Math 1320 Lab6 Name: ______________

Instructor: Bridget Fan Due: 03/11/16 in class

Show all work including necessary graphs to get full credits!

1. Use Alternating Series Estimation Theorem or Taylor's Inequality to answer the following questions:

   (a) Approximate function $f(x) = \ln(1 + 2x)$ by a Taylor polynomial of degree 3 at $x = 1$. How accurate is the estimation when $0.4 \le x \le 1.6$?

$f(x) = \ln(1+2x)$ $\quad f(1) = \ln(3)$

$f'(x) = \frac{2}{1+2x}$ $\quad f'(1) = \frac{2}{3}$

$f''(x) = \frac{-4}{(1+2x)^2}$ $\quad f''(1) = -\frac{4}{9}$

$f'''(x) = \frac{16}{(1+2x)^3}$ $\quad f'''(1) = \frac{16}{27}$

$f^{(4)}(x) = \frac{-16 \cdot 6}{(1+2x)^4}$

$$\boxed{f(x) \approx \ln(3) + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3}$$

Alternating Series implies

$|\text{Error}| \le a_{n+1} = \frac{|f^{(4)}(1)|}{4!}(x-1)^4$

$= \frac{4}{81}(x-1)^4$

Since $0.4 \le x \le 1.6$, $0 \le |x-1| \le 0.6$

$\Rightarrow |\text{Error}| \le \frac{4}{81} 0.6^4 = \boxed{0.0064}$

   (b) Approximate function $f(x) = \arctan x$ by a Taylor polynomial of degree 5 at $x = 0$. Estimate the range of x such that the accurate of the estimation is around 0.05.

$f(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5}$

Alternating Series implies

$|\text{Error}| \le a_3 = \frac{|x|^7}{7} \le 0.05 \Rightarrow |x|^7 \le 0.35 \Rightarrow |x| \le \sqrt[7]{0.35} \approx 0.8607$

2. **Maclaurin Series** Consider the differential equation $\frac{dy}{dt} = ky^2$.

(a) Write the first few terms of the Maclaurin expansion for $y(t)$.

$y(t) = y(0) + y'(0)t + \frac{y''(0)}{2!} t^2 + \cdots$

(b) Suppose that $y^{(n)}(t) = n!k^n y^{n+1}(t)$. Show that $y^{(n+1)}(t) = (n+1)!k^{n+1} y^{n+2}(t)$.

$\frac{d}{dt} y^{(n)}(t) = y^{(n+1)}(t) = n!\, k^n\, (n+1) y^n(t) \cdot \frac{dy}{dt}$

$= n!\, k^n (n+1) y^n(t) \cdot ky^2$

$= (n+1)!\, k^{(n+1)}\, y^{n+1}(t)$

(c) The argument you have just used is called *induction*. If for the base case $n = 1$ some relation holds, and assuming the relation holds for $n = m$ implies that the relation holds for $n = m + 1$, then it is true for all n. Write the Maclaurin series of $y(t)$.

$y'(0) = 1! \cdot k \cdot y^2(0) = ky^2(0)$

$y''(0) = 2 \cdot k^2\, y^3(0) = 2k^2 y^3(0)$

$\vdots$

$y(t) = y(0) + ky^2(0) \cdot t + k^2 y^3(0) \cdot t^2 + \cdots$

$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n$

$= \sum_{n=0}^{\infty} k^n \cdot y^{n+1}(0) \cdot t^n$

(d) Observe the pattern in the Maclaurin series. Given $y(0)$, what is the solution of the differential equation? Use what you know about geometric series to find the time $t > 0$ for which $y(t)$ ceases to exist.

$\sum_{n=0}^{\infty} k^n y^{n+1}(0)\, t^n = \frac{y(0)}{1 - ky(0)t}$

Geometric series converges when $|r| < 1 \Rightarrow |ky(0)t| < 1$

$\Rightarrow |t| < \frac{1}{|ky(0)|}$

3. Find the equation for the following object in $\mathbb{R}^3$

(a) Find the equation of a sphere if one of its diameters has endpoints $(0,3,2)$ and $(7,-1,-2)$.

Midpoint $\leftrightarrow$ center

$\hookrightarrow \left(\frac{0+7}{2}, \frac{3-1}{2}, \frac{2-2}{2}\right) = \left(\frac{7}{2}, 1, 0\right)$

$R^2 = \left(\frac{7}{2}-0\right)^2 + (1-3)^2 + (0-2)^2 = \frac{81}{4}$

$$\left(x-\frac{7}{2}\right)^2 + (y-1)^2 + z^2 = \frac{81}{4}$$

(b) Find the equation of all points such that is equal distance from points $A(-1,0,3)$ and $B(1,1,-1)$

Let point to be (x, y, z)

$(x+1)^2 + y^2 + (z-3)^2 = (x-1)^2 + (y-1)^2 + (z+1)^2$

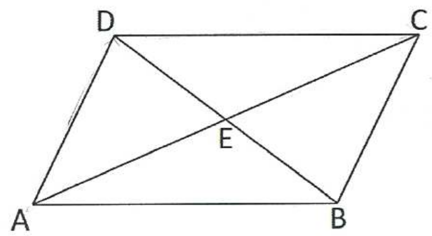
$(x+1)^2 - (x-1)^2 + y^2 - (y-1)^2 + (z-3)^2 - (z+1)^2 = 0$

$(2x)\cdot 2 + (2y-1) + (2z-2)(-4) = 0$

$4x + 2y - 1 - 8z + 8 = 0$

$4x + 2y - 8z = -7$

4. (a) Given the parallelogram:



Find $\overrightarrow{AB} + \overrightarrow{AD} - \overrightarrow{DC}$.

$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$

$\overrightarrow{AC} - \overrightarrow{DC} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

(b) Find the unit vectors that has the same direction as $\vec{a} = 2i - j$ and find $|\vec{a}|$

$|\vec{a}| = \sqrt{2^2+(-1)^2} = \sqrt{5}$

$\vec{u} = \frac{1}{\sqrt{5}}\vec{a} = \frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j$

(c) Repeat part (b) for vector $\vec{a} =< 1, 2, 3 >$.

$|\vec{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$\vec{u} = \frac{1}{\sqrt{14}}\vec{a} = \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$