\mathrm{depth}(\mathfrak{c}) \geq \dim(S) - r + 1$ (where we view $\mathfrak{c}$ as an $S-$module). On the other hand, we have the isomorphisms

$$\mathfrak{b}/\mathfrak{c} \cong \langle b_{r+1} \rangle / \langle b_{r+1} \rangle \cap \mathfrak{c} \cong S/(\mathfrak{c} \colon \langle b_{r+1} \rangle) = S/(\mathfrak{c} \colon \mathfrak{b}).$$

Since $(\mathfrak{c} \colon \mathfrak{a}) \subset (\mathfrak{c} \colon \mathfrak{b})$, in addition to $b_{r+1}(\mathfrak{c} \colon \langle b_{r+1} \rangle) = b_{r+1}(\mathfrak{c} \colon \mathfrak{b}) \subset \mathfrak{c} \subset (\mathfrak{c} \colon \mathfrak{a})$ and that b_{r+1} is a non zero-divisor of $S/(\mathfrak{c} \colon \mathfrak{a})$, we can conclude that $(\mathfrak{c} \colon \mathfrak{a}) = (\mathfrak{c} \colon \mathfrak{b})$. Thus $\mathfrak{b}/\mathfrak{c} \cong S/(\mathfrak{c} \colon \mathfrak{a})$, which has depth $\dim(S) - r$. Therefore, if we look at the exact sequence $0 \to \mathfrak{c} \to \mathfrak{b} \to \mathfrak{b}/\mathfrak{c} \to 0$, we can conclude that $\mathrm{depth}(\mathfrak{b}) \geq \dim(S) - r$ (where we view $\mathfrak{b}$ as an $S-$module). Thus, by looking at the exact sequence $0 \to \mathfrak{b} \to S \to S/\mathfrak{b} \to 0$, we can conclude that $\dim(S) - (r+1) \leq \mathrm{depth}(S/\mathfrak{b})$ and the result follows.

□

Remark 22. *1.* In Proposition 21, since S is a regular local ring, if we look at the exact sequence of modules $0 \to \mathfrak{b} \to \mathfrak{a} \to \mathfrak{a}/\mathfrak{b} \to 0$ and recall that localization is an exact functor, then condition 3 is in fact equivalent to the condition that $\dim(\mathrm{Supp}(\mathfrak{a}/\mathfrak{b})) \leq \dim(S) - (r+1)$. This gives us an alternative proof of Proposition 3.3 in [20] (in the case where S is a regular local ring).

2. Since $\mathfrak{b}$ can be generated by $r+1$ and $\dim(S/\mathfrak{b}) = \dim(S) - r$, then if $\mathfrak{b}$ is not a complete intersection, it is an almost complete intersection. The connection between almost complete intersections and non-isolated singularities (in the real case, in addition to their connection to Pellikaan's work) have been studied by van Straten and Warmt in [25].

Lemma 23. *Let f_t be a relative morsification of f, and let $\mathfrak{a}$ be a generification of $Jac(f_t)$ over $\mathfrak{p}$. Then $M(\mathfrak{a})$ is a Cohen-Macauly module over R.*

Proof. Let $f \in I^p \setminus I^{p+1}$ and let $f_t = f + tg$ be a relative morsification of f (which exists by Theorem 14). If $M(\mathfrak{a}) = 0$ we are done, so we can assume that $M(\mathfrak{a}) \neq 0$. We first show that $\dim(\mathrm{Supp}(M(\mathfrak{a}))) \leq 1$.

First, since we can write $f_t = f + tg$ where $f, g \in \mathbb{C}\{\underline{x}\}$, we have that $Jac(f_t) + \langle t \rangle = Jac(f) \cdot R + \langle t \rangle$. Therefore, we have that

$$R/Jac(f_t) \otimes R/\langle t \rangle = R/Jac(f) \cdot R + \langle t \rangle \cong \mathbb{C}\{\underline{x}\}/Jac(f),$$

where $Jac(f) = \langle \partial_1(f), \ldots, \partial_n(f) \rangle \subset \mathbb{C}\{\underline{x}\}$, which gives us that

$$\dim(R/Jac(f_t)) \leq \dim(\mathbb{C}\{\underline{x}\}/Jac(f)) + 1 = 2,$$

as $V(Jac(f)) = \mathrm{Sing}(V(f))$. Yet, because f_t is a morsification, we have that $Jac(f_t) \subset \mathfrak{p}$ and so we can conclude that $\dim(R/Jac(f_t)) \geq \dim(R/\mathfrak{p}) = 2$. Therefore we have that $\dim(R/Jac(f_t)) = 2$. Now, from the exact sequence $0 \to M(\mathfrak{a}) \to R/Jac(f_t) \to R/\mathfrak{a} \to 0$ we can conclude that $\dim(\mathrm{Supp}(M(f_t))) \leq \dim(\mathrm{Supp}(R/Jac(f_t))) = 2$, and since we know that $Jac(f_t)_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$, we get that $\mathfrak{p} \notin \mathrm{Supp}(M(\mathfrak{a}))$.

Following Notation 19, consider the quotient sheaf $\mathcal{O}_U/\mathcal{J}_F$ and the sheaf $\mathcal{M}$. Note that $\mathcal{M}_0 = M(\mathfrak{a})$ and $(\mathcal{O}_U/\mathcal{J}_F)_0 = R/Jac(f_t)$. It is enough to show that $\dim(\mathrm{Supp}(\mathcal{M})) \leq 1$ as a complex germ, since we have that $\dim(\mathrm{Supp}(M(\mathfrak{a}))) = \dim(\mathrm{Supp}(\mathcal{M}_0)) \leq \dim(\mathrm{Supp}(\mathcal{M}))$.

Since $V(\mathfrak{p}) \subset V(Jac(f_t))$ we have that $V(x_1, \ldots, x_{n-1}) \subset \mathrm{Supp}(\mathcal{O}_U/\mathcal{J}_F)$. Therefore, because we have that $\mathrm{Supp}(\mathcal{M})$ is a closed subset of $\mathrm{Supp}(\mathcal{O}_U/\mathcal{J}_F)$ with $\dim(\mathrm{Supp}(\mathcal{O}_U/\mathcal{J}_F)) = 2$, we need to prove that $\mathrm{Supp}(\mathcal{O}_U/\mathcal{J}_F)$ contains no irreducible components of dimension 2 except for $V(x_1, \ldots, x_{n-1})$.

Yet, since f_t is a relative morsification of f_t, then for every small enough t_0 we have that $(\mathrm{Crit}(f_{t_0}) \cap U_n) \setminus (V(x_1, \ldots, x_{n-1}))$ is composed of a finite set of discrete points (which are exactly the Morse points of f_{t_0}). Thus for every small enough t_0 we have that $(\mathrm{Supp}(\mathcal{O}_U/\mathcal{J}_F|_{\pi^{-1}(t_0)}) \cap U_n) \setminus (V(x_1, \ldots, x_{n-1}))$ is zero dimensional and from Exercise 1.3.2 in [12] we can conclude that indeed $\mathrm{Supp}(\mathcal{O}_U/\mathcal{J}_F)$ contains no irreducible components of dimension 2 except for $V(x_1, \ldots, x_{n-1})$.

Now, since $\dim(\mathrm{Supp}(M(\mathfrak{a}))) \leq 1$, $R/\mathfrak{a}$ is a Cohen-Macaulay ring of dimension 2, and $Jac(f_t) = \langle \partial_1(f_t), \ldots, \partial_n(f_t) \rangle$, we can apply Proposition 21 and the first part of Remark 22 where $\mathfrak{b} = Jac(f_t)$, $\dim(R) = n + 1$, and $r = n - 1$, and get that $M(\mathfrak{a})$ is Cohen-Macaulay over R with $\dim(\mathrm{Supp}_R(M(\mathfrak{a}))) = 1$, as desired.

□

Remark 24. *1.* Looking at the second part of the proof of Proposition 23 from a geometric point of view, the irreducible decomposition of $V(Jac(f_t))$ is of the form $V(\mathfrak{p}) \cup V(\mathfrak{q}_1) \cup \cdots \cup V(\mathfrak{q}_s)$ where $V(\mathfrak{q}_i)$ is a curve for every i and s is exactly the number of A_1 points in this morsification. This statement is elaborated in the proof of Theorem 18.

2. In Lemma 23, the fact that f_t is a relative morsification plays a crucial role here. For example, $f_t(x, y, z) = x^{p+1} + y^p + tx^p$ is not a relative morsification of $f(x, y, z) = x^{p+1} + y^p$ (as we saw in Example 12) and $\mathrm{Supp}({}^{\mathcal{O}_U}/_{\mathcal{J}_F})$ has more than one irreducible component of dimension 2.

The following proposition, Proposition 25, plays a crucial role in the proof of Theorem 18, Theorem 54, and other related results. It is a restatement of Proposition 7.2 from [18] and of Theorem 6.4.7 in [7].

Proposition 25. *Let $U \subset \mathbb{C}^{n+1}$ be a small open neighborhood of the origin of the form $U = U_1 \times U_n$ where $U_1 \subset \mathbb{C}_t$ and $U_n \subset \mathbb{C}^n_{\underline{x}}$ are both small open neighborhood of their respect origins. Denote by $\pi \colon U \to U_1$ be the projection map. Let $\mathcal{N}$ be a coherent $\mathcal{O}_U$-module such that $\mathcal{N}_0$, the stalk of $\mathcal{N}$ at the origin, is Cohen-Macaulay over $\mathbb{C}\{\underline{x}, t\}$ of dimension 1 and of finite rank over $\mathbb{C}\{t\}$. Then for every small t_0 we have that*

$$\dim_{\mathbb{C}}((\mathcal{N}|_{\pi^{-1}(0)})_0) = \sum_{s \in \pi^{-1}(t_0)} \dim_{\mathbb{C}}((\mathcal{N}|_{\pi^{-1}(t_0)})_p).$$

Here the fibre of the module is defined via the restriction with respect to U_1, that is, $\mathcal{N}|_{\pi^{-1}(t_0)} := \mathcal{N}|_{\pi^{-1}(t_0)} \otimes {}^{\mathcal{O}_U}/_{\langle t - t_0 \rangle}$ for $t_0 \in U_1$.

Remark 26. The following result, Proposition 27, relies heavily on the Lê-Ramanujan theorem, which is a result from [24], and plays a crucial role in understanding how the hypersurface with an empty transversal discriminant behaves. This result is an important component in the proof of Theorem 18, and especially in the proof of Lemma 29. As the Lê-Ramanujan theorem is still open in dimension 3, we assume that $n - 1 \neq 3$. For a deeper discussion on this theorem and related results, see article titled "Equisingularity" in [8].

Proposition 27. *If ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac(f)}$ is Cohen-Macaulay over $\mathbb{C}\{\underline{x}\}$ then $\Delta^{\perp}(f) = \emptyset$.*

Proof. Following the convention of Notation 19, we consider f as an analytic function $f \colon U \to \mathbb{C}$, where $U \subset \mathbb{C}^n$ is a small bounded open set of the form $U = U_{n-1} \times U_1$ where $U_{n-1} \subset \mathbb{C}^{n-1}$ and $U_1 \subset V(I)$, where we view $V(I)$ as a copy of $\mathbb{C}^1$. Consider the quotient sheaf $\mathcal{N} = {}^{\mathcal{O}_U}/_{\mathcal{J}}$, where $\mathcal{J}$ is the sheaf generated by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$, and let $\pi \colon U \to U_1$ be the projection map.

Since $({}^{\mathcal{O}_U}/_{\mathcal{J}})_0 = {}^{\mathbb{C}\{\underline{x}\}}/_{Jac(f)}$ is a Cohen-Macaulay module, then by Proposition 25 we have that, up to shrinking U, the stalks of $(\pi_* \mathcal{N})|_{\tilde{U}_1}$ are free over $\mathbb{C}\{x_n\}$ and of the same rank. Note that for every $z_0 \in U_1$ we have that as module over $\mathbb{C}\{x_n\}$,

$$((\mathcal{N}|_{\pi^{-1}(z_0)})_{(0,\ldots,0,z_0)} \cong {}^{\mathbb{C}\{x_1, \ldots, x_{n-1}\}}/_{Jac(f_{z_0})},$$

where $f_{z_0}(x_1,\ldots,x_{n-1}) = f(x_1,\ldots,x_{n-1},z_0) \in \mathbb{C}\{x_1,\ldots,x_{n-1}\}$ and where $Jac(f_{x_0})$ is the ideal generated by the derivatives of f_{z_0} with respect to the variables $x_1,\ldots,x_{n-1}$. We can conclude that for every $z_1, z_2 \in U_1$,

$$\mu(f_{z_1}) = \dim_{\mathbb{C}}(\pi_*((\mathcal{N}|_{\pi^{-1}(z_1)})_{(0,\ldots,0,z_1)}) = \dim_{\mathbb{C}}((\mathcal{N}|_{\pi^{-1}(z_2)})_{(0,\ldots,0,z_2)}) = \mu(f_{z_2}).$$

Thus the Milnor number of f_{z_0} does not on the choice of $z_0 \in U_1$, and the result follows from the Lê-Ramanujan theorem (recall that $n-1 \neq 3$) and Remark 8. □

Remark 28. The converse of of Proposition 27 is not true. For example, take $f(x,y,z) = 2k(x^{3k}+y^{3k}) - 3kz^2x^{2k}y^{2k}$ where $k \geq 2$. Then $\Delta^{\perp}(f) = \emptyset$, but $S = {}^{\mathbb{C}\{x,y,z\}}/_{Jac(f)}$ is not a Cohen-Macaulay module, since

$$Jac(f) = \langle x^{3k-1} - z^2x^{2k-1}y^{2k}, y^{3k-1} - z^2x^{2k}y^{2k-1}, zx^{2k}y^{2k}\rangle$$

and by direct computation we get that $\dim(S) = 1$ but $\operatorname{depth}(S) = 0$, as both x, y and z are all zero divisors of S.

Lemma 29. *Let $f \in I^p \setminus I^{p+1}$ with $\operatorname{Sing}(V(f)) = V(I)$, and let $\mathfrak{a}$ be a generification of $Jac(f_t)$ over $\mathfrak{p}$.*

1. *For every relative morsification of f, $j_0(\mathfrak{a}) = 0$ if and only if $M(\mathfrak{a}) = 0$.*
2. *If there exists some relative morsification f_t of f such that $j_0(\mathfrak{a}) = 0$, then $\Delta^{\perp}(f) = \emptyset$.*

Proof. 1. Let f_t be a relative morsification of f. First, if $M(\mathfrak{a}) = 0$ then obviously $j_0(\mathfrak{a}) = 0$. Second, assume that $j_0(\mathfrak{a}) = 0$. Then $M(\mathfrak{a}) \otimes_R {}^R/_{\langle t\rangle} = 0$, and thus $\operatorname{Supp}(M(\mathfrak{a})) \cap V(\langle t\rangle) = \emptyset$. We can conclude that $M(\mathfrak{a}) = 0$, since otherwise, by Lemma 23, we would have that $\dim(\operatorname{Supp}(M(\mathfrak{a}))) = 1$, and thus the intersection $\operatorname{Supp}(M(\mathfrak{a})) \cap V(\langle t\rangle)$ would contain the maximal ideal $\mathfrak{m}$, as R is a local ring.

2. Assume that $j_0(f_t) = 0$ for some relative morsification f_t of f. Then we get that $M(\mathfrak{a}) = 0$ and thus $Jac(f_t) = \mathfrak{a}$. Yet, since $\mathfrak{a}$ is generification of $Jac(f_t)$ over $\mathfrak{p}$, we have that ${}^R/_{\mathfrak{a}}$ is Cohen-Macaulay of dimension 2, and so is ${}^R/_{Jac(f_t)}$. Thus, since t is a non zero-disivor of ${}^R/_{\mathfrak{a}}$ we have that ${}^R/_{Jac(f_t)} \otimes {}^R/_{\langle t\rangle} \cong {}^{\mathbb{C}\{\underline{x}\}}/_{Jac(f)}$ is a Cohen-Macaulay ring of dimension 1. Hence the result follows from Proposition 27.

□

Remark 30. From Lemma 29 we can conclude that there exists a generification $\mathfrak{a}$ of $Jac(f_t)$ over $\mathfrak{p}$ such that $j_0(\mathfrak{a}) = 0$ if and only if ${}^R/_{Jac(f_t)}$ is a Cohen-Macaulay ring.

Lemma 31. *Let f_t be a relative morsification of f and let $\mathfrak{a}$ be a generification of $Jac(f_t)$ over $\mathfrak{p}$ such that $j_0(\mathfrak{a}) \neq 0$. Then $j_0(\mathfrak{a})$ is finite if and only if for every m we have that $t^m \cdot \mathfrak{a} \not\subset Jac(f_t)$.*

Proof. Since $j_0(\mathfrak{a}) \neq 0$, then by Lemma 29, we have that $M(\mathfrak{a}) \neq 0$, which tells us that $\mathrm{Supp}_R(M(\mathfrak{a})) \neq \emptyset$. Thus by Lemma 23 we get that $M(\mathfrak{a})$ is a module of dimension 1 over R. Therefore, since $\mathrm{Supp}_R(M(\mathfrak{a}) \otimes_R {}^{R}/\langle t\rangle) = \mathrm{Supp}_R(M(\mathfrak{a})) \cap V(\langle t\rangle)$, we have that either the intersection is exactly the unique maximal ideal or $\mathrm{Supp}(M(\mathfrak{a})) \subset V(\langle t\rangle)$.

If their intersection is the maximal ideal, that is, $\mathrm{Supp}_R(M(\mathfrak{a}) \otimes_R {}^{R}/\langle t\rangle) = \{\mathfrak{m}\}$, then since $M(\mathfrak{a}) \otimes_R {}^{R}/\langle t\rangle$ is a finitely generated module over $\mathbb{C}\{\underline{x}\}$, from Nakayama's Lemma (see Corollary 4.8 in [10]) we can conclude that $j_0(\mathfrak{a})$ is finite.

Otherwise, $\mathrm{Supp}(M(\mathfrak{a})) \subset V(\langle t\rangle)$ is equivalent to the fact $\sqrt{Ann(M(\mathfrak{a}))} \supset \langle t\rangle$, which itself is equivalent to the fact that there exists some m such that $t^m \cdot \mathfrak{a} \subset Jac(f_t)$. In this case we have that $\mathrm{Supp}_R(M(\mathfrak{a})) = \mathrm{Supp}_R(M(\mathfrak{a}) \otimes {}^{R}/\langle t\rangle)$, which is of dimension 1, and $j_0(\mathfrak{a})$ can not be finite. $\square$

Remark 32. The assumption in Lemma 31 that $j_0(\mathfrak{a}) \neq 0$ is essential. Take for example $f(x, y, z) = x^p + y^p$ and consider the trivial deformation $f_t = f$. Since f is Morse outside $V(\langle x, y\rangle)$ and since $\Delta^{\perp}(f) = \emptyset$ we have that f_t is a relative morsification of f and that $\mathfrak{a} = Jac(f_t)$ is a generification of itself over $\mathfrak{p}$. Yet we have that $j_0(f_t) = 0$, which is finite, and that $t^m \cdot \mathfrak{a} \subset Jac(f_t)$ for every m.

Lemma 33. *Let $f \in I^p \setminus I^{p+1}$ with $\mathrm{Sing}(V(f)) = V(I)$, let f_t be a relative morsification of f, and let $\mathfrak{a}$ be a generification of $Jac(f_t)$ over $\mathfrak{p}$ such that $j_0(\mathfrak{a})$ is finite and non-zero. Then, following the notations of Notation 19, for every t_0 small enough we have that $j(\mathfrak{a}, p, t_0)$ is finite for every $p \in \pi^{-1}(t_0)$ and*

$$j_0(\mathfrak{a}) = \sum_{p \in \pi^{-1}(t_0)} j(\mathfrak{a}, p, t_0).$$

Proof. Since $j_0(\mathfrak{a})$ is non-zero and finite then from Lemma 29 and Lemma 23, $M(\mathfrak{a})$ is a Cohen-Macaulay module of dimension 1 over R, and the result follows from Proposition 25. $\square$

We are now ready to prove Theorem 18.

Proof of Theorem 18. First, if $j_0(\mathfrak{a})$ is infinite then the statement is true vacuously, and if $j_0(\mathfrak{a}) = 0$ then from Lemma 29 we have that $\deg(\Delta^{\perp}(f)) = 0$, and from Remark 13 we can conclude that $\#A_1(f_{t_0}) = 0$ for every small t_0. Hence we can assume that $j_0(\mathfrak{a})$ is finite and non zero.

Following Notation 19, let $t_0 \in U_1$ and let $p \in \pi^{-1}(t_0)$. If $p \notin \mathrm{Crit}(f_{t_0})$ then $j(\mathfrak{a}, p, t_0) = 0$ since ${}^{\mathcal{O}_{\mathbb{C}^{n+1}}}/\mathcal{I}_F$ and ${}^{\mathcal{O}_{\mathbb{C}^{n+1}}}/\mathcal{J}$ are supported inside the critical locus of F with respect to $x_1, \ldots, x_n$. If p is a non singular critical point of f_{t_0} then p is a Morse point, since f_{t_0} is Morse outside $V(I)$. As $p \notin V(\mathfrak{p}) \cap U$ and $\sqrt{\mathfrak{a}} = \mathfrak{p}$, then we have that $(\mathcal{I}_F)_{(p,t)} \cong \mathbb{C}\{\underline{x} - \underline{p}\}$. By applying Morse's Lemma (Theorem

2.46 in [12]) we have that $j(\mathfrak{a}, p, t_0) = 1$. So, combined with Lemma 33, we can conclude that

$$j_0(\mathfrak{a}) = \sum_{p \in \mathrm{Crit}(f_{t_0})} j(\mathfrak{a}, p, t_0) = \#A_1(f_t) + \sum_{p \in \mathrm{Sing}(f_{t_0})} j(\mathfrak{a}, p, t_0).$$

From Lemma 29, we get that for every $p \in \Delta^{\perp}(f_{t_0})$ we have $j(\mathfrak{a}, p, t_0) \geq 1$. Since f_t is a relative morsification of f, then $\Delta^{\perp}(f_{t_0})$ is reduced, and from Remark 9 we can conclude that the size of $\Delta^{\perp}(f_{t_0})$, as a set, is exactly $\deg(\Delta^{\perp}(f))$. Thus, we get that

$$j_0(\mathfrak{a}) = \#A_1(f_t) + \sum_{p \in \mathrm{Sing}(f_{t_0})} j(\mathfrak{a}, p, t_0) \geq \#A_1(f_t) + \sum_{p \in \Delta^{\perp}(f_{t_0})} j(\mathfrak{a}, p, t_0),$$

and since $\sum_{p \in \Delta^{\perp}(f_{t_0})} j(\mathfrak{a}, p, t_0) \geq \deg(\Delta^{\perp}(f))$, the result follows.

□

Remark 34. From the proof of Theorem 18 and from part 1 of Remark 24, we can conclude that for every relative morsification f_t of f, the number of $\#A_1(f_{t_0})$ does not depend on t_0 and is finite (as long as t_0 is small enough).

We now discussion finding an explicit generifications of $Jac(f_t)$ over $\mathfrak{p}$. Since we are interested in $M(\mathfrak{a}) = {}^{\mathfrak{a}}/_{Jac(f_t)}$, one would like to find a "nice" ideal $Jac(f_t) \subset \mathfrak{a}$ which is a generification of $Jac(f_t)$ over $\mathfrak{p}$. The following definitions and two propositions would give us some motivation for such an ideal.

Definition 35. *The* ***generic Jacobian ideal of*** f ***(over*** I***)*** *is defined as* $Jac_{gen}(f) = \mathbb{C}\{\underline{x}\} \cap Jac(f)_{\langle x_1, \dots, x_{n-1} \rangle}$ *and* ***the Jacobi number of*** f ***(over*** I***)*** *is defined to be* $j(f) = \dim_{\mathbb{C}}({}^{Jac_{gen}(f)}/_{Jac(f)})$.

Remark 36. Note that

$$Jac_{gen}(f) = \{g \in \mathbb{C}\{\underline{x}\} \colon \exists s \notin \langle x_1, \dots, x_{n-1} \rangle \textit{ such that } sg \in Jac(f)\}.$$

In addition, we can compare the definition of $Jac_{gen}(f)$ with the definition of I_f, as presented in Section 5 of [19].

Proposition 37. ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac_{gen}(f)}$ *is a Cohen-Macaulay ring of dimension* 1*. In addition,* $Jac_{gen}(f)$ *is the smallest such ideal which contains* $Jac(f)$*.*

Proof. First, x_n is a non zero-divisor of ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac_{gen}(f)}$ since if $x_n h \in Jac_{gen}(f)$, then there exists some $s \notin \langle x_1, \dots, x_{n-1} \rangle$ such that $s x_n h \in Jac(f)$, but since $s x_n \notin \langle x_1, \dots, x_{n-1} \rangle$, we get that $h \in Jac_{gen}(f)$. Thus the depth of ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac_{gen}(f)}$ is at least 1. Yet we have that $Jac(f) \subset Jac_{gen}(f)$, and thus $V(Jac_{gen}(f)) \subset V(Jac(f))$. So we have that $\dim({}^{\mathbb{C}\{\underline{x}\}}/_{Jac_{gen}(f)}) \leq \dim({}^{\mathbb{C}\{\underline{x}\}}/_{Jac(f)}) = 1$, and we can conclude that ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac_{gen}(f)}$ is a Cohen-Macaulay ring of dimension 1.

Now, let $Jac(f) \subset \mathfrak{a}$ be an ideal such that ${}^{\mathbb{C}\{\underline{x}\}}/_{\mathfrak{a}}$ is a Cohen Macaulay ring of dimension 1, and let $h \in Jac_{gen}(f)$. Then there exists some $s \notin \langle x_1, \ldots, x_{n-1} \rangle$ such that $sh \in Jac(f)$. But since $Jac(f) \subset \mathfrak{a}$ we get that $sh \in \mathfrak{a}$. Yet, since ${}^{\mathbb{C}\{\underline{x}\}}/_{\mathfrak{a}}$ is a Cohen Macaulay ring of dimension 1 we have that s must be a non zero-divisor of ${}^{\mathbb{C}\{\underline{x}\}}/_{\mathfrak{a}}$, and thus $h \in \mathfrak{a}$. □

Proposition 38. *$j(f)$ is finite, and equals to zero if and only if ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac(f)}$ is Cohen-Macaulay over $\mathbb{C}\{\underline{x}\}$.*

Proof. Since ${}^{Jac_{gen}(f)}/_{Jac(f)}$ is finitely generated over $\mathbb{C}\{\underline{x}\}$, it is enough to prove that its support is only $\mathfrak{m}$. First, since $V(Jac(f)) = V(Jac_{gen}(f)) = V(I)$ then for every $\mathfrak{q} \notin V(I)$ we have that $(Jac_{gen}(f))_{\mathfrak{q}} = (Jac(f))_{\mathfrak{q}} = 0$. Now, since as a subset of $\mathrm{Spec}(\mathbb{C}\{\underline{x}\})$ we have that $V(I) = \{I, \mathfrak{m}\}$, and since $(Jac(f))_I = (Jac_{gen}(f))_I$, we indeed get that the support is exactly the maximal ideal. The second part follows from Proposition 37. □

Therefore, a natural idea would be to look at $j_0(Jac_{gen}(f_t))$ where we set $Jac_{gen}(f_t) = R \cap Jac(f_t)_{\mathfrak{p}}$ since (as with $Jac_{gen}(f)$) we have that $(Jac_{gen}(f_t))_{\mathfrak{p}} = Jac(f_t)_{\mathfrak{p}}$, $\dim({}^{R}/_{Jac_{gen}(f_t)}) \leq 2$, and t is a non zero-divisor of ${}^{R}/_{Jac_{gen}(f_t)}$. But, as we see in the next example, the quotient ring ${}^{R}/_{Jac_{gen}(f_t)}$ need not be a Cohen-Macaulay ring for any f_t.

Example 39. Let $f_t(x, y, z) = tx^p + x^{p+1}y + zy^{p-2}x^3 + y^p$. We show that the depth of ${}^{R}/_{Jac_{gen}(f_t)}$ is 1 by showing that z is a zero divisor of ${}^{R}/_{Jac_{gen}(f_t) + \langle t \rangle}$, since t is a non zero-divisor of ${}^{R}/_{Jac_{gen}(f_t)}$. First, we have that

$$Jac(f_t) = \langle ptx^{p-1} + (p+1)x^p y + 3zx^2 y^{p-2}, x^{p+1} + (p-2)zy^{p-2}x^3 + py^{p-2}, y^{p-2}x^3 \rangle.$$

Thus we get that $ptx^p + (p+1)x^{p+1}y \in Jac(f_t)$. But since $ptx^p + (p+1)x^{p+1}y = x^p(pt + (p+1)xy)$ and $(pt + (p+1)xy) \notin Jac_{gen}(f_t)$, we get that $x^p \in Jac_{gen}(f_t)$. Thus we can conclude that $ptx^{p-1} + zx^2y^{p-2} \in Jac_{gen}(f_t)$ and thus $zx^2y^{p-2} \in Jac_{gen}(f_t) + \langle t \rangle$. Yet we have that $x^2y^{p-2} \notin Jac_{gen}(f_t) + \langle t \rangle$.

Yet, if we assume that $j(f) \neq 0$ then this is true, as we see in the following lemmata.

Lemma 40. *Let $\mathfrak{a} \subset R$ be any ideal and let $\mathfrak{a} = \cap_{i=1}^k \mathfrak{a}_i$ be the prime decomposition of $\mathfrak{a}$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_l \subset \mathfrak{p}$ and $\mathfrak{a}_{l+1}, \ldots, \mathfrak{a}_k \not\subset \mathfrak{p}$. Denote $\sqrt{\mathfrak{a}_i} = \mathfrak{q}_i$ for every i and $\mathfrak{b} = \cap_{i=1}^l \mathfrak{a}_i$. Then $\cap_{i=l+1}^k \mathfrak{p}_i \not\subset \mathfrak{p} \cup \langle t \rangle$ if and only if for every $r \in \mathfrak{b}$ there exists some $s \notin \mathfrak{p} \cup \langle t \rangle$ such that $sr \in \mathfrak{a}$.*

Proof. First, assume that $\cap_{i=l+1}^k \mathfrak{p}_i \not\subset \mathfrak{p} \cup \langle t \rangle$. Then by the Prime Avoidance Lemma (see Section 3.2 in [10]) there exists some $s \in (\cap_{i=l+1}^k \mathfrak{p}_i) \setminus (\mathfrak{p} \cup \langle t \rangle)$. Therefore there exists some N such that $s^N \in \cap_{i=l+1}^k \mathfrak{a}_i$ and $s^N \notin \mathfrak{p} \cup \langle t \rangle$. So for every $r \in \mathfrak{b}$ we have that $r \in \cap_{i=1}^l \mathfrak{a}_i$ and so $s^N r \in \cap_{i=1}^k \mathfrak{a}_i = \mathfrak{a}$.

Second, assume that for every $r \in \mathfrak{b}$ there exists some $s \notin \mathfrak{p} \cup \langle t \rangle$ such that $sr \in \mathfrak{a}$. Let $c \in \mathfrak{b} \setminus \mathfrak{a}$, and let $s \notin \mathfrak{p} \cup \langle t \rangle$ such that $sc \in \mathfrak{a}$. Then for every i we have that $sc \in \mathfrak{a}$. But since for every i, $\mathfrak{a}_i$ is primary with $r \notin \mathfrak{a} \subset \mathfrak{a}_i$, we have that $s \in \sqrt{\mathfrak{a}_i} = \mathfrak{p}_i$. Thus we can conclude that $s \in \cap_{i=l+1}^{k} \mathfrak{p}_i$ but $s \notin \mathfrak{p} \cup \langle t \rangle$.

□

Lemma 41. *Assume that $j(f) \neq 0$. Then for every $r \in Jac_{gen}(f_t)$ there exists some $s \notin \mathfrak{p} \cup \langle t \rangle$ such that $sr \in Jac(f_t)$.*

Proof. Let $Jac(f_t) = \cap_{i=1}^{k} \mathfrak{a}_i$ be the prime decomposition of $Jac(f_t)$. Now, since $Jac(f_t)_\mathfrak{p} = Jac_{gen}(f_t)_\mathfrak{p}$ we can assume that $\sqrt{\mathfrak{a}_1} = \mathfrak{p}$. Since $j(f) \neq 0$, then $Jac_{gen}(f_t) \neq Jac(f_t)$, and so from the proof of Lemma 23 the only 2−dimensional irreducible component of $Jac(f_t)$ is $V(\mathfrak{p})$. Therefore we have that $\mathfrak{a}_i \not\subset \mathfrak{p}$ for every $i \neq 1$. Now, since $V(\mathfrak{a}_i)$ is a curve for every $i > 1$ and $\cup_{i>1} V(\mathfrak{a}_i) = V(\cap_{i>1} \mathfrak{a}_i)$, we get that $\cap_{i>1} \mathfrak{a}_i \not\subset \langle t \rangle$ in addition to $\cap_{i>1} \mathfrak{a}_i \not\subset \mathfrak{p}$, and the result follows from Lemma 40. □

Lemma 42. *Denote $\mathfrak{q} = \langle x_1, \ldots, x_{n-1}, t \rangle$ and for every $\mathfrak{a} \subset R$ define $s_\mathfrak{q}(\mathfrak{a}) = R \cap \mathfrak{a}_\mathfrak{q}$. Then $s_\mathfrak{q}(Jac(f_t) + \langle t \rangle) = s_\mathfrak{q}(Jac(f_t)) + \langle t \rangle$.*

Proof. First, note that $s_\mathfrak{q}(Jac(f_t) + \langle t \rangle) = s_\mathfrak{q}(Jac(f_t)) + s_\mathfrak{q}(\langle t \rangle)$ (as $Jac(f_t), \langle t \rangle \subset \mathfrak{q}$ but they are not contained in each other) and $\langle t \rangle \subset s_\mathfrak{q}(\langle t \rangle)$. Now, let $r \in s_\mathfrak{q}(\langle t \rangle)$. Then there exists some $s \notin \mathfrak{q}$ such that $rs \in \langle t \rangle$. But since $\langle t \rangle \subset \mathfrak{q}$ is a prime ideal, then $r \in \langle t \rangle$. □

Proposition 43. *Assume that $j(f) \neq 0$. Then $Jac_{gen}(f_t) + \langle t \rangle = Jac_{gen}(f) \cdot R + \langle t \rangle$.*

Proof. First, let $h \in Jac_{gen}(f_t) + \langle t \rangle$. Then there exists some $b \in R$ such that $h - bt \in Jac_{gen}(f_t)$, and we can assume that $h \notin \langle t \rangle$. By Lemma 41, there exists some $s \notin \mathfrak{p} \cup \langle t \rangle$ such that $s(h - bt) \in Jac(f_t)$. Now, we can write $s = s_0 + t\tilde{s}$ where $0 \neq s_0 \in \mathbb{C}\{\underline{x}\}$, and we have that $s_0 h + t(\tilde{s}h - bh) = \sum_{i=1}^{n} a_i \partial_i(f_t)$ for some $a_1, \ldots, a_n \in R$. Now recall that $f_t = f + tg$ for $f, g \in \mathbb{C}\{\underline{x}\}$, so if we write $a_i = c_i + td_i$ for some $c_1, \ldots, c_n \in \mathbb{C}\{\underline{x}\}$ and some $d_1, \ldots, d_n \in R$, then we have that $s_0 h + t(\tilde{s}h - bh + \sum_{i=1}^{n} a_i \partial_i(g) + \sum_{i=1}^{n} d_i \partial_i(f)) = \sum_{i=1}^{n} c_i \partial_i(f) \in Jac(f) \subset \mathbb{C}\{\underline{x}\}$. If $c_i = 0$ for every i then $s_0 h = 0$, and thus $h = 0$. Otherwise, by comparing coefficients we have that $s_0 h \in Jac(f)$, which tells us that $h \in Jac_{gen}(f)$.

Second, let $h \in Jac_{gen}(f) \cdot R + \langle t \rangle$. Then there exists some $b \in R$, some $r_1, \ldots, r_n \in R$, and some $c_1, \ldots, c_n \in Jac_{gen}(f)$ such that $b - ht = \sum_i r_i c_i$, and we can assume that $h \in \mathbb{C}\{\underline{x}\}$. Thus there exists some $s \notin I \subset \mathbb{C}\{\underline{x}\}$ such that $sr_i \in Jac(f)$ for every i. Therefore there exists some $d_1, \ldots, d_n \in R$ such that $s(h - bt) = \sum_{i=1}^{n} d_i \partial_i(f)$, and so $sh + t(\sum_{i=1}^{n} c_i \partial_i(g) - sb) = \sum_{i=1}^{n} d_i \partial_i(f_t) \in Jac(f_t)$. Hence $sh \in Jac(f_t) + \langle t \rangle$, which gives us that $h \in s_\mathfrak{q}(Jac(f_t) + \langle t \rangle)$. But, from Lemma 42 we have that $s_\mathfrak{q}(Jac(f_t) + \langle t \rangle) = s_\mathfrak{q}(Jac(f_t)) + \langle t \rangle \subset Jac_{gen}(f_t) + \langle t \rangle$, and the result follows. □

Proposition 44. *If $j(f) \neq 0$ then ${}^{R}/_{Jac_{gen}(f_t)}$ is a Cohen-Macaulay module over R of dimension 2 and $j_0(Jac_{gen}(f_t)) = j(f)$.*

Proof. First, recall that from Proposition 37 we have that $\mathbb{C}\{\underline{x}\}/Jac_{gen}(f)$ is a Cohen-Macaulay module over $\mathbb{C}\{\underline{x}\}$ of dimension 1. Therefore, it is enough to prove that t is a non zero-divisor of $R/Jac_{gen}(f_t)$ and that $R/Jac_{gen}(f_t) \otimes_R R/\langle t\rangle \cong \mathbb{C}\{\underline{x}\}/Jac_{gen}(f)$. First, assume that $th \in Jac_{gen}(f_t)$. Then there exists some $s \notin \mathfrak{p}$ such that $sth \in Jac(f_t)$, but since $t \notin \mathfrak{p}$ as well, then we have that $h \in Jac_{gen}(f_t)$. Therefore t is a non zero-divisor of $R/Jac_{gen}(f_t)$. Second, from Proposition 43 we have that $Jac_{gen}(f_t) + \langle t\rangle = Jac_{gen}(f)\cdot R + \langle t\rangle$, and so $R/Jac_{gen}(f_t) \otimes_R R/\langle t\rangle \cong R/Jac_{gen}(f_t) + \langle t\rangle \cong R/Jac_{gen}(f)\cdot R + \langle t\rangle \cong \mathbb{C}\{\underline{x}\}/Jac_{gen}(f)$.

Now in order to prove that $j_0(Jac_{gen}(f_t)) = j(f)$ it is enough to prove that $Jac_{gen}(f)/Jac(f) \cong Jac_{gen}(f_t)/Jac(f_t) \otimes_R R/\langle t\rangle$. Since $Jac_{gen}(f_t) + \langle t\rangle = Jac_{gen}(f)\cdot R + \langle t\rangle$, then every element of $h \in Jac(f) \subset \mathbb{C}\{\underline{x}\}$ can be written as $h = h_0 + tb$ for $h_0 \in Jac_{gen}(f_t)$ and $b \in R$. Therefore we can define a map $\Phi\colon Jac_{gen}(f) \to Jac_{gen}(f_t)/Jac(f_t) \otimes_R R/\langle t\rangle$ by $\Phi(h) = (h_0 + Jac(f_t)) \otimes (1 + \langle t\rangle)$. Then by direct computation we can conclude that Φ is a well defined map. surjective, and $\ker(\Phi) = Jac(f)$. Thus, the result follows from the first isomorphism theorem. □

Theorem 45. *For every relative morsification f_t of f and for every small enough t_0 we have that*

$$j(f) \geq \#A_1(f_{t_0}) + \deg(\Delta^{\perp}(f)).$$

Proof. If $\Delta^{\perp}(f) = \emptyset$ then from Remark 13 we have that $\#A_1(f_t) = 0$ and the result is true vacuously. Otherwise, From Proposition 44 we have that $Jac_{gen}(f_t)$ is a generification of $Jac(f_t)$ and the result follows from Theorem 18. □

Example 46. One can calculate the following examples:

f	$j(f)$	$\deg(\Delta^{\perp}(f))$
$x^p + y^p z$	$p-1$	$p-1$
$x^p + y^p z^q + y^{p+1}$	$(2q-1)(p-1)$	$q(p-1)$
$x^p z^{q_1} + y^p z^{q_2} + y^{p+1} + x^{p+1}$	$(p-2)(2(q_1+q_2)-1) + 2(q_1+q_2)$	$(q_1+q_2)(p-1)$
$\prod_{i=1}^{n}(x^{p_i} + y^{p_i} z)$	$\sum_{i=1}^{n}(p_i - 1)$	$\sum_{i=1}^{n}(p_i - 1)$
$\prod_{i=1}^{n}(x^{p_i} + y^{p_i} z^{q_i} + y^{p_i+1})$	$\sum_{i=1}^{n}(2q_i - 1)(p_i - 1)$	$\sum_{i=1}^{n} q_i(p_i - 1)$
$\sum_{i=1}^{n-2}(x_i^p) + x_{n-1}^p x_n^q + x_{n-1}^{p+1}$	$(2q-1)(p-1)^{n-2}$	$q(p-1)^{n-2}$
$\sum_{i=1}^{n-2}(x_i^p x_n^{q_i} + x_i^{p+1})$	$2^{n-2}(\sum_{i=1}^{n-2} q_i)(p-1)^{n-2} + (2^{n-2}-1)p$	$(\sum_{i=1}^{n} q_i)(p-1)^{n-2}$

Comparing the table above with Example 12 we can see that Theorem 45 is true in all of these cases (and we even have an equality). Yet, the following example shows that the inequality in Theorem 45 need not be an equality.

Example 47. Let $f(x,y,z) = 2k(x^{3k} + y^{3k}) - 3kz^2x^{2k}y^{2k}$ for $k \geq 2$. As we saw in Remark 28, $\Delta^{\perp}(f) = \emptyset$ but $Jac(f) \neq Jac_{gen}(f)$, since $\mathbb{C}\{\underline{x}\}/Jac(f)$ is not a Cohen-Macaulay ring, in contrast to $\mathbb{C}\{\underline{x}\}/Jac_{gen}(f)$, as we have seen in Proposition 37. Thus we get that $j(f) > 0$ but $\deg(\Delta^{\perp}(f)) = 0 = \#A_1(f_{t_0})$ for every small t_0.

5 Milnor Number of Yomdin-Type Singularities

In this section we discuss how we can apply the results from Section 4 to generalize some of the results in [21], which in addition gives us a bound for $\deg(\Delta^{\perp}(f))$ by an algebraic invariant. We start by recalling a few results from Section 3 of [21], which we use throughout this section.

Proposition 48. *Let $f \in I^p \setminus I^{p+1}$ with* $\mathrm{Sing}(V(f)) = V(I)$*. Denote $J_{n-1}(f) = \langle \partial_1(f), \ldots, \partial_{n-1}(f) \rangle \subset \mathbb{C}\{\underline{x}\}$. Then*

1. *$R/J_{n-1}(f)$ is a Cohen-Macaulay ring of dimension* 1.
2. *For large enough $k \in \mathbb{N}$ we have that $f + x_n^k$ has an isolated singularity at the origin and its Milnor number is*

$$\mu(f + x_n^k) = j(f) + (k-1)(p-1)^{n-1} + \dim_{\mathbb{C}}(Jac(f)/Jac_{gen}(f) \cap Jac(f + x_n^k)).$$

3. *For large enough $k \in \mathbb{N}$ we have that $Jac_{gen}(f) \cap Jac(f + x_n^k) = J_{n-1}(f) + \langle \partial_n(f) \rangle \cdot Jac_{gen}(f)$.*

Remark 49.
1. Since $f \in I^p \setminus I^{p+1}$, the generic multiplicity of $Jac_{gen}(f)$ is $(p-1)$, and thus, since set theoretically, $V(Jac_{gen}(f)) = V(x_1, \ldots, x_{n-1})$, we get that $\dim_{\mathbb{C}}(\mathbb{C}\{\underline{x}\}/Jac_{gen}(f) + \langle x_n \rangle) = \mathrm{mult}(Jac_{gen}(f), \langle x_n \rangle) = (p-1)^{n-1}$.
2. Recall that the Milnor number of some $g \in \mathbb{C}\{\underline{x}\}$ is defined to be $\mu(g) = \dim_{\mathbb{C}}(\mathbb{C}\{\underline{x}\}/Jac(f))$. The milnor number plays a crucial role in the theory of isolated singularities, as discussed in length in Section 2 of Chapter 1 of‘[12].

We are now interested in computing and bounding the complex dimension of $Jac(f)/Jac_{gen}(f) \cap Jac(f + x_n^k)$ (for large enough k), which we denote by $\delta(f)$. Note that $\delta(f)$ is finite since $\mu(f + x_n^k)$ is finite for large enough k and $\delta(f) \leq \mu(f + x_n^k)$, from Proposition 48. In addition, note that $\delta(f)$ does not depend on k, since by Proposition 48 we have that $Jac(f)/Jac_{gen}(f) \cap Jac(f + x_n^k) \cong Jac(f)/J_{n-1}(f) + \langle \partial_n(f) \rangle \cdot Jac_{gen}(f)$ (for large enough k).

Proposition 50. $\mathbb{C}\{\underline{x}\}/(Jac(f + x_n^k) \colon \langle \partial_n(f) \rangle) \cong Jac(f)/Jac_{gen}(f) \cap Jac(f + x_n^k)$

Proof. Note that $Jac(f)/Jac_{gen}(f) \cap Jac(f + x_n^k)$ is generated by $\partial_n(f)$, since $\partial_i(f) \in Jac(f + x_n^k)$ for every $i < n$. Thus we have a surjective map

$$\varphi \colon \mathbb{C}\{\underline{x}\} \to Jac(f)/Jac_{gen}(f) \cap Jac(f + x_n^k)$$

defined by $\varphi(r) = r \cdot \partial_n(f)$ with $\ker(\varphi) = \mathrm{Ann}({}^{Jac(f)}/_{Jac_{gen}(f) \cap Jac(f+x_n^k)})$. Since $\partial_i(f) \in Jac(f + x_n^k)$ for every $i < n$, we get that $\ker(\varphi) = (Jac(f + x_n^k) \colon \langle \partial_n(f) \rangle)$. Hence, from the first isomorphism theorem we get that

$$^{Jac(f)}/_{Jac_{gen}(f) \cap Jac(f+x_n^k)} \cong {}^{\mathbb{C}\{\underline{x}\}}/_{(Jac(f+x_n^k) \colon \langle \partial_n(f) \rangle)},$$

as desired. □

Lemma 51. *Let f_t be a relative morsification of f. Then for a large enough k we have that ${}^{R}/_{(Jac(f_t+x_n^k) \colon \langle \partial_n(f_t) \rangle)}$ is a Cohen-Macaulay module over R of dimension 1 and*

$${}^{R}/_{(Jac(f_t+x_n^k) \colon \langle \partial_n(f_t) \rangle)} \otimes {}^{R}/_{\langle t \rangle} \cong {}^{\mathbb{C}\{\underline{x}\}}/_{(Jac(f+x_n^k) \colon \langle \partial_n(f) \rangle)},$$

where, as in Notation 19, we denote $R = \mathbb{C}\{\underline{x}, t\}$.

Proof. First, from Proposition 48 we have that ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac(f+x_n^k)}$ is zero dimensional for large enough k. Thus ${}^{R}/_{Jac(f_t+x_n^k)}$ is one dimensional in this case, and since we know that $Jac(f_t + x_n^k) \subset (Jac(f_t + x_n^k) \colon \langle \partial_n(f_t) \rangle)$, we can conclude that that $\dim({}^{R}/_{(Jac(f_t+x_n^k) \colon \langle \partial_n(f_t) \rangle)}) \leq 1$. Therefore it is enough to show that t is a non zero-divisor of ${}^{R}/_{(Jac(f_t+x_n^k) \colon \langle \partial_n(f_t) \rangle)}$. Assume that $tr \in (Jac(f_t + x_n^k) \colon \langle \partial_n(f_t) \rangle)$, and then $tr\partial_n(f_t) \in Jac(f_t + x_n^k)$. But ${}^{\mathbb{C}\{\underline{x}\}}/_{Jac(f+x_n^k)}$ is zero dimensional and $Jac(f + x_n^k)$ is generated by n elements in R, therefore ${}^{R}/_{Jac(f_t+x_n^k)}$ is a Cohen-Macaulay module and in particular, t is a non zero-divisor of ${}^{R}/_{Jac(f_t+x_n^k)}$. Hence since $tr\partial_n(f_t) \in Jac(f_t + x_n^k)$ we get that $r\partial_n(f_t) \in Jac(f_t + x_n^k)$, which gives us that $r \in (Jac(f_t + x_n^k) \colon \langle \partial_n(f_t) \rangle)$. Thus t is a non zero-divisor and we can conclude that ${}^{R}/_{(Jac(f_t+x_n^k) \colon \langle \partial_n(f_t) \rangle)}$ is a Cohen-Macaulay module of dimension 1.

Second, in order to finish the proof it is enough to show that that

$$(Jac(f_t + x_n^k) \colon \langle \partial_n(f_t) \rangle) + \langle t \rangle = (Jac(f + x_n^k) \colon \langle \partial_n(f) \rangle) \cdot R + \langle t \rangle.$$

Let $r \in (Jac(f_t + x_n^k) \colon \langle \partial_n(f_t) \rangle) + \langle t \rangle$. Then there exists some $b \in R$ and $r_0 \in \mathbb{C}\{\underline{x}\}$ such that $r = r_0 - bt \in (Jac(f_t + x_n^k) \colon \langle \partial_n(f_t) \rangle)$. Then $\partial_n(f_t) \cdot (r_0 - bt) \in Jac(f_t + x_n^k)$, and so there exists some $c_1, \ldots, c_n \in R$ such that $\partial_n(f_t) \cdot (r_0 - bt) = \sum_{i=1}^n c_i \partial_i(f_t) + c_n k x_n^{k-1}$. Therefore, if we write $c_i = a_i + \tilde{c}_i t$ where $a_1, \ldots, a_n \in \mathbb{C}\{\underline{x}\}$ and $\tilde{c}_1, \ldots, \tilde{c}_n \in R$, then there exists some $\tilde{b} \in R$ such that $\partial_n(f) \cdot r_0 - \tilde{b}t = \sum_{i=1}^n a_i \partial_i(f) + c_n k x_n^{k-1}$. By comparing coefficients we have that $\partial_n(f) r_0 \in Jac(f + x_n^k)$. So $r_0 \in (Jac(f + x_n^k) \colon \langle \partial_n(f) \rangle)$ and we get that $r \in (Jac(f + x_n^k) \colon \langle \partial_n(f) \rangle) \cdot R + \langle t \rangle$. The other inclusion follows from stability of colon ideal under such a base change (see similarity with Exercise 15.41 in [10]). □

Lemma 52. $\delta(f) = 0$ *if and only if* $\partial_n(f) \in J_{n-1}(f)$.

Proof. From Proposition 50, if $\delta(f) = 0$ then $(Jac(f + x_n^k) \colon \langle \partial_n(f) \rangle) = \mathbb{C}\{\underline{x}\}$, which is equivalent to $\partial_n(f) \in Jac(f + x_n^k)$. Therefore there exists some

$c_1, \ldots, c_n \in \mathbb{C}\{\underline{x}\}$ such that $\partial_n(f) = \sum_{i=1}^{n} c_i\partial_i(f) + c_n k x_n^{k-1}$. Therefore $c_n x_n^k \in Jac(f)$, and since $\sqrt{Jac(f)} = I$ we have that $c_n \in I$. Yet, we have that $(1-c_n)\partial_n(f) = \sum_{i=1}^{n-1} c_i\partial_i(f) + c_n k x_n^{k-1}$, and since $1-c_n$ is invertible we can conclude that $\partial_n(f) \in J_{n-1}(f) + \langle x_n^{k-1}\rangle$. Hence $Jac_{gen}(f+x_n^k) = Jac(f) + \langle x_n^{k-1}\rangle = J_{n-1}(f) + \langle x_n^{k-1}\rangle$, and in particular $\partial_n(f) \in Jac(f + x_n^k)$. By applying Proposition 48 we can conclude that $\partial_n(f) \in J_{n-1}(f) + \langle \partial_n(f)\rangle \cdot Jac_{gen}(f)$. Therefore there exists some $g \in Jac_{gen}(f)$ such that $\partial_n(f) + g \cdot \partial_n(f) \in J_{n-1}(f)$, but since $1 + g$ is invertible we can conclude that $\partial_n(f) \in J_{n-1}(f)$ as desired. On the other hand, if $\partial_n(f) \in J_{n-1}(f)$ then from Proposition 48 we get that $Jac_{gen}(f) \cap Jac(f + x_n^k) = J_{n-1}(f) = Jac(f)$, and so $\delta(f) = 0$. □

Remark 53. From Lemma 52 we can conclude that if $\delta(f) = 0$ then $\Delta^\perp(f) = \emptyset$. This is true since $\partial_n(f) \in J_{n-1}(f)$ is true if and only if up to a coordinate change, f is a function of $x_1, \ldots, x_{n-1}$ (for more details see Proposition 1.11 in [21] and Section 9 of [18]), and we can combine this result with Remark 8.

Theorem 54. $\delta(f) \geq \deg(\Delta^\perp(f))$.

Proof. Let f_t be a relative morsification of f, which exists from Theorem 14. Then by Proposition 51 we have that $R/(Jac(f_t + x_n^k) \colon \langle \partial_n(f_t)\rangle)$ is a Cohen-Macaulay module of dimension 1 for a large enough k, and from Remark 53 we have that if $\Delta^\perp(f) \neq \emptyset$ then $\delta(f) > 0$. Observe that in addition, if f is either smooth or a Morse function, then $\delta(f) = 0$. Therefore, if we look at the corresponding sheaf, we get that the support of the corresponding fiber is finite, is contained in $V(I)$, and contains the corresponding transversal discriminant. Therefore the result follows by applying the same technique based upon Proposition 25 which we used in the proof of Theorem 18. □

Corollary 55. *Assume that $n \neq 4$. Then for any relative morsification f_t of f we have that $\mu(f + x_n^k) \geq \#A_1(f_{t_0}) + (k-1)(p-1)^{n-1} + 2\deg(\Delta^\perp(f))$ for every large enough k and for every small enough t_0.*

Proof. Follows from Theorem 45 and Theorem 54. □

As with Theorem 45, the inequality in Theorem 54 need not be an equality, as we see in the following example.

Example 56. As in Example 47, we have that for $f(x, y, z) = 2k(x^{3k} + y^{3k}) - 3kz^2x^{2k}y^{2k}$ with $k \geq 2$, $\Delta^\perp(f) = \emptyset$ but $\partial_n(f) \notin Jac(f + x_n^k)$, and so from Proposition 50 we get $\delta(f) > 0$.

We end this section with a proposition and a corollary which examine the relation between a relative morsification f_t of f and the deformation $f_t + x_n^k$ of $f + x_n^k$ (for a large enough k).

Proposition 57. *Let f_t be a relative morsification of f. Then for large enough $k \in \mathbb{N}$ and for every small enough t_0 we have that $\mu(f + x_n^k) = \#A_1(f_{t_0}) + \mu(f_{t_0} + x_n^k)$.*

Proof. By Proposition 48, $f+x_n^k$ is an isolated singularity for a large enough k, and so $f_{t_0}+x_n^k$ is an isolated singualrity for every small enough t_0 as well. Now, for every A_1 point z_0 of f_{t_0} we have that $\text{Hess}(f_{t_0})(z_0) \in GL_n(\mathbb{C})$, and we have that $\text{Hess}(f_{t_0}+x_n^k)(z_0) = \text{Hess}(f_{t_0})(z_0)+\text{Hess}(x_n^k)(z_0)$. Yet, $GL_n(\mathbb{C}) \subset \mathbb{C}^{n\times n}$ is an open set (with respect to the Euclidean topology), and f_{t_0} has only a finite number of A_1 critical points (as mentioned in Remark 34). Therefore there exists some K such that for every $k \geq K$ we have that $\text{Hess}(f+x_n^k)(z) \in GL_n(\mathbb{C})$ for every A_1 point z of f_{t_0}. Hence we can conclude that every A_1 point of f_{t_0} is an A_1 point of $f_{t_0}+x_n^k$, and vice versa. Thus the result follows semi-continuity of the Milnor number (See Theorem 2.6 in [12]). □

Corollary 58. *Let f_t be a relative morsification and assume that $n \neq 4$. Then $\mu(f_{t_0}+x_n^k) \geq (k-1)(p-1)^{n-1} + 2\deg(\Delta^{\perp}(f))$ for every large enough k and for every small enough t_0.*

Proof. Follows directly from Corollary 55 and Proposition 57. □

6 Closing Remarks and Questions

In the case where $\text{Sing}(V(f))$ is a reduced scheme, that is, if $p=2$, we can write $f = \sum_{i,j<n} a_{i,j}x_ix_j$ where $a_{i,j} \in \mathbb{C}\{\underline{x}\}$ with $a_{i,j} = a_{j,i}$ for every i,j. Then, as explained in Example 4.2 in [16], we have that

$$\Delta^{\perp}(f) = \{\det(\{a_{i,j}|_{V(I)}\}_{i,j}) = 0\}.$$

Therefore, if the transversal discriminant of $V(f)$ is reduced, we have that $\Delta^{\perp}(f)$ is exactly the set of D_∞ type points, and the degree of $\Delta^{\perp}(f)$ (as a Cartier divisor) is exactly the number of such D_∞ points. In addition, if $\Delta^{\perp}(f) = \emptyset$ then f has only A_∞ points on its singular locus. Therefore, studying the transversal discriminant generalizes the study of D_∞ points in [23, 6, 22].

Thus, in the reduced case we have that $Jac_{gen}(f) = I$. So, for f with $\text{Sing}(V(f)) = V(I)$ such that its generic transversal type is an ordinary multiple point, we can conclude that:

1. $j(f) = 0$ if and only if $\delta(f) = 0$ if and only if f is a A_∞ singularity
2. $j(f) = 1$ if and only if $\delta(f) = 1$ if and only if f is a D_∞ singularity.

This reproves Lemma 1.10 and Remark 2.16 in [22].

In addition, if we look back at the definition of a relative morsification as in Definition 10, then the remark above, in addition to Theorem 14, reproves the existence of a relative morsification in the case $p=2$, as presented in [23, 22]. Therefore, if we reconsider the proof of Theorem 45 (based upon Theorem 18) and the proof of Theorem 54, we can conclude that we in fact have equality in the reduced case ($p=2$), that is, $j(f) = \#A_1(f_t) + \deg(\Delta^{\perp}(f))$ and

$\delta(f) = \deg(\Delta^{\perp}(f))$. This reproves Proposition 2.19 in [22] and Theorem 4.2 in [21], respectively.

Yet, as we have seen in Example 47 and in Example 56, in general we do not have equalities in Theorem 45 and in Theorem 54. One can note that in this case we only have an inequality since the transversal discriminant only measures the topological equaisingularity of the transversal sections of $V(f)$ (as seen in Remark 8), but not the analytical equisingularity. Therefore, it would be interesting to find an alternative definition for the transversal discriminant than the one presented in [15, 16], which would be empty if and only if the transversal sections of $V(f)$ along $V(I)$ are analytically equisingular.

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