Final Prep: Early Material

1. What is an arithmetic sequence?
   \[ a_1, a_2, a_3, \ldots \quad a_{n+1} = a_n + d \]  
   \[ (d \in \mathbb{R} \text{ is constant}) \]

2. What is a geometric sequence?
   \[ a_{n+1} = r \cdot a_n \]  
   \[ (r \in \mathbb{R} \text{ is constant}) \]

3. What's the 170th term of 3, 5, 7, 9, ... ?
   \[ a_n = a_1 + (n-1)d \]
   \[ (n = 170) \]
   \[ (d = 2) \]
   \[ a_{170} = 3 + 169(2) = 3 + 338 = 341 \]

4. What's the 214th term of -5, 15, -45, 135, ... ?
   \[ a_n = a_1 \cdot r^{n-1} \]
   \[ (n = 214) \]
   \[ (r = -3) \]
   \[ a_{214} = (-5)(-3)^{213} \]

5. For which kinds of sequences do you know how to find the sum of the first k terms?
   Arithmetic
   \[ \text{Sum} = \frac{\frac{7}{15}}{1 - \frac{5}{3}} = \frac{7}{5} \]

6. For which kinds of sequences have we learned how to find the sum of all the terms?
   Geometric, when \(-1 < r < 1\)

7. What's the sum of the first 40 terms of -7, -4, -12, ... ?
   \[ \text{Sum of the first k terms} = \frac{k(a_1 + a_k)}{2} \]
   \[ (k = 40) \]
   \[ (a_1 = -7) \]
   \[ \frac{40}{2} [(-7 + (-7 + 39(3))] \]
   \[ = 20(-14 + 117) \]
   \[ = 20(103) = 2060 \]

8. What's \( \sum_{i=1}^{\infty} \left(\frac{7}{5}\right)^i \) ?
   \[ a_1 = \frac{7}{5} \]
   \[ r = \frac{1}{5} \]
   \[ \frac{7}{5^2} = \frac{7}{25} \]
   \[ \text{(see answer under (#6))} \]
9. Find \[ \sum_{i=0}^{3} (1 - i^2) = (1 - (1)^2) + (1 - (2)^2) + (1 - (3)^2) \]
   \[= 0 + (-3) + (-8) = -11 \]

10. Find \[ \sum_{i=1}^{30} 5 \]
    \[= 5 + 5 + 5 + \ldots + 5 = 5(30) = 150 \]

11. How many ways are there to order a set of 48 objects?
    \[ 48! = 48(47)(46) \ldots (3)(2)(1) \]
    "48 factorial"

12. How many ways can you choose and then order 17 objects from a set of 58 objects?
    \[ \frac{58!}{(58 - 17)!} = \frac{58!}{41!} \]

13. How many subsets of a set of 98 objects contain exactly 23 objects?
    \[ \frac{98!}{(98 - 23)! 23!} = \frac{98!}{75! 23!} = \binom{98}{23} \]
    "binomial coefficient"

14. What does "options multiply" mean?
    To find the number of ways to make a sequence of decisions: Count how many ways each decision can be made, and then multiply all these numbers together.

15. Write \( \binom{7}{4} \) as a natural number in standard form.
    \[ \binom{7}{4} = \frac{7!}{(7-4)! 4!} = \frac{7!}{3! 4!} = \frac{7(6)(5)(4!)}{3! 4!} = \frac{7(6)(5)}{3!(2)(1)} = 7(5) = 35 \]
Function composition

16. \( f(x) = x^2 + 2 \), \( g(x) = 3x - 1 \). Find \( f \circ g(x) \) and \( g \circ f(x) \).

\[
\begin{align*}
\text{\( f \circ g(x) = f(g(x)) = f(3x - 1) \)} \\
\text{\( g \circ f(x) = g(f(x)) \)}
\end{align*}
\]

"\( f \) composed with \( g \)" \( = (3x - 1)^2 + 2 \)

17. \( f(x) = 2(x - 4)^3 + 1 \). Find \( f^{-1}(y) \).

\[
\begin{align*}
y &= 2(x - 4)^3 + 1 \\
\rightarrow \text{Solve for} \ x & \text{ for } y \\
y - 1 &= 2(x - 4)^3 \\
\frac{y - 1}{2} &= (x - 4)^3 \\
\sqrt[3]{\frac{y - 1}{2}} &= x - 4
\end{align*}
\]

\[ f^{-1}(y) = 4 + \sqrt[3]{\frac{y - 1}{2}} \]

18. What are the implied domains of the following functions:

\[
\begin{align*}
f(x) &= x \quad \mathbb{R} \\
f(x) &= x^2 \quad \mathbb{R} \\
f(x) &= x^3 \quad \mathbb{R} \\
f(x) &= x^4 \quad \mathbb{R} \\
f(x) &= \sqrt[3]{x} \quad x \geq 0 \quad \text{Implied domain } = [0, \infty) \\
f(x) &= \sqrt[3]{x} \quad \mathbb{R} \\
f(x) &= \frac{1}{x} \quad \mathbb{R} - \{0\} \\
f(x) &= e^x \quad \mathbb{R} \\
f(x) &= \log_e(x) \quad (0, \infty)
\end{align*}
\]

The subset of \( \mathbb{R} \) for which the function is defined.
19. What's the implied domain of \( f(x) = \frac{2x^5}{3} - 3x^2 + 27 \)?

\( \mathbb{R} \) (a polynomial is defined for every \( x \in \mathbb{R} \))

20. What's the implied domain of \( g(x) = \frac{3x-7}{x^2-4} \)?

Not defined when \( x^2 - 4 = 0 \)
\[ x = 2 \text{ or } -2 \]

Implied domain = \( \mathbb{R} - \{2, -2\} \)

21. What's the implied domain of \( h(x) = \frac{1}{e^x} \)?

Not defined if \( e^x = 0 \)
However, \( e^x \) never equals 0,
\[ \Rightarrow \text{Implied domain} = \mathbb{R} \]

22. What's the implied domain of \( \sqrt[3]{7-x} \)?

\[ 7 - x > 0 \]
\[ 7 > x \]
\[ x < 7 \]

\[ \Rightarrow (-\infty, 7] \]

23. What's the implied domain of \( 5x^2 - 3\sqrt{2x-3} \)?

This function is well-defined for any \( x \in \mathbb{R} \) (nothing)

Implied domain = \( \mathbb{R} \)

24. What's the implied domain of \( 2x - \log_e(3x+4) \)?

\[ 3x + 4 > 0 \]
\[ 3x > -4 \]
\[ x > -\frac{4}{3} \]

\[ \Rightarrow \left(-\frac{4}{3}, \infty\right) \]

25. What's the implied domain of \( e^{\sqrt[5]{5x-2}} + 3x^2 - 5 \)?

Same as (23) : no issues like dividing by zero, square root of a negative number, etc.

Implied domain = \( \mathbb{R} \)