Math 1320 Lab2 Name: ______________

Instructor: Bridget Fan Due: 01/28/16 in class

Show all work including necessary graphs to get full credits!

1. Calculate the length of the curve defined parametrically by $y = e^{-t}\cos(t)$, $x = e^{-t}\sin(t)$ from $t = 0$ to $t = 2\pi$.

$$\int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2e^{-2t}}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, e^{-t}\, dt$$

$$= -\sqrt{2}\, e^{-t} \Big|_0^{2\pi}$$

$$= -\sqrt{2}\left[e^{-(2\pi)} - 1\right]$$

$$\frac{dy}{dt} = -e^{-t}\cos(t) + e^{-t}(-\sin t)$$

$$\frac{dx}{dt} = -e^{-t}\sin(t) + e^{-t}(\cos t)$$

$$\left(\frac{dy}{dt}\right)^2 = e^{-2t}\left[-\cos t - \sin t\right]^2$$

$$\left(\frac{dx}{dt}\right)^2 = e^{-2t}\left[-\sin t + \cos t\right]^2$$

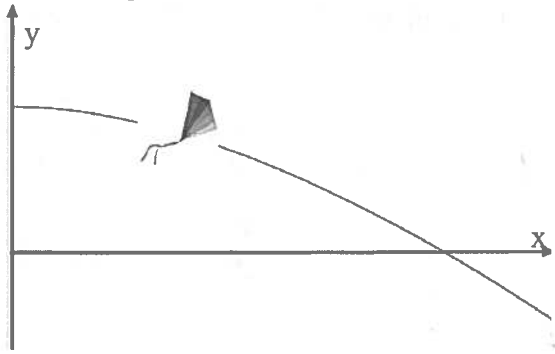
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{-2t}\left[\cos^2 t + \sin^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t \sin t\right.$$

$$= e^{-2t} \cdot 2$$

2. A steady wind blows a kite due west. The height above the ground from horizontal position $x \geq 0$ is given by

$$y = (x+1)^{\frac{1}{2}} - \frac{1}{3}(x+1)^{\frac{3}{2}}.$$

(a) Find the position where the kite falls onto the ground.



When the kite falls onto the ground, $y=0$

$$(x+1)^{\frac{1}{2}} - \frac{1}{3}(x+1)^{\frac{3}{2}} = 0$$

$$(x+1)^{\frac{1}{2}} = \frac{1}{3}(x+1)^{\frac{3}{2}}$$

$$(x+1)^{\frac{1}{2}-\frac{3}{2}} = \frac{1}{3}$$

$$\frac{1}{x+1} = \frac{1}{3}$$

$$x+1 = 3$$

$$x = 2$$

(b) Find the distance traveled by the kite before it fell on the ground. (Hint: $1 + (\frac{dy}{dx})^2$ is a perfect square.)

$$D = \int_0^2 \sqrt{1+(\tfrac{dy}{dx})^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} - \frac{1}{2}(x+1)^{\frac{1}{2}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{2\sqrt{1+x}} - \frac{\sqrt{1+x}}{2}\right)^2$$

$$= 1 + \frac{1}{4(1+x)} + \frac{1+x}{4} - \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{4(1+x)} + \frac{1+x}{4}$$

$$= \left(\frac{1}{2\sqrt{1+x}} + \frac{\sqrt{1+x}}{2}\right)^2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2\sqrt{1+x}} + \frac{\sqrt{1+x}}{2}$$

$$D = \int_0^2 \frac{1}{2\sqrt{1+x}} + \frac{\sqrt{1+x}}{2}\, dx$$

$$= (x+1)^{\frac{1}{2}} + \frac{1}{3}(x+1)^{\frac{3}{2}} \Big|_0^2$$

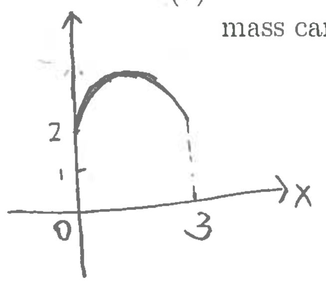
$$= \sqrt{3} + \frac{3\sqrt{3}}{3} - 1 - \frac{1}{3}$$

$$= 2\sqrt{3} - \frac{4}{3}$$

3. Consider a thin wire of length 3 cm, positioned so that it lies in the region $0 \le x \le 3$. The wire then has a density in mg cm^{-1} given by:

$$\lambda(x) = 2 + \frac{\pi}{3}\sin\left(\frac{\pi x}{3}\right)$$

(a) Draw the graph of the density function. Find the mass of the wire in mg. (Hint:the mass can be viewed as the area of the region bounded by the density function.)



$$M = \int_0^3 \lambda(x)\,dx$$
$$= \int_0^3 2 + \frac{\pi}{3}\sin(\frac{\pi}{3}x)\,dx$$
$$= 2x - \cos(\frac{\pi}{3}x)\Big|_0^3$$
$$= 6 - [(-1) - 1]$$
$$= 8$$

(b) Use the symmetry of $\lambda(x)$ to guess where is the center of the mass, $\bar{x}$.

Since the density is symmetric, the center of the mass will be at its center position, which is $x = \frac{3}{2}$

(c) Write down an integral expressing $\bar{x}$ and evaluate the integral to check if it matches with your guess from part (*b*).

I.B.P

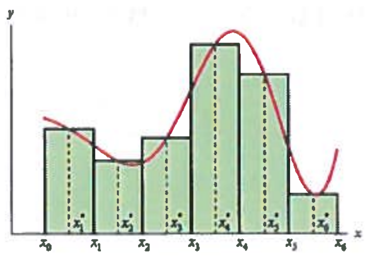
$$\bar{x} = \frac{1}{M}\int_0^3 x\cdot\lambda(x)dx = \frac{1}{8}\int_0^3 x\cdot[2+\frac{\pi}{3}\sin(\frac{\pi}{3}x)]\,dx$$
$$= \frac{1}{8}\int_0^3 2x + \frac{\pi}{3}x\sin(\frac{\pi}{3}x)dx$$
$$= \frac{1}{8}\int_0^3 2x\,dx + \frac{1}{8}\int_0^3 \frac{\pi}{3}x\sin(\frac{\pi}{3}x)dx$$

$$\frac{1}{8}\int_0^3 2x\,dx = \frac{1}{8}x^2\Big|_0^3 = \frac{9}{8}$$

$$\frac{1}{8}\int_0^3 \frac{\pi}{3}x\sin(\frac{\pi}{3}x)dx = \frac{1}{8}\left[x(-\cos\frac{\pi}{3}x)\Big|_0^3 - \int_0^3 -\cos(\frac{\pi}{3}x)dx\right]$$
$$= \frac{1}{8}\left\{3\cdot-(-1)-0 - \left[-\sin(\frac{\pi}{3}x)\cdot\frac{3}{\pi}\Big|_0^3\right]\right\}$$
$$= \frac{1}{8}(3+0) = \frac{3}{8}$$

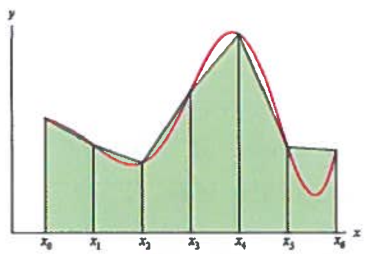
$$\Rightarrow \bar{x} = \frac{9}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$$ Match!

4. Recall that there are three methods to estimate definite integral: Midpoint Rule, Trapezoid Rule and Simpson's Rule. Assume that we want to evaluate the integral of function $f(x)$ on interval $[a, b]$. First, we will divide the interval into n subintervals with length $\Delta x = \dfrac{b-a}{n}$. Then, we will estimate the area under the curve on each subinterval A_i as following:



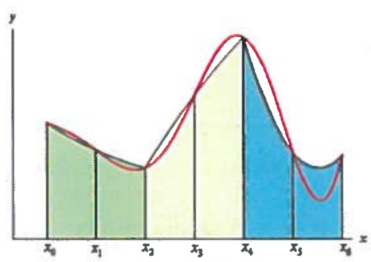
Midepoint Rule

$$\bar{x} = \frac{x_i - x_{i-1}}{2}$$
$$A_i = f(\bar{x}_i)\Delta x$$
$$\int_a^b f(x)dx \approx \Delta x[f(\bar{x}_1)+f(\bar{x}_2)+\ldots+f(\bar{x}_n)]$$



Trapezoid Rule

$$A_i = \frac{1}{2}[f(x_{i-1}) + f(x_i)]\Delta x$$
$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(A_1 + A_2 + \ldots + A_n)$$
$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \ldots + f(x_n)]$$



Simpson's Rule

$$A_i = \frac{1}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})]\Delta x$$
$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(A_1 + A_3 + \ldots + A_{n-1})$$
$$= \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\ldots+f(x_n)]$$

(a) Use $n = \overset{4}{\cancel{8}}$ and all three different methods to estimate the following integral: 4 decimal

$$\int_0^2 cos(1+x)dx$$

$\Delta x = \frac{1}{2}$

Make sure Calculator is in the mode of Rad. instead of degree.

Midpoint:

$$\int_0^2 \cos(1+x)dx \approx \frac{1}{2}\left[f(\tfrac{1}{4})+f(\tfrac{3}{4})+f(\tfrac{5}{4})+f(\tfrac{7}{4})\right]$$

$$=\frac{1}{2}\left[\cos(\tfrac{5}{4})+\cos(\tfrac{7}{4})+\cos(\tfrac{9}{4})+\cos(\tfrac{11}{4})\right]$$

$$=-0.7076998473$$

Trapezoid:

$$\int_0^2 \cos(1+x)dx \approx \frac{1}{2}\cdot\frac{1}{2}\left[f(0)+2f(\tfrac{1}{2})+2f(1)+2f(\tfrac{3}{2})+f(2)\right]$$

$$=\frac{1}{4}\left(\cos(1)+2\cos(\tfrac{3}{2})+2\cos(2)+2\cos(\tfrac{5}{2})+\cos(3)\right)$$

$$=-0.68569917289$$

Simpson's:

$$\int_0^2 \cos(1+x)dx \approx \frac{1}{3}\cdot\frac{1}{2}\left[f(0)+4f(\tfrac{1}{2})+2f(1)+4f(\tfrac{3}{2})+f(2)\right]$$

$$=\frac{1}{6}\left[\cos(1)+4\cos(\tfrac{3}{2})+2\cos(2)+4\cos(\tfrac{5}{2})+\cos(3)\right]$$

$$=-0.70060158655$$

(b) Calculate the integral and compare the result with part (a), which method gives the closest estimation?

$$\int_0^2 \cos(1+x)dx = \sin(1+x)\Big|_0^2 = \sin(3)-\sin(1)$$

$$=-0.70035097614$$

5. Suppose that you are gathering water from a well by pulling a rope attached to a bucket of water. As the force of gravity F_G acts on the bucket, this requires a work W given by

$$W = \int_0^d F_G(y)dy$$

where $d = 8$ m is the depth of the well and $y = 0$ corresponds to the bottom of the well. If the mass of the rope is negligible, the force of gravity on the bucket is given by $F_G(y) = m(y)g$, where $g = 9.81$ m s^{-2} and $m(y)$ is the mass of the bucket.

(a) Suppose that the mass of the bucket is constant, $m(y) = 5$ kg. Find W.

$$W = \int_0^8 F_G(y)dy = \int_0^8 m(y)\cdot g\,dy = \int_0^8 5\cdot 9.81\,dy$$
$$= 5\cdot 9.81\cdot 8\,\text{J}$$
$$= 392{,}4\,\text{J}$$

(b) Now suppose that the bucket has the same initial mass $m(0) = 5$ kg, but is continuously leaking so that its mass is reduced by a constant rate of 0.25 kg for every meter it is raised. Find $m(y)$ and W.

$$m(y) = 5 - 0{,}25y$$

$$W = \int_0^8 m(y)\cdot g\,dy = \int_0^8 \left(5 - \frac{y}{4}\right)9{,}81\,dy$$
$$= 9{,}81\cdot\left[5y\Big|_0^8 - \frac{1}{8}y^2\Big|_0^8\right] = 313{,}92\,\text{J}$$
$$\cancel{= 9{,}81\cdot(40-32)\,\text{J}}$$
$$\cancel{= 78{,}48\,\text{J}}$$