NETWORK MODELING OF FLUID TRANSPORT THROUGH SEA ICE WITH ENTRAINED EXOPOLYMERIC SUBSTANCES∗

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Abstract. Sea ice hosts a rich ecosystem of flora and fauna, from microscale to macroscale. Algae living in its porous brine microstructure, such as the diatom Melosira arctica, secrete gelatinous exopolymeric substances (EPS) which are thought to protect these communities from their cold and highly saline environment. Recent experimental work has shown significant changes in the structure and properties of young sea ice with entrained Melosira EPS, such as increased brine volume fraction, salt retention, pore tortuosity, and decreased fluid permeability. In particular, we find that the cross-sectional areas of the brine inclusions are described by a bimodal lognormal distribution, which generalizes the classic lognormal distribution of Perovich and Gow. We propose a model for the effective fluid permeability of young, EPS-laden sea ice, consisting of a random network of pipes with cross-sectional areas chosen from this bimodal distribution. We consider an equilibrium model posed on a square lattice, incorporating only the most basic features of the geometry and connectivity of the brine microstructure, and find good agreement between our model and the observed drop in fluid permeability. Our model formulation suggests future directions for experimental work, focused on measuring the inclusion size distribution and fluid permeability of sea ice with entrained EPS as functions of brine volume fraction. The drop in fluid permeability observed in experimental work and predicted by the model is significant, and should be taken into account, for example, in physical or ecological process models involving fluid or nutrient transport.

Key words. Sea ice, porous media, fluid permeability, exopolymeric substances, network model

AMS subject classifications. 00A69, 76S05, 90B15

1. Introduction. Sea ice that forms on the surface of high latitude oceans hosts a rich ecosystem, from autotrophs (algae) and other microorganisms that dwell within the ice, to small crustaceans (e.g., krill) that feed below it, and the macrofauna (e.g., penguins or polar bears in the Southern or Northern hemispheres, respectively) that forage from it. Indeed, the higher trophic levels of current polar ecosystems largely depend on sea ice as a platform on which to live, forage, and reproduce [35]. The areal extent and physical properties of sea ice also figure significantly in global climate models [15]. The growth, structure, and properties of ice formed from seawater containing the major ions (Na+, K+, Ca2+, Cl−, SO42−, CO32−) have been studied for decades [28, 35, 38, 39]. On the macroscale, much of what is known comes from remote sensing of sea ice via airborne platforms such as satellites, planes, and helicopters [3, 4, 10, 13, 22], as well as expeditions into Earth’s sea ice packs [23, 24, 26, 27, 31]. On the microscale, much has been learned from analysis of both natural sea ice [25, 28, 35] as well as artificially grown sea ice, which is generally devoid of life and its organic products [11, 16, 28, 35]. In broad strokes, we can say that sea ice is a porous medium exhibiting structure over many length scales, principally composed of a solid matrix of pure ice, with inclusions of brine (including microorganisms and their exudates), salt, air, and other impurities. Moreover, the microstructure of sea ice evolves with

∗Submitted to the editors DATE.

Funding: This work was funded by the National Science Foundation through Grants DMS-0602219, ARC-0934721, DMS-0940249, and DMS-1413454, the Office of Naval Research through Grant N00014-13-10291, and the Karl M. Bause Endowed Professorship (to J.W.D.).

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time, as fluid flowing through the porous microstructure tends to modify inclusion
cnectivity and channel structure.

The effect of algae or their cell-free organic matter on the growth, structure, and
properties of sea ice is not as well understood. Recent work of Krembs, Eicken, and
Deming [21] compared artificial sea ice grown from seawater containing exopolymeric
substances (EPS) with several controls, and observed that the artificial EPS-laden
ice had a more tortuous microstructure, larger brine volume fraction, greater salt
retention, and a net drop in fluid permeability. In sea ice, larger volume fractions and
larger salinities typically lead to larger fluid permeabilities. On the other hand, scaling
considerations for general porous media indicate that fluid permeability decreases
with the square of tortuosity [6]. As proposed in [21], one possible explanation for
the observed net drop in fluid permeability is that the observed increases in brine
volume fraction and salt retention were not enough to overcome the observed increase
in tortuosity.

Models of fluid flow through porous media, and fluid flow through sea ice, vary
widely in complexity. General studies (unrelated to sea ice) include Koplik [19], who
posed a network (pipe) model for linear Stokes flow in a regular periodic network;
Koplik, et al. [20], who applied the network model of [19] to porous media, in par-
ticular Massillon sandstone; Torquato and Pham [37], who derived “void bounds” on
the fluid permeability of hierarchical porous media, including coated parallel, circular
pipe geometries; and Hyman, et al. [17], who numerically integrated the Navier-
Stokes equations in the pore space of a stochastically-generated porous medium, and
studied the heterogeneities of flow. Studies specific to sea ice include [9, 11, 12, 40]
(the latter two to be described below).

Zhu, et al. [40] posed and analyzed a model for fluid flow through sea ice consisting
of a random network of pipes, followed by small but important modifications in [11].
While the model is a two-dimensional pipe network based on a square lattice, and
assumes a given equilibrium state, the effective fluid permeability of the pipe network
agreed well with the data of Freitag [7] for the fluid permeability of artificially-grown,
young sea ice.

In this work, we extend the two dimensional model of [11, 40], based on the
findings in [21], to consider the effects of micro-scale biochemistry in young sea ice, in
particular the presence of algal exudates, on the larger scale fluid transport properties
of the ice. In the remainder of this introductory section, we summarize the original
random network (pipe) model of [11, 40], and recall the void bounds of [12, 37] for
fluid transport in sea ice; in Section 2, we conclude the discussion of the original
model, including all the details such as parameter selection; in Section 3, we develop
our new model and state the main results; and we conclude in Section 4.

1.1. Random network model for fluid transport through sea ice. In this
section, we recall the random pipe network model of [40], including a synopsis of the
derivations of the linear system and the effective parameter \( k \).

Consider a block of sea ice, with a given brine volume fraction \( \phi \in [0, 1] \), and given
dimensions \( L \times D \times h \) m\(^3\), where \( D \) is the vertical depth, \( L \) the horizontal span, and
\( h \) the horizontal thickness. Note that \( h \ll D \) (and \( L \)) can be viewed as the dimension
of a cell in which a typical brine inclusion is contained. The random pipe network
model is formulated as follows.

Consider a square lattice

\[
L = \{(hi, hj) \in \mathbb{R}^2 : 0 \leq i \leq m, 0 \leq j \leq n\},
\]
where $h = L/m = D/n$ for some given $m, n \in \mathbb{Z}$. The parameter $h$ can be viewed as the size of a typical brine inclusion. We form a random pipe network from $L$ by connecting a given node $(i, j)$ (shorthand for the node located at the point $(hi, hj)$) to its four nearest neighbors $\{(i \pm 1, j), (i, j \pm 1)\}$ with fluid filled pipes. Choose the cross-sectional area of each pipe from a random distribution $A$ comparable to the brine inclusions found in young sea ice. Next, induce an upward flow through the network by a pressure drop $p_b - p_t$, where $p_b > p_t$ are the pressures at the bottom and top of the network, respectively; see Figure 1.

Denote $R_v^{i,j}$ and $R_h^{i,j}$ as the radii of the pipes connecting the nodes with indices $(i, j)$, $(i, j + 1)$ and $(i + 1, j)$, respectively. (Similarly, denote $A_v^{i,j}$ and $A_h^{i,j}$ as the cross-sectional area of these pipes.) For each pipe of radius $R$, the fluid flow within is assumed to be a classic Poiseuille flow, with flux $Q$ given by

$$Q = -\frac{\pi R^4}{8\mu} \nabla P,$$

where $\nabla P$ is the constant pressure gradient in the pipe, and $\mu$ is the fluid viscosity.

Let $p_{i,j}$ denote the pressure in the fluid at the $(i, j)$-th node of the network; since the pipe length $h$ is small, in [40] the pressure gradient $\nabla P$ is approximated by a standard finite difference:

$$\nabla P \approx \frac{p_{i+1,j} - p_{i,j}}{h} \text{ or } \nabla P \approx \frac{p_{i,j+1} - p_{i,j}}{h},$$

depending on whether the pipe is oriented horizontally or vertically. Assuming that the fluid is incompressible, the fluxes $Q$ converging on the $(i, j)$-th node must sum to zero; combining Equation (2) and the approximation (3) leads to a linear equation for each unknown $p_{i,j}$:

$$\begin{align*}
(R_v^{i,j})^4(p_{i,j+1} - p_{i,j}) - (R_v^{i,j-1})^4(p_{i,j} - p_{i,j-1}) + \\
(\cdots) + (R_h^{i,j})^4(p_{i+1,j} - p_{i,j}) - (R_h^{i,j-1})^4(p_{i,j} - p_{i-1,j}) &= 0.
\end{align*}$$

We impose Dirichlet boundary conditions on the top and bottom: $p_{i,n} = p_t$ and $p_{i,0} = p_b$, with $p_b, p_t$ defining the pressure drop $p_b - p_t > 0$, as discussed previously in this subsection, and periodic boundary conditions on the sides. To be more precise, the $(0, j)$-th and $(m, j)$-th nodes are connected, with the consequence that the linear

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equations for \( p_{0,j} \) and \( p_{m,j} \) (with \( j = 1, \ldots, n - 1 \)) vary from Equation (4). For example, the linear equation for \( p_{0,j} \) becomes instead:

\[
(R_{0,j}^v)^4 (p_{0,j+1} - p_{0,j}) - (R_{0,j-1}^v)^4 (p_{0,j} - p_{0,j-1}) + \cdots
\]

\[
+ (R_{0,j}^h)^4 (p_{1,j} - p_{0,j}) - (R_{m,j}^n)^4 (p_{0,j} - p_{m,j}) = 0.
\]

Let \( Q_{i,j} \) be the flux through the vertical pipe between the \((i,j)\)-th and \((i,j+1)\)-th nodes. In view of the upward flow through the random pipe network, Zhu et al. [40] define the total flux \( \overline{Q} \) as the sum of the fluxes through the topmost row of vertical pipes in the network:

\[
\overline{Q} = \sum_{i=0}^{m} Q_{i,n-1} = -\frac{\pi}{8\mu} \sum_{i=0}^{m} (R_{i,n-1}^v)^4 \frac{p_t - p_{i,n-1}}{h}.
\]

On the other hand, when the network is viewed instead as a model of a porous medium, the average velocity \( \overline{U} \) depends linearly on the pressure drop \((p_t - p_b)/D\),

\[
\overline{U} = -\frac{k}{\mu} \frac{p_t - p_b}{D},
\]

where \( k \) is the effective fluid permeability in the vertical direction and \( \mu \) is the fluid viscosity. The usual definition of flux means that \( U \) and \( \overline{Q} \) are linearly related by the cross-sectional area through which the flow occurs,

\[
\overline{U} = \frac{\overline{Q}}{Lh}.
\]

Substituting Equations (6) and (7) into (8), and solving for \( k \), leads to an equation for \( k \) depending on the model parameters and the solution of the linear system of equations (4):

\[
k = \frac{\pi D}{8Lh^2} \sum_{i=0}^{m} (R_{i,n-1}^v)^4 \frac{p_t - p_{i,n-1}}{p_t - p_b}.
\]

Indeed, Equation (9) is the effective permeability of the network, and is the key quantity of interest in the model of [40] for the effective fluid permeability of young sea ice.

### 1.2. Void bounds for fluid transport in sea ice.

In this section, we discuss rigorous bounds for the fluid permeability of sea ice, derived in [12, 37], as the basis for our new bounds in the case of a bimodal inclusion size distribution. In order to do so, we first consider the formulation and definition of the effective fluid permeability tensor \( k \) of a random porous medium. Then we will define the trapping constant \( \gamma \), since there is a rigorous bound on the effective permeability in terms of this related, homogenized parameter which also characterizes the random porous medium. We will also compute exactly the trapping constant for a parallel, circular cylinder geometry, which is relevant to our bounds. In our formulation we will emphasize the multiscale nature of the homogenization problem that one faces in this geophysical context of fluid transport through sea ice.

We are interested in sea ice as a porous medium for a given temperature \( T \) and salinity \( S \), which determine the brine volume fraction \( \phi \) [28, 35, 38, 39]. Within a given vertical depth range in a sea ice sheet, perhaps up to tens of centimeters or so, the
microstructural characteristics can be quite uniform over many meters horizontally. In such layers the porous brine microstructure is statistically homogeneous. However, we are also interested in how the bulk properties of the ice vary with depth, where variations in temperature and salinity, as well as possibly ice type and age, affect brine microstructural features and transport properties. We think of the submillimeter scale set by the porous microstructure of the ice as the “fast” scale, and the much larger scale variations in the temperature and salinity, and thus in the bulk properties, on the order of tens of centimeters to meters, as the “slow” scale.

Consider a random porous medium occupying a region $V \subset \mathbb{R}^d$ of volume $V = |V|$, partitioned into two sub-domains: the void phase $V_1 \subset V$, and solid phase $V_2 \subset V$. We will be interested in the infinite volume limit. Let $(\Omega, P)$ be a probability space characterizing the pore microstructure, where $\Omega$ is the set of realizations $\omega$ of the random medium and $P$ is a probability measure on $\Omega$. For any realization $\omega \in \Omega$, let $\chi(x, \omega)$ be the characteristic or indicator function of the void or brine phase $V_1$,

$$\chi(x, \omega) = \begin{cases} 1, & x \in V_1, \\ 0, & x \in V_2. \end{cases}$$

We first assume that $\chi(x, \omega)$ is a stationary random field such that $P$ has translation invariant statistics, corresponding to the infinite medium in all of $\mathbb{R}^d$. Then the medium is statistically homogeneous, and satisfies an ergodic hypothesis, where ensemble averaging over realizations $\omega \in \Omega$ is equivalent to an infinite volume limit $V \to \infty$ of an integral average over $V \subset \mathbb{R}^3$, denoted by $\langle \cdot \rangle$ [36]. This and related limits have been shown to exist and to be equal to the ensemble average in some situations, thus establishing the ergodic hypothesis [8, 14].

For many porous media [12, 36], there is typically a characteristic, microscopic length scale $\ell$ associated with the medium, such as the “typical” size of the brine inclusions in sea ice. For example, the scale over which the two point correlation function for the void phase varies is a good measure of this length. It is small compared to a typical macroscopic length scale $L$, where by $L$ here we mean sample size or thickness of a statistically homogeneous layer, on the order of $\sqrt{|V|}$ in three dimensions. Then the parameter $\epsilon = \ell/L$ is small, and one is interested in obtaining the effective fluid transport behavior in the limit as $\epsilon \to 0$. To obtain such information, the method of 
two-scale homogenization or 
two-scale convergence [1, 2, 14, 18, 32, 33, 34, 36] has been developed in various forms, based on the identification of two scales: a slow scale $x$ and a fast scale $y = x/\epsilon$.

The velocity and pressure fields in the pore space, $u^s(x)$ and $p^s(x)$, for $x \in V_1$, are assumed to depend on these two scales $x$ and $y$. The idea is to average, or homogenize over the fast microstructural scale $y$, leading to a simpler equation in the slow variable $x$ describing the overall behavior of the flow, namely, Darcy’s law. Variations of average microstructural properties on the slower $x$ scale can then be incorporated through dependence of the effective permeability tensor on $x$. For example, the bulk properties of sea ice in situ typically vary with depth, particularly when there is a large temperature gradient between the top and bottom of the sea ice layer.

The slow (creeping) flow of a viscous fluid with velocity field $u^s(x)$ and pressure field $p^s(x)$ in the void phase $V_1$ is governed by the Stokes equations,

$$\nabla p^s = \mu \Delta u^s, \quad x \in V_1, \quad \nabla \cdot u^s = 0, \quad x \in V_1, \quad u^s(x) = 0, \quad x \in \partial V_1.$$  

A force acting on the medium such as gravity can be incorporated into $p^s$. From left to right in (11), we have the steady state fluid momentum equation in the zero
Reynolds number limit, the incompressibility condition, and the no-slip boundary condition on the pore surface. The macroscopic equations can be derived through a two-scale expansion \[1, 2, 14, 18, 32, 33, 34, 36\]

\[
\mathbf{u}(\mathbf{x}) = \varepsilon^2 \mathbf{u}_0(\mathbf{x}, \mathbf{y}) + \varepsilon^3 \mathbf{u}_1(\mathbf{x}, \mathbf{y}) + \ldots
\]

\[
p(\mathbf{x}) = p_0(\mathbf{x}, \mathbf{y}) + \varepsilon p_1(\mathbf{x}, \mathbf{y}) + \ldots
\]

Note that the leading term in the velocity expansion is \(O(\varepsilon^2)\), while the leading term in the pressure expansion is \(O(1)\). This physical effect was handled in \[1, 2\] analytically by scaling the viscosity of the fluid by \(O(\varepsilon^2)\), which balances the friction of the fluid from the no-slip boundary condition on the solid boundaries of the pores.

Substitution of the two-scale expansion into the Stokes equations yields systems of equations involving both \(x\) and \(y\) derivatives.

The leading order system is analyzed by considering a second order tensor velocity field \(\mathbf{w}(\mathbf{y})\) and a vector pressure field \(\mathbf{\pi}(\mathbf{y})\) \[36\], both varying on the fast scale, which satisfy

\[
\Delta \mathbf{w} = \nabla \mathbf{\pi} - I, \quad \mathbf{y} \in \mathcal{V}_1, \quad \nabla \cdot \mathbf{w} = 0, \quad \mathbf{y} \in \mathcal{V}_1, \quad \mathbf{w} = 0, \quad \mathbf{y} \in \partial \mathcal{V}_1,
\]

where \(I\) is the identity matrix, and \(\mathbf{w}\) and \(\mathbf{\pi}\) are extended to all of \(\mathcal{V}\) by taking their values in the solid phase \(\mathcal{V}_2\) (ice) to be 0. In these equations the \((i,j)\) component of \(\mathbf{w}\) is the \(j\)th component of the velocity due to a unit pressure gradient in the \(i\)th direction, and \(\pi_j\) is the \(j\)th component of the associated scaled pressure. By averaging the leading order term of the velocity \(\mathbf{u}_0\) over \(y\), we obtain the macroscopic equations governing the flow through the porous medium,

\[
\mathbf{u}(\mathbf{x}) = -\frac{1}{\mu} \mathbf{k} \cdot \nabla p(\mathbf{x}), \quad \mathbf{x} \in \mathcal{V},
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in \mathcal{V},
\]

where \(p(\mathbf{x}) = p_0(\mathbf{x})\) and

\[
\mathbf{k} = \langle \mathbf{w} \rangle
\]

is the effective fluid permeability tensor. Equation (15) is known as Darcy’s law and equation (16) is the macroscopic incompressibility condition. These macroscopic equations were obtained in \[1, 2\] for periodic media through an appropriate limit as \(\epsilon \to 0\). We shall be interested in the permeability in the vertical direction \(k_{zz} = k\), in units of \(m^2\).

We now consider the steady-state trapping problem with perfectly absorbing traps \[12, 36, 37\], where diffusion of a passive tracer occurs in \(\mathcal{V}_1\) and trapping occurs on the surface of the solid phase \(\mathcal{V}_2\) (or the boundary of the pore space \(\partial \mathcal{V}_1\)). The tracer concentration field \(c(\mathbf{x})\) is governed by

\[
\mathcal{D} \Delta c(\mathbf{x}) = -G, \quad \mathbf{x} \in \mathcal{V}_1, \quad c = 0, \quad \mathbf{x} \in \partial \mathcal{V}_1,
\]

with diffusion coefficient \(\mathcal{D}\) and generation rate per unit trap-free volume \(G\). For an ergodic medium, two-scale homogenization \[1, 36\] shows that \(\gamma\) obeys the first
order rate equation $G = \gamma D C$, with average concentration $C = \langle c(x) \rangle$, and trapping constant defined via

\begin{equation}
\gamma^{-1} = \langle u \rangle = \lim_{V \to \infty} \left( \frac{1}{V} \int_V u(x) dx \right),
\end{equation}

with volume $V = |V|$, and scaled concentration field $u(x)$ solving

\begin{equation}
\Delta u(x) = -1, \quad x \in V_1, \quad u(x) = 0, \quad x \in \partial V_1.
\end{equation}

For dimensions $d = 2, 3$, $\gamma^{-1}$ has units of length squared. A key result that we use is a bound on the permeability in terms of the trapping constant, as in Theorem 23.5 of [36],

\begin{equation}
k \leq \gamma^{-1},
\end{equation}

in the sense that $\gamma^{-1} - k$ is always a positive semidefinite matrix, with equality in the case of transport through parallel channels of constant cross-section.

Consider now the case of parallel circular cylinders, with radii given by a random distribution $R_I$, and define the $n$-th moment as

\begin{equation}
\langle R^n_I \rangle = \frac{1}{\rho} \sum_{k=1}^{\infty} \rho_k R^n_{I_k},
\end{equation}

where $\rho_k$ is the number density of the $k$-th size $R_{I_k}$, and $\rho$ the characteristic density. Recall now that $\gamma$ is defined in terms of Equations (19) and (20). For a given cylinder with radius $r_i$, the solution of (20) is

\begin{equation}
\langle R^n_I \rangle = \frac{1}{4} r_i^2 - r^2.
\end{equation}

Substituting (23) into (19) yields

\begin{equation}
\gamma^{-1} = \lim_{V \to \infty} \left( \frac{1}{V} \sum_{i=1}^{\infty} \int_0^{2\pi} \int_0^{r_i} \frac{1}{4} (r_i^2 - r^2) r dr d\theta \right)
\end{equation}

\begin{equation}
= \lim_{V \to \infty} \left( \frac{1}{V} \sum_{i=1}^{\infty} \frac{\pi r_i^4}{8} \right),
\end{equation}

where the sum is over all the cylindrical inclusions in the void space $V_1$, indexed by $i \in \mathbb{N}$. In the infinite volume limit, (24) can be expressed in a form similar to (22) as a sum over $k$, involving the number density $\rho_k$:

\begin{equation}
\gamma^{-1} = \frac{\pi}{8} \sum_{k=1}^{\infty} \rho_k R^4_{I_k}.
\end{equation}

The volume fraction of the inclusions we are considering can be defined as $\phi = \frac{\pi}{8} \frac{\rho}{\Gamma(1+d/2)} \langle R^d_I \rangle$ [29], where $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ is the gamma function. Here we consider $d = 2$, in which case the volume fraction is given by

\begin{equation}
\phi = \rho \pi \langle R^2_I \rangle.
\end{equation}
Solving for $\pi$ in (26), and substituting into (25) yields:

\[ \gamma^{-1} = \frac{\phi}{8(R_f^2)} \frac{1}{\rho} \sum_{k=1}^{\infty} \rho_k R_{ik} = \frac{\phi(R_f^4)}{8(R_f^2)}. \]  

Equation (27) defines the effective trapping constant $\gamma$ for the special case of diffusion occurring in parallel, circular cylinders. Recalling the discussion surrounding (21), in this special case we have

\[ k = \frac{\phi(R_f^4)}{8(R_f^2)}. \]

Moreover, because $k \leq \gamma^{-1}$ in general geometries, the upper bound

\[ k \leq \frac{\phi(R_f^4)}{8(R_f^2)} \]

applies for general random porous media, from which it is straightforward to recover the void upper bound stated in [12, 37].

2. Previous results. In this section we recall the previous results of [11, 40].

2.1. Choice of random distribution. A random distribution which governs the choices of the radii $R_{i,j}^h, R_{i,j}^v$ of each pipe in the network is still required (alternatively, the cross-sectional areas $A_{i,j}^h, A_{i,j}^v$ of each pipe). In [25], the observed distribution for the cross-sectional area of brine inclusions in young sea ice (among other ice types) was best fit by a lognormal distribution, i.e.,

\[ A = e^X, \quad f_X(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \]

Recall that, for a lognormal random variable $A$, we have

\[ \text{E}[A] = e^{\mu + \frac{\sigma^2}{2}} \quad \text{and} \quad \text{Var}[A] = (e^{\sigma^2} - 1)(e^{\mu + \frac{\sigma^2}{2}})^2. \]

In [12] it was found that the function

\[ a(\phi) = \pi(7 \times 10^{-5} + 1.6 \times 10^{-4}\phi)^2 \text{ m}^2 \]

approximated the dependence of the mean cross-sectional areas on $\phi$ observed by [25]. Indeed, in [40], the cross-sectional areas of each pipe are lognormally distributed, with expectation given by Equation (31),

\[ \text{E}[A] = a(\phi). \]

A short calculation, after substituting (30) and (31) into (32), yields

\[ \mu + \frac{\sigma^2}{2} = \ln a(\phi). \]

The parameter model considered in [40] for the lognormal random variable $A$ is then as follows: let $\sigma$ be a free parameter, and let $\mu = \ln a(\phi) - \frac{\sigma^2}{2}$.  

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2.2. Upper bound on the fluid permeability. As discussed in Subsection 1.1
for random porous media, the upper bound on the effective permeability tensor $k$ is
given by the inverse of the trapping constant $\gamma^{-1}$, in the case of parallel cylinders of
random radii.

In the random pipe network model, we are interested in the effective vertical
permeability $k_{zz} = \hat{z}^\top k \hat{z}$ (where $\hat{z} = [0, 0, 1]^T$ is the usual unit vector in $\mathbb{R}^3$), which
we will denote as $k := k_{zz}$. As a discrete model for the general random medium
considered in (28), we will use the general bound

\begin{equation}
 (34) \quad k \leq \frac{\phi(R_i^2)}{8(R_i^2)}
\end{equation}

for the network model vertical permeability. Recall that Equation (34) was derived
in the context of circular, parallel cylinders, in which case we can reformulate (34) as

\begin{equation}
 (35) \quad k \leq \frac{\phi(A_i^2)}{8\pi(A_i)}.
\end{equation}

For the random pipe network model of [40], we have a specific random distribution
in mind—the lognormal distribution. Recalling Equation (29), the $n$-th moment of a
lognormal random variable $A$ with parameters $(\mu, \sigma^2)$ is given by

\begin{equation}
 (36) \quad \langle A^n \rangle = \int_{-\infty}^{\infty} e^{ny}(2\pi\sigma^2)^{-1/2} \exp[-(y - \mu)^2/(2\sigma^2)]dy.
\end{equation}

Some algebra yields

\begin{equation}
 (37) \quad ny - \frac{(y - \mu)^2}{2\sigma^2} = \left(y - \frac{(\mu + n\sigma^2)}{2}\right)^2 + \frac{n(2\mu + n\sigma^2)}{2}.
\end{equation}

Let $\mu' = \mu + n\sigma^2$, then combining (36) and (37) yields

\begin{equation}
 (38) \quad \langle A^n \rangle = \exp \left[\frac{n(2\mu + n\sigma^2)}{2}\right].
\end{equation}

Based on (35), we need to compute $\langle A^n \rangle$ for $n = 1, 2$. For $n = 1$, $\langle A \rangle = \text{E}[A]$ is given
in (32),

\begin{equation}
 (39) \quad \langle A \rangle = a(\phi),
\end{equation}

while for $n = 2$,

\begin{equation}
 (40) \quad \langle A^2 \rangle = \exp [2\mu + 2\sigma^2] = \exp [2\mu + \sigma^2] e^{\sigma^2} = \left(\exp \left[\frac{\sigma^2}{2}\right]\right)^2 e^{\sigma^2},
\end{equation}

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Thus, combining (39) and (40) with (35) yields

\[ k(\phi) \leq \frac{\phi}{8\pi} a(\phi) e^{\sigma^2}, \]

which is precisely the upper bound stated in [11, 40].

2.3. Numerical results. Two figures of [11, 40], showing the key results of the original random pipe network model, are reconstructed in Figure 2. Note that the second of the two figures, Figure 2(b), shows the results of a slight modification of the model, to be described in this subsection.

Indeed, while the results of the original model, shown in Figure 2(a), agree well with the laboratory data of [7] for large \( \phi \), they disagree with the lab data for small \( \phi \) by more than an order of magnitude. Note that the logarithm of the lab data decreases somewhat linearly for large \( \phi \), and drops precipitously as \( \phi \to 0.05^+ \). Physically, this precipitous drop can be understood in terms of the “Rules of Fives” [9], whereby columnar sea ice undergoes a temperature-driven transition from an impermeable porous medium to one where the pores have connected up to form channels through which fluid can flow at around \( \phi = 0.05 \) (when temperature \( T = -5^\circ C \) and bulk salinity \( S = 5 \) ppt). Keeping in mind the Rule of Fives, the random pipe network formulation, and the discrepancy between the numerical results and the data shown in Figure 2(a), leads one to the conclusion that a means by which the network can become largely disconnected as \( \phi \to 0.05^+ \) must be introduced.

In [11], two additional parameters were introduced, to allow for the requisite disconnection to occur: we will refer to these as “disconnection probabilities,” and denote them as \( p_h \in [0, 1] \), the probability that a horizontal pipe will be “broken” (removed), and \( p_v \in [0, 1] \), the probability that a vertical pipe will be broken. Conceptually, nonzero \((p_h, p_v)\) will cause the random pipe network to have “gaps” through which fluid cannot flow. In terms of the model discussed in Subsection 1.1, this impedance of flow is achieved by drawing a sample of random numbers \( U_{i,j}^h \) and \( U_{i,j}^v \) from a uniform random variable \( U = \text{unif}(0,1) \), with \( i, j = 0, 1, \ldots, N + 1 \), and then setting \( R_{i,j}^a = 0 \) if \( U_{i,j}^a < p_a \) for \( a = h, v \). The choice of \((p_h, p_v)\) in [11] varied with volume fraction \( \phi \), so that the network was largely disconnected as \( \phi \to 0.05^+ \), with the result reconstructed in Figure 2(b) showing excellent agreement with the laboratory data of [7].

3. Random pipe model for sea ice with entrained EPS. Consider now a model for the effective permeability \( k \) of a block of sea ice with entrained EPS (e.g., due to the presence of algae). Recall from Section 1 that the key differences between a block of young sea ice with and without entrained EPS, as observed in [21], are: increased tortuosity, increased volume fraction, increased salt retention, and a net drop in fluid permeability.

In the context of our random pipe model, which is posed on a square lattice with randomly-sized pipes, increased geometric complexity of individual pores is not represented, while increased tortuosity of the fluid pathways can be thought of as being reflected in the variation of the parameters \((p_h, p_v)\). Similar to the original model [40] (Subsections 1.2, 2.1 and 2.3), we regard our random network as a simplified model of the pore space of a statistically homogeneous block of ice (in equilibrium) with entrained EPS, and are at this point primarily interested in the vertical effective permeability of the network as a model for the vertical effective permeability of the block of ice.
It will be useful for later comparisons between numerics and data to expand on the description of the observed net drop [21] in fluid permeability in young sea ice with entrained EPS versus EPS-free ice. Indeed, specific data for fluid permeability were not presented, although the text made clear that the decrease in fluid permeability was by at most an order of magnitude.

From a modeling perspective, the components are largely the same as described in Subsection 1.1 and Section 2. We consider a block of young sea ice (with entrained EPS) at a given equilibrium state. We model the pore space of the block of sea ice as a square lattice Equation (1) with circular pipes connecting a given node \((i,j)\) to its nearest neighbors \(\{(i\pm1,j), (i,j\pm1)\}\). Assuming classic Poiseuille flow, approximation by finite differences, and incompressibility, we derive the linear system (4). Defining the total flux, average velocity, and their linear relationship (6)–(8), and solving for \(k\) leads to the definition for the effective permeability (9) of the network. These components constitute our model for the effective permeability of a block of young sea ice with entrained EPS. The main differences from the original model are the choices of cross-sectional area distribution and parameters.

While it was observed in [25] that a lognormal distribution (29) models well the observed data on brine inclusion cross-sectional area \(A\) in young sea ice, based on data in [21], here we find that a better model for \(P\) (and thus \(A\), assuming circular cross-sections) in the case of young sea ice with entrained EPS is given by a bimodal–lognormal distribution,

\[
A = e^Y, \quad f_Y(x; \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = pf_X(x; \mu_1, \sigma_1^2) + (1 - p)f_X(x; \mu_2, \sigma_2^2).
\]  

We call this distribution “bimodal” due to the shape of the graph of \(f_Y: \mathbb{R} \to \mathbb{R}^+\) as seen in Figure 3, which has this “double bump” appearance in general under certain assumptions on the relative magnitudes of the parameters. We also include the histogram of the underlying data in Figure 3. We now briefly recall the discussion in [21] surrounding the data, and we also discuss our method for estimating its distribution.

Indeed, from [21], the data were collected as follows: 1.) The diatom *Melosira arctica* var. *krembsii* was isolated from the bottom of sea ice in the Chukchi Sea.
Fig. 3. Bimodal lognormal distribution for the cross-sectional areas of the brine inclusions in EPS-laden sea ice, using data from [21]. The histogram of log(A) for observed values of cross-sectional area A of young sea ice with entrained EPS (scaled with the sum of box areas equal to unity) is shown in gray. Superimposed are the best fit probability density functions (PDF), with the normal PDF (dashed, in blue) corresponding to the classical lognormal distribution in [25], and the new bimodal PDF (solid, in red).

In order to use this information in our random pipe network, we need to assume that the brine cross-sections are circular, and calculate the area of the assumed circular cross-sections from their measured perimeters. The validity of this assumption is unfortunately questionable, but is an artifact of the model used. Indeed, in [21] it is observed that the pore space in artificial sea ice grown with Melosira EPS is more geometrically complex than in controls (artificial and natural sea ice grown without Melosira EPS), i.e., not circular. Nonetheless, we proceed with the calculation as these are the only data available, and acknowledge as important future work the need to: a.) better quantify the geometrical complexity experimentally, and b.) better model this complexity.

Based on the histogram of the data, shown in Figure 3, and the classical result of [25], we postulated that the data of [21] –converted from measurements of perimeter to estimates of area, according to $A = P/(4\pi)$ – might be distributed according to a generalization of the classical lognormal distribution. One of the simplest such generalizations is when $\ln A$ is distributed according to a mixture of two normal distributions, i.e., Equation (42). Assuming this form for the random distribution of estimated areas of the $n = 234$ brine inclusions, we estimated the parameters using a maximum-likelihood estimate:

\[
(p, \mu_1, \mu_2, \sigma_1, \sigma_2) = (0.48, -6.56, 0.02, 1.30, 2.30).
\]

3.1. Upper bound on fluid permeability in EPS-laden sea ice. Next, we explicitly calculate the new void bound in Equations (35) and (41), i.e., the upper
bound on fluid permeability $k$ for our new model using the bimodal–lognormal distribution. Indeed, the moments of $A = e^Y$ can be computed explicitly. The first step is to observe that

$$
\langle A^n \rangle = E[A^n] = E[\exp(nY)] = \int_{-\infty}^{\infty} e^{ny} \sum_{i=1}^{2} w_i f(y; \mu_i, \sigma_i^2)dy,
$$

where $w_1 = p$, $w_2 = 1 - p$, and $f(y; \mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp[-(y - \mu)^2/(2\sigma^2)]$. The remainder of the calculation follows Subsection 2.2, Equations (36)–(40),

$$
\langle A^n \rangle = p \exp \left[ \frac{n(2\mu_1 + n\sigma_1^2)}{2} \right] + (1 - p) \exp \left[ \frac{n(2\mu_2 + n\sigma_2^2)}{2} \right].
$$

Then, combining (35) and (45), with $n = 1, 2$, yields

$$
k \leq \frac{\phi \left( p \exp \left[ 2\mu_1 + 2\sigma_1^2 \right] + (1 - p) \exp \left[ 2\mu_2 + 2\sigma_2^2 \right] \right)}{8\pi \left( p \exp \left[ \frac{2\mu_1 + \sigma_1^2}{2} \right] + (1 - p) \exp \left[ \frac{2\mu_2 + \sigma_2^2}{2} \right] \right)}.
$$

Including the parameters discussed in Subsection 3.2 simplifies (46) considerably (see (49)).

### 3.2. Numerical results

We now discuss parameter selection, the simplified form of the void upper bound, numerical simulations of the new random pipe network model, and computational considerations.

Recalling from Section 3 (Figure 3 and Equations (42) and (43)), the data of [21] were measured from photomicrographs of $n = 234$ brine inclusions at $-10^\circ C$. The data therefore give a quantitative understanding of the brine inclusions in one sample of sea ice with entrained EPS, at some given brine volume fraction $0 < \phi < 1$. In order to compare and contrast with the previous results [11] and data [7], however, we wish to have a series of samples, with varying brine volume fraction $\phi$, in particular $\phi \to 0.05^+$. In the absence of conclusive data to make mathematical modeling decisions with regard to the parameters, we proceed as follows.

As before (Subsections 2.1 and 2.3), we require $E[A] = a(\phi)$, where $a(\phi)$ is given by (31), but $A$ is given by the new bimodal distribution (42). As $\phi$ varies, we suppose that $\epsilon = \frac{\mu_2 - \mu_1}{2}$ is fixed. Additionally, we let $\sigma = \sigma_1 = \sigma_2$ be a free parameter, let $\mu_1 = \ln a(\phi) - \frac{\sigma^2}{2} - \epsilon$, and let $\mu_2 = \ln a(\phi) - \frac{\sigma^2}{2} + \epsilon$. In summary, we have

$$
\begin{align*}
\epsilon &= \frac{\mu_2 - \mu_1}{2}, \\
\mu_1 &= \ln a(\phi) - \frac{\sigma^2}{2} - \epsilon, \\
\mu_2 &= \ln a(\phi) - \frac{\sigma^2}{2} + \epsilon.
\end{align*}
$$

Given the parameters (47), and the requirement that $E[A] = a(\phi)$, a straightforward but tedious calculation begins by substituting (47) into (45) (with $n = 1$), and concludes with the realization that $p$ is, in fact, a dependent parameter,

$$
p(\epsilon) = \frac{1}{1 + e^{-\epsilon}}.
$$

We also incorporate the disconnection probabilities $(p_h, p_c)$, which are chosen as in [11], so that the network is largely disconnected as $\phi \to 0.05^+$. The specific parameters used are given in Table 1. To both reiterate and expand on the discussion of $(p_h, p_c)$ from Subsection 2.3, the choice of these parameters acts as a model of the
percolation transition [9] observed in sea ice (commonly called “The Rule of Fives” in the literature), were observed as necessary in [11], and are similar (but not the same) as the parameters used in [11].

With the choice of parameters as in (47) and (48), (46) reduces to

\[ k(\phi) \leq \left(2 \cosh \epsilon - 1\right) \frac{\phi}{8\pi} a(\phi) e^{\sigma^2}. \]  

Assuming \( \mu_2 \geq \mu_1 \), then \( \epsilon \geq 0 \). When \( \epsilon = 0 \), the separation \( \mu_2 - \mu_1 = 0 \), and the underlying probability distribution is the classical lognormal distribution, as in the original model [40] (Subsections 1.1, 2.1 and 2.3). In this case, Equation (49) reduces to (41), reinforcing that our model is an extension of the existing model.

As discussed at the start of Section 3, the observed drop in fluid permeability [21] of young sea ice with entrained EPS versus EPS-free ice was by at most an order of magnitude. When we choose \( \epsilon = 3.3 \), as suggested by Equations (43) and (47), we see too severe a drop in \( k \) — by three orders of magnitude, instead of one. Choosing instead \( \epsilon = 1.6 \), we see by comparing Figure 2(b) and Figure 4 that the drop in fluid permeability is in much better agreement, with only a slight overestimate when \( \phi < 0.15 \). The severe drop by the natural choice of \( \epsilon = 3.3 \) may be due to a lack of precision in the original data, the lack of geometric information in the data, or the lack of geometric complexity in the model, while the slight overestimate when \( \epsilon = 1.6 \) and \( \phi < 0.15 \) may be due to the choices for \((p_h, p_v)\) — which were not quite the same as the original numerical simulations reconstructed in Figure 2(b) (see Table 1).

Considering now the new, rigorous upper bound (49), when \( \epsilon > 0 \) we have \( 2 \cosh \epsilon - 1 > 1 \), so the upper bound (49) is similar to (41), up to a multiplicative constant \( C > 1 \). With the choice of \( \epsilon = 1.6 \) as discussed above, \( C = 2 \cosh \epsilon - 1 \approx 4.15 \), which means that our void bound for the new bimodal-lognormal distribution is approximately four times larger than the void bound for the original lognormal distribution — opposite the previously observed and now numerically-simulated drop in fluid permeability \( k \). Ideally we would have a somewhat tight (if not optimal) upper bound, which suggests the need for additional analysis.

When the original classical lognormal distribution is used (as in Subsections 1.1, 2.1 and 2.3) and \( N = 1024 \), the computing time for the iterative multigrid solver of [40] (implemented in Fortran) increases rapidly with \( \sigma > 1 \). Increasing \( \sigma \) from 1.0 to 1.5, for example, increases the required number of iterations and thus the required time by several orders of magnitude. Similar timing increases arise when the classical lognormal distribution is replaced with the new bimodal-lognormal distribution. The issue is magnified by the choice of \((p_h, p_v)\) for \( \phi < 0.15 \), which causes the underlying matrix to become indefinite.

---

**Table 1**

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( p_h )</th>
<th>( p_v )</th>
<th>( \phi )</th>
<th>( p_h )</th>
<th>( p_v )</th>
</tr>
</thead>
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</tr>
<tr>
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<td>0.175</td>
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<td>0</td>
</tr>
<tr>
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<td>0.25</td>
<td>0.2</td>
<td>0.20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.125</td>
<td>0.15</td>
<td>0.1</td>
<td>0.225</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The choice of parameters \((p_h, p_v)\) used in new simulations (Figure 4), to model the percolation transition [9] as \( \phi \to 0.05^+ \), similar to [11].
These convergence issues and increases in computing times are perhaps not wholly unexpected, given the corresponding increase in variance, and thus the increasingly rough (random) coefficients involved. In lieu of augmenting the existing multigrid solver of [40] to better handle the real, symmetric, possibly indefinite linear systems with rough coefficients, we have instead designed a MATLAB implementation and used built-in direct solvers – in particular a Cholesky solver when the matrix is positive definite, and the MA57 routine [5] when the matrix is numerically indefinite – for which convergence is not an issue.

To illustrate the increase in variance more clearly, we show in Table 2 the means and variances of the random lognormal or bimodal–lognormal distributions (denoted \( A_{\text{orig}} \) and \( A_{\text{new}} \)) used in the original and the new random pipe network models, respectively. Recall that \( E[A_{\text{orig}}] = E[A_{\text{new}}] = a(\phi) \) (31), by assumption. The variances \( \text{Var}[A_{\text{orig}}] \) and \( \text{Var}[A_{\text{new}}] \), however, are not equal. Indeed, Table 2 shows that for all four values of \( \phi \) we have \( \text{Var}[A_{\text{new}}] > \text{Var}[A_{\text{orig}}] \), by around an order of magnitude.

Indeed, in either case (the original or the new model), the effect of increasing the parameter \( \phi \) is to increase the value of \( a(\phi) \) and thus \( \mu \) (for the original model) or \( (\mu_1, \mu_2) \) (for the the new model). In the new model, however, the density function \( f_Y \) (42) has two peaks, each approximately the same width, and separated by a distance \( 2\epsilon \). Thus the increase in variance is expected.

Continuing, we can consider now how the model informs our understanding of the actual physical systems involved. Indeed, the graph of the bimodal–lognormal distribution in Figure 3 suggests that this drop in fluid permeability is not unexpected. Indeed, this drop should be evident from the dominance of the left bump centered on very small inclusions with cross-sections an order of magnitude smaller than, say, the mean of the classical lognormal distribution, leading to a higher probability of these constrictive pathways, and lowering the effective fluid transport properties of the porous medium.

4. Conclusions. While the effective fluid permeability of sea ice is a critical parameter affecting the properties of sea ice, and thus affecting polar ecosystems and global climate models, the effects of biogeochemistry on this parameter are not yet well understood. The random pipe network model presented herein is a mathematical model for the effective fluid permeability of young sea ice with entrained EPS, under the simplifying assumption that the given block of sea ice is at equilibrium. As far as the authors are aware, this paper presents the first work to consider a two-dimensional model of the effects of microscale biochemistry, and particularly the presence of algal exudates, on the larger scale physical properties of sea ice. We find good agreement between observations [21] and our numerical simulations, analyze this result, and dis-
Future work is needed to understand the effects of biology and chemistry on the properties of the sea ice. Indeed, direct improvements to this model could be achieved by studying data related to a.) observations of the average cross-sectional area of brine inclusions in young, EPS-laden sea ice, as a function of \( \phi \); and b.) observations of the fluid permeability of this type of ice, as a function of \( \phi \). Formulating and analyzing a discrete, nonequilibrium, random pipe network model for fluid permeability of young sea ice represents an exciting new direction, which would require judicious modeling of salinity, temperature, phase change, and connectivity. A possible application of considerable importance to large-scale studies of sea ice, would be to mathematically model percolation blockage in young sea ice, as studied in [30]. This type of modeling could help to explain why the presence of EPS in sea ice extends the lifetime of the ice [21].

Acknowledgments. We gratefully acknowledge support from the Division of Mathematical Sciences and the Division of Polar Programs at the U.S. National Science Foundation (NSF) through Grants DMS-0602219, ARC-0934721, DMS-0940249, and DMS-1413454. We are also grateful for support from the Office of Naval Research (ONR) through Grant N00014-13-10291. We are grateful to Christopher Krembs for providing the detailed measurements used in the new model analysis, and the Karl M. Banse Endowed Professorship for support to J.W.D. Finally, we would like to thank the Math Climate Research Network (MCRN) for their support.

REFERENCES


FLUID FLOW IN SEA ICE WITH EXOPOLYMERIC SUBSTANCES


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