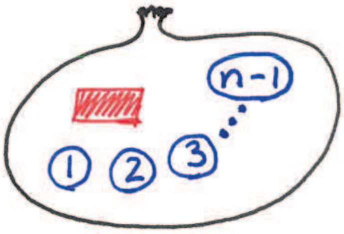
Why $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-1}{k+1}$

Suppose you have a bag of n rocks, namely 1 brick and $n-1$ marbles.



$$\binom{n}{k+1} = \left(\begin{array}{l}\text{\# of ways to choose}\\ k+1 \text{ rocks from bag}\\ \text{of } n \text{ rocks}\end{array}\right)$$

$$= \left(\begin{array}{l}\text{\# of ways to choose}\\ k+1 \text{ rocks that include}\\ \text{the brick}\end{array}\right) + \left(\begin{array}{l}\text{\# of ways to choose}\\ k+1 \text{ rocks that don't}\\ \text{include the brick}\end{array}\right)$$

$$= \left(\begin{array}{l}\text{\# of ways to choose}\\ k \text{ marbles from the}\\ n-1 \text{ marbles in the bag}\end{array}\right) + \left(\begin{array}{l}\text{\# of ways to choose}\\ k+1 \text{ marbles from the}\\ n-1 \text{ marbles in the bag}\end{array}\right)$$

$$= \binom{n-1}{k} + \binom{n-1}{k+1}$$

Why the Binomial Theorem is true

Using the distributive law,

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

$$= xxx + xxy + xyx + xyy$$

$$+ yxx + yxy + yyx + yyy$$

Each of the eight terms in this sum is obtained by choosing a black variable, a blue variable, and a red variable (x or y).

$$\begin{pmatrix}\text{\# of ways to choose}\\ \text{three } x\text{'s}\end{pmatrix} = \begin{pmatrix}\text{\# of ways to choose}\\ \text{zero } y\text{'s from } \{y, y, y\}\end{pmatrix} = \binom{3}{0}$$

$$\begin{pmatrix}\text{\# of ways to choose}\\ \text{two } x\text{'s and one } y\end{pmatrix} = \begin{pmatrix}\text{\# of ways to choose}\\ \text{one } y \text{ from } \{y, y, y\}\end{pmatrix} = \binom{3}{1}$$

$$\begin{pmatrix}\text{\# of ways to choose}\\ \text{one } x \text{ and two } y\text{'s}\end{pmatrix} = \begin{pmatrix}\text{\# of ways to choose}\\ \text{two } y\text{'s from } \{y, y, y\}\end{pmatrix} = \binom{3}{2}$$

$$\begin{pmatrix}\text{\# of ways to choose}\\ \text{three } y\text{'s}\end{pmatrix} = \begin{pmatrix}\text{\# of ways to choose}\\ \text{three } y\text{'s from } \{y, y, y\}\end{pmatrix} = \binom{3}{3}$$

So $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$