Two questions about local cohomology

Anurag K. Singh

We will discuss two questions on local cohomology. These have a few things in common:

• Thanks to the Frobenius endomorphism $F(r) = r^p$, there are stronger results in prime characteristic p.

• The polynomial rings $\mathbb{Z}[x_1, \ldots, x_d]$ and $\mathbb{Q}[x_1, \ldots, x_d]$ have endomorphisms satisfying $\varphi_k(x_i) = x_i^k$. These come in useful.

• Both questions are unanswered for defining ideals of $E \times \mathbb{P}^1$, where E is an elliptic curve.

Let R be a commutative ring. For an ideal $\mathfrak{a} \subset R$, the local cohomology modules $H^i_{\mathfrak{a}}(R)$ are constructed as

$$H^i_{\mathfrak{a}}(R) = \varinjlim_t \operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) \qquad \text{for } i \geqslant 0\,,$$

where $\{\mathfrak{a}_t\}$ is a decreasing chain of ideals cofinal with the chain $\{\mathfrak{a}^t\}$.

Any chain of ideals cofinal with $\{\mathfrak{a}^t\}$ yields the same direct limit. Eisenbud, Mustaţă, and Stillman raised the following question:

Question. Let R be a polynomial ring over a field, and $\mathfrak{a}$ an ideal of R. Is there a chain $\{\mathfrak{a}_t\}$ as above, such that each $\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R)$ injects into $H^i_{\mathfrak{a}}(R)$, i.e.,

$$H^i_{\mathfrak{a}}(R) \cong \bigcup_t \operatorname{Ext}^i_R(R/\mathfrak{a}_t, R)\,?$$

Example. Let K be a field, $R = K[w,x,y,z]$ and

$$\mathfrak{a} = \ker\left(R \twoheadrightarrow K[s^4, s^3t, st^3, t^4]\right) .$$

Then $H^i_{\mathfrak{a}}(R) = 0$ for all $i \geqslant 3$.

If K has prime characteristic, Hartshorne proved that $\mathfrak{a}$ is a set-theoretic complete intersection.

Hence $\mathfrak{a} = \operatorname{rad}(f,g)$, and the ideals $\mathfrak{a}_t = (f^t, g^t)$ form a cofinal chain with $\operatorname{Ext}^i_R(R/\mathfrak{a}_t, R) \hookrightarrow H^i_{\mathfrak{a}}(R)$.

Question. If K has characteristic 0, does there exist an ideal $\mathfrak{b}$ with $\operatorname{rad}\mathfrak{b} = \mathfrak{a}$ such that

$$\operatorname{Ext}^i_R(R/\mathfrak{b}, R) \longrightarrow H^i_{\mathfrak{a}}(R)$$

is injective for each i? If so, then

$$\operatorname{Ext}^3_R(R/\mathfrak{b}, R) = 0 = \operatorname{Ext}^4_R(R/\mathfrak{b}, R)$$

implies that $R/\mathfrak{b}$ is Cohen-Macaulay, i.e., that $\mathfrak{a}$ is "set-theoretically Cohen-Macaulay."

Positive answers

Mustaţă (2002). If $\mathfrak{a}$ is generated by square-free monomials $m_1, \ldots, m_r$, let $\mathfrak{a}^{[t]} = (m_1^t, \ldots, m_r^t)$. The natural maps

$$\text{Ext}_R^i(R/\mathfrak{a}^{[t]}, R) \longrightarrow H_{\mathfrak{a}}^i(R)$$

are injective for all i and all t.

S.-Walther: If R is regular of characteristic $p > 0$ and $R/\mathfrak{a}$ is F-pure, then the natural maps

$$\text{Ext}_R^i(R/\mathfrak{a}^{[p^t]}, R) \longrightarrow H_{\mathfrak{a}}^i(R)$$

are injective for all i and all t.

Remark. These can be viewed as cases of a more general statement about rings R and $R/\mathfrak{a}$ with suitable endomorphisms.

A ring A of prime characteristic is F-*pure* if

$$F \otimes 1 \colon A \otimes_A M \longrightarrow A \otimes_A M$$

is injective for all M, where F is the Frobenius.

Examples of F-pure rings include:

regular rings,

determinantal rings,

Plücker embeddings of Grassmannians,

coordinate rings of non-supersingular elliptic curves,

polynomial rings mod square-free monomial ideals,

normal affine semigroup rings,

direct summands of F-pure rings,

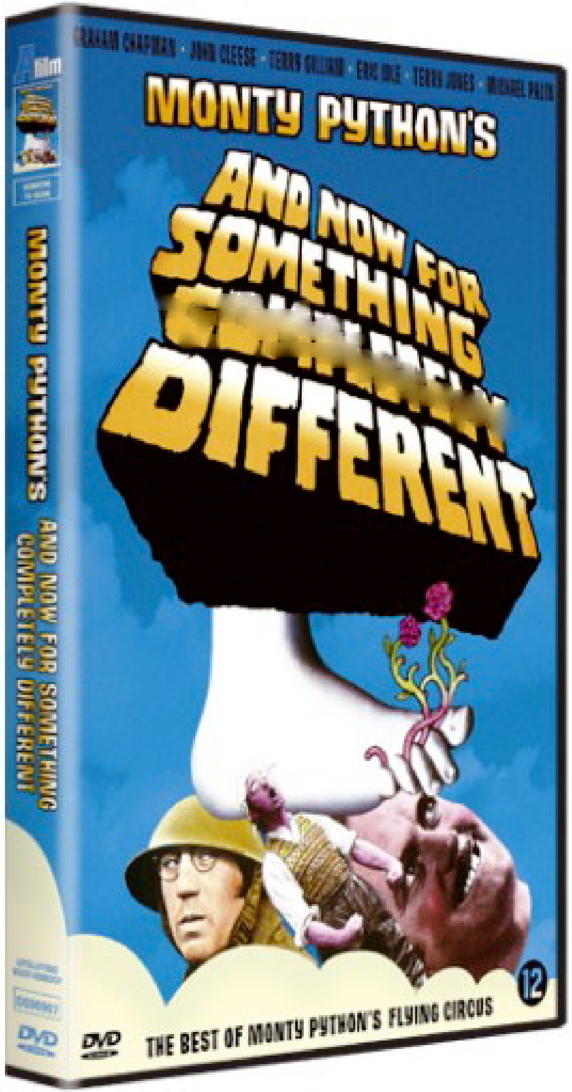
tensor products $A \otimes_K B$, where A and B are F-pure algebras over a perfect field K.

Let $E \subset \mathbb{P}^2_{\mathbb{Q}}$ be an elliptic curve, and E_p denote its reduction mod p for large p. Then $E_p \times \mathbb{P}^1_{\mathbb{Z}/p}$ has a homogeneous coordinate ring of the form $R/\mathfrak{a}$, where $R = \mathbb{Z}/p[x_0, \dots, x_5]$. Consider the maps

$$\begin{array}{ccc} \operatorname{Ext}^4_R(R/\mathfrak{a}^{[p^t]}, R) & \xrightarrow{\alpha_t} & \operatorname{Ext}^4_R(R/\mathfrak{a}^{[p^{t+1}]}, R) \\ \| & & \| \\ R/\mathfrak{m}^{[p^t]} & \xrightarrow{\alpha_t} & R/\mathfrak{m}^{[p^{t+1}]} \end{array}$$

If E_p is not supersingular, then $R/\mathfrak{a}$ is F-pure and α_t is injective for each t. In particular, $H^4_{\mathfrak{a}}(R) \neq 0$.

If E_p is supersingular, then $R/\mathfrak{a}$ is not F-pure and $\alpha_t = 0$. In particular, $H^4_{\mathfrak{a}}(R) = 0$. The Eisenbud-Mustaţă-Stillman question essentially asks if $\mathfrak{a}$ is set-theoretically Cohen-Macaulay.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Question. (Huneke) Does $H^i_{\mathfrak{a}}(R)$ have finitely many associated primes, where R is Noetherian?

Affirmative answers

Huneke and Sharp (1993). Regular rings containing a field of prime characteristic.

Lyubeznik (1993). Regular local rings containing a field of characteristic 0.

Lyubeznik (2000). Unramified regular local rings of mixed characteristic.

S. (2000). The module $H^3_{(x,y,z)}(R)$ has p-torsion for each prime integer p, where

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)\,.$$

Hence $H^3_{(x,y,z)}(R)$ has infinitely many associated primes.

Katzman (2002). The module $H^2_{(x,y)}(R)$ has infinitely many associated primes, where

$$R = K[s, t, u, v, x, y]/((sux - tvy)(ux - vy))\,.$$

S.-Swanson (2004). Families of examples where the ring is a local/graded hypersurface, which is a UFD, and has rational singularities (in characteristic 0) or is F-regular (in prime characteristic).

Conjecture. (Lyubeznik) For regular rings R, the set of associated primes of $H^i_{\mathfrak{a}}(R)$ is finite.

This conjecture is open for $R = \mathbb{Z}[x_1, \ldots, x_d]$. Take an ideal $\mathfrak{a} \subset R$. For each prime integer p, we have

$$0 \longrightarrow R \xrightarrow{p} R/p \longrightarrow R/pR \longrightarrow 0\,,$$

which induces a long exact sequence

$$\begin{aligned}\cdots \longrightarrow H^{i-1}_{\mathfrak{a}}(R) \xrightarrow{p} H^{i-1}_{\mathfrak{a}}(R) \longrightarrow H^{i-1}_{\mathfrak{a}}(R/pR) \\ \longrightarrow H^{i}_{\mathfrak{a}}(R) \xrightarrow{p} H^{i}_{\mathfrak{a}}(R) \longrightarrow H^{i}_{\mathfrak{a}}(R/pR) \longrightarrow \cdots\,.\end{aligned}$$

Lyubeznik's conjecture implies that $H^i_{\mathfrak{a}}(R)$ has p-torsion for at most finitely many primes p, and hence that the connecting homomorphisms are zero for almost all p.

This is easily seen to be true if $\mathfrak{a}$ is a set-theoretic complete intersection.

Example. Let $R = \mathbb{Z}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$ and $\mathfrak{a} = (\Delta_1, \Delta_2, \Delta_3)$ be the ideal generated by the 2×2 minors.

In the following exact sequence, is the connecting homomorphism zero?

$$0 \longrightarrow H^2_{\mathfrak{a}}(R) \xrightarrow{p} H^2_{\mathfrak{a}}(R) \longrightarrow H^2_{\mathfrak{a}}(R/pR) \longrightarrow H^3_{\mathfrak{a}}(R) \xrightarrow{p} H^3_{\mathfrak{a}}(R) \longrightarrow 0$$

Yes: the module $H^2_{\mathfrak{a}}(R/pR)$ is generated by elements corresponding to the equations

$$u^q\Delta_1^q + v^q\Delta_2^q + w^q\Delta_3^q \equiv 0 \mod p$$
$$x^q\Delta_1^q + y^q\Delta_2^q + z^q\Delta_3^q \equiv 0 \mod p$$

where $q = p^e$. These generators belong to the image of $H^2_{\mathfrak{a}}(R) \longrightarrow H^2_{\mathfrak{a}}(R/pR)$ because . . .

$$u^{k+1}\Delta_1^{2k+1}\sum_{n=0}^{k}\sum_{i=0}^{n}\binom{k}{n}\binom{k+i}{k}\binom{k+n-i}{k}$$
$$\times(-1)^i w^i v^{n-i} x^n \Delta_2^{k-i}\Delta_3^{k-n+i}$$

$$+\,v^{k+1}\Delta_2^{2k+1}\sum_{n=0}^{k}\sum_{i=0}^{n}\binom{k}{n}\binom{k+i}{k}\binom{k+n-i}{k}$$
$$\times(-1)^i u^i w^{n-i} y^n \Delta_3^{k-i}\Delta_1^{k-n+i}$$

$$+\,w^{k+1}\Delta_3^{2k+1}\sum_{n=0}^{k}\sum_{it=0}^{n}\binom{k}{n}\binom{k+i}{k}\binom{k+n-i}{k}$$
$$\times(-1)^i v^i u^{n-i} z^n \Delta_1^{k-i}\Delta_2^{k-n+i} = 0\,.$$

Example. Let $R = \mathbb{Z}\begin{bmatrix} s & u & v & w \\ t & x & y & z \end{bmatrix}$ and take the ideal $\mathfrak{a} = (ut - xs, vt - ys, wt - zs)$. In addition to the Koszul relations, we have the Plücker relation

$$(vz - wy)(ut - xs) + (wx - uz)(vt - ys) \\ + (uy - vx)(wt - zs) = 0\,.$$

Consequently we have elements of $H^2_{\mathfrak{a}}(R/pR)$ corresponding to the equations

$$(vz - wy)^q(ut - xs)^q + (wx - uz)^q(vt - ys)^q \\ + (uy - vx)^q(wt - zs)^q \equiv 0 \mod p\,.$$

The map $H^2_{\mathfrak{a}}(R) \longrightarrow H^2_{\mathfrak{a}}(R/pR)$ is surjective since . . .

$$(ut-xs)^{2k+1}\sum_{r=0}^{k}\sum_{n=0}^{r}\frac{d(k+1)}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}$$
$$\times (st)^{k-r}(vz-wy)^{2k+1-r-n}(wy)^{n}(vt-ys)^{r}(wt-zs)^{r}$$

$$+(vt-ys)^{2k+1}\sum_{r=0}^{k}\sum_{n=0}^{r}\frac{d(k+1)}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}$$
$$\times (st)^{k-r}(wx-uz)^{2k+1-r-n}(uz)^{n}(wt-zs)^{r}(ut-xs)^{r}$$

$$+(wt-zs)^{2k+1}\sum_{r=0}^{k}\sum_{n=0}^{r}\frac{d(k+1)}{2k+1-r}\binom{2k+1-r}{n}\binom{k-n}{r-n}$$
$$\times (st)^{k-r}(uy-vx)^{2k+1-r-n}(vx)^{n}(ut-xs)^{r}(vt-ys)^{r}$$
$$=0$$

Example. Let $R = \mathbb{Z}[x_0, \ldots, x_5]$, and take $\mathfrak{a} \subset R$ such that $R/\mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the coordinate ring of $E \times \mathbb{P}^1$, for E an elliptic curve. Our exact sequence now is

$$
\begin{aligned}
0 \longrightarrow H^3_{\mathfrak{a}}(R) \xrightarrow{p} H^3_{\mathfrak{a}}(R) \longrightarrow H^3_{\mathfrak{a}}(R/pR) \\
\longrightarrow H^4_{\mathfrak{a}}(R) \xrightarrow{p} H^4_{\mathfrak{a}}(R) \longrightarrow H^4_{\mathfrak{a}}(R/pR) \longrightarrow 0\,.
\end{aligned}
$$

As we saw earlier, $H^4_{\mathfrak{a}}(R/pR) = 0$ if and only if E_p is supersingular.

Hence $H^4_{\mathfrak{a}}(R) \xrightarrow{p} H^4_{\mathfrak{a}}(R)$ is **surjective** for infinitely many p, and **not surjective** for infinitely many p.

Lyubeznik's conjecture implies that this map is **injective** for almost all p. Is that true?

For an ideal $\mathfrak{a} \subset R$, consider as before

$$H^i_{\mathfrak{a}}(R/pR) \xrightarrow{\delta^i} H^{i+1}_{\mathfrak{a}}(R) \xrightarrow{p} H^{i+1}_{\mathfrak{a}}(R) \xrightarrow{\pi^{i+1}} H^{i+1}_{\mathfrak{a}}(R/pR)\,.$$

The *Bockstein homomorphism* β^i is the composition

$$\beta^i = \pi^{i+1} \circ \delta^i \colon H^i_{\mathfrak{a}}(R/pR) \longrightarrow H^{i+1}_{\mathfrak{a}}(R/pR)\,.$$

S.-Walther: Let $R = \mathbb{Z}[x_1, \dots, x_d]$, and take $\mathfrak{a} \subset R$. Then the Bockstein homomorphisms

$$\beta^i \colon H^i_{\mathfrak{a}}(R/pR) \longrightarrow H^{i+1}_{\mathfrak{a}}(R/pR)$$

are zero for almost all primes p.

The proof uses endomorphisms φ with $\varphi(x_i) = x_i^p$.

Remark. If $R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$ and $\mathfrak{a} = (x, y, z)$, then the Bockstein homomorphisms

$$\beta^2 \colon H^2_{\mathfrak{a}}(R/pR) \longrightarrow H^3_{\mathfrak{a}}(R/pR)$$

are nonzero for all primes p.