Math 1320 Lab4 Name: ______________

Instructor: Bridget Fan Due: 02/11/16 in class

Show all work including necessary graphs to get full credits!

1. Use the method of separation of variables to solve the following problems:

   (a)
   $$(y + xy)y' = 1$$
   Solve for y, where $y' = \dfrac{dy}{dx}$.

   $y(1+x)\frac{dy}{dx} = 1$

   $y\,dy = \frac{1}{1+x}dx$

   $\int y\,dy = \int \frac{1}{1+x}dx$

   $\frac{y^2}{2} = \ln|1+x| + C$

   $y = \sqrt{\ln(1+x)^2} + C$

   (b) Find an equation of the curve that passes through $(0, 1)$ and whose slope at (x, y) is $6x(y-1)^{2/3}$.

   $\frac{dy}{dx} = 6x(y-1)^{2/3}$

   $(y-1)^{-\frac{2}{3}}dy = 6x\,dx$

   $\int (y-1)^{-\frac{2}{3}}dy = \int 6x\,dx$

   $3(y-1)^{\frac{1}{3}} = 3x^2 + C$

   $(y-1)^{\frac{1}{3}} = x^2 + C$

   $y-1 = (x^2+C)^3$

   $y = (x^2+C)^3 + 1$

   $(0,1) \Rightarrow 1 = C^3 + 1 \Rightarrow C = 0$

   $\Rightarrow y = x^6 + 1$

2. A glass of hot water is cooling down with surrounding temperature of 72 degrees. The rate of change of the water temperature $T(t)$ is proportional to the difference between $T(t)$ and the surrounding temperature. Suppose at $t = 0$ the water temperature is 100 degrees and drops to 82 degrees after 10 minutes.

   (a) Set up the differential equation describing $T(t)$.

   $$T'(t) = k(T-72)$$

   (b) Solve the differential equation from part (a).(Hint: Your answer of $T(t)$ should have two unknowns: the constant of proportionality and the arbitray constant c from integration.)

   $$\frac{dT}{dt} = k(T-72)$$

   $$\int \frac{1}{T-72}\, dT = \int k\, dt$$

   $$\ln|T-72| = kt + C$$

   $$e^{(kt+c)} = T-72$$

   $$T = e^{(kt+c)} + 72$$

   (c) Use the two conditions given by the problem to solve the two unknowns from part (b).

   $$T(0) = e^{c} + 72 = 100 \Rightarrow e^{c} = 28 \Rightarrow c = \ln 28$$

   $$T(10) = e^{(10k+c)} + 72 = 82 \Rightarrow e^{10k} e^{c} = 10 \Rightarrow e^{10k} \cdot 28 = 10$$

   $$\Rightarrow e^{10k} = \frac{5}{14}$$

   $$10k = \ln\frac{5}{14}$$

   $$k = \frac{1}{10}\ln\frac{5}{14}$$

3. A population $P(t)$ has constant relative birth and death rates α and β, respectively, and a constant emigration rate m (α, β and m are positive constants). Assume $\alpha > \beta$. The rate of change of the population at time t is modeled by

$$\frac{dP}{dt} = kP - m, \quad \text{where} \quad k = \alpha - \beta.$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.

$\frac{1}{kP-m}\,dP = dt$

$\int \frac{1}{kP-m}\,dP = \int 1\,dt$

$\frac{1}{k}\ln|kP-m| = t + C$

$Ce^{kt} = kP - m$

$P = \frac{1}{k}(Ce^{kt} + m)$

(b) What condition on m relative to kP_0 will lead to exponential expansion of the population? What condition on m relative to kP_0 will result in constant population? Population decline?

$kP_0 > m$ population expansion

$kP_0 = m$ constant population

$kP_0 < m$ population decline

(c) In 1847, the population of Ireland was 8 million and the difference between the relative birth and death rates was 1.6%. As a result of the potato famine in the 1840s and 1850s, 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining?

$k = 1.6\% = 0.016$

$m = 210{,}000$

$P_0 = 8{,}000{,}000$

$kP_0 = 128{,}000 < m = 210{,}000$

Population Decline

4. Determine whether the sequence is convergent or divergent and explain why.

(a) $a_n = e^{\frac{1}{n}}$

Because function e^x is continuous

$\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty}\frac{1}{n}} = e^0 = 1$ $\boxed{\text{Convergent}}$

(b) $a_n = (-1)^n \dfrac{n}{n^2+1}$

Let $b_n = \frac{n}{n^2+1}$

$0 \leq \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} \leq \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} b_n = 0$

We know that $-b_n \leq a_n \leq b_n$, by squeeze thm, we know $\lim_{n\to\infty} a_n \to 0$

$\boxed{\text{Convergent}}$

(c) $a_n = \dfrac{\sin(n\pi)}{n}$

Since $\sin(n\pi) = 0$, $a_n = \frac{0}{n} = \underset{\text{constant}}{\underbrace{0}}$ $\boxed{\text{Convergent}}$

(d) $a_n = \ln(n) - \ln(n-1)$

$a_n = \ln\left(\frac{n}{n-1}\right)$

Since $\ln(x)$ is continuous,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n}{n-1}\right) = \ln\left(\lim_{n\to\infty}\frac{n}{n-1}\right) = \ln\left(\lim_{n\to\infty}\frac{1}{1-\frac{1}{n}}\right) = \ln(1) = 0$ $\boxed{\text{Convergent}}$

(e) $a_n = \sqrt{n} - \sqrt{n-1}$

$$a_n = \sqrt{n} - \sqrt{n-1} = \frac{(\sqrt{n}-\sqrt{n-1})\cdot(\sqrt{n}+\sqrt{n-1})}{\sqrt{n}+\sqrt{n-1}}$$

$$= \frac{n-(n-1)}{\sqrt{n}+\sqrt{n-1}} = \frac{1}{\sqrt{n}+\sqrt{n-1}}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}} = 0$ $\boxed{\text{Convergent}}$