Numerical equivalence and the vanishing conjecture

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Let (R, m) be a d-dim. Noetherian local ring. Assume that R is an image of a RLR.

Vanishing Conjecture

Let M and N be finitely generated R-modules such that $\mathrm{pd}_R M < \infty$, $\mathrm{pd}_R N < \infty$ and $\ell_R(M \otimes_R N) < \infty$.

If $\dim M + \dim N < \dim R$, then

$$\chi_R(M, N) = 0.$$

$(\chi_R(M, N) := \Sigma_{i \geq 0} (-1)^i \ell_R(\mathrm{Tor}_i^R(M, N))$

Roberts, Gillet-Soulé (1985)

If R is a complete intertsection, then the conjecture is true.

Recall the proof

$G_0(R) :=$ the Grothendieck group of f. g. R-modules.

$A_*(R) = \oplus_{i=0}^d A_i(R)$ is the Chow group of $\mathrm{Spec}(R)$, where

$$A_i(R) := \oplus_{\dim R/P = i} \mathbf{Z}[\mathrm{Spec}(R/P)] / \sim_{\mathrm{rat}} .$$

Singular Rieman-Roch theorem (Baum-Fulton-MacPherson)

We have an $\mathbf{Q}$-isomorphism:

$$G_0(R)_{\mathbf{Q}} \xrightarrow{\tau_R} A_*(R)_{\mathbf{Q}}$$

$\tau_R(R) = q_d + q_{d-1} + \cdots + q_0$ is sometimes called the *todd class* of R. ($q_i \in A_i(R)_{\mathbf{Q}}$)

Remark

(1) $q_d = \Sigma_{\dim R/P=d}\, \ell_{R_P}(R_P)[\mathrm{Spec}(R/P)] \neq 0$.

(2) R : a complete intersection $\Longrightarrow$ $q_i = 0$ for $i < d$

(3) R : Cohen-Macaulay $\Longrightarrow$ $\tau_R(\omega_R) = q_d - q_{d-1} + q_{d-2} - \cdots$

(4) R : Gorenstein $\Longrightarrow$ $q_{d-i} = 0$ for each odd i

(5) R : normal $\Longrightarrow$ $q_{d-1} = -\frac{\mathrm{cl}(\omega_R)}{2}$

($q_{d-1} \in A_{d-1}(R) = \mathrm{Cl}(R) \ni \mathrm{cl}(\omega_R)$)

$C^m(R)$:= the category of bounded finite R-free complex $\mathbf{F}.$ s. t. $\ell_R(H_i(\mathbf{F}.)) < \infty$ for any i

If $\mathbf{F}. \in C^m(R)$, then we have

$$\begin{aligned} \chi_{\mathbf{F}.} : \ & G_0(R) \longrightarrow \mathbf{Z} \\ & \alpha \mapsto \sum_i (-1)^i \ell_R(H_i(\mathbf{F}. \otimes \alpha)). \end{aligned}$$

<u>Localized Chern character ch($\mathbf{F}.$)</u>

ch($\mathbf{F}.$) makes the following diagram commutative

$$\begin{array}{ccc} G_0(R)_{\mathbf{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbf{Q}} \\ \chi_{\mathbf{F}.} \downarrow & & \downarrow \mathrm{ch}(\mathbf{F}.) \\ \mathbf{Q} & = & \mathbf{Q} \end{array}$$

Assume that M, N satisfy the assumption of the vanishing conjecture. Let $\mathbf{G}.$ (resp. $\mathbf{H}.$) be finite R-free resolutions of M (resp. N).

$\Longrightarrow \mathbf{G}.\otimes\mathbf{H}. \in C^m(R)$, $\chi_{\mathbf{G}.\otimes\mathbf{H}.}(R) = \chi_R(M,N)$.

Roberts $\mathrm{ch}(\mathbf{G}. \otimes \mathbf{H}.)\,(q_d) = 0$ if $\dim M + \dim N < \dim R$. (R may not be a c. i..)

$$\chi_{\mathbf{G}.\otimes\mathbf{H}.}(R) = \mathrm{ch}(\mathbf{G}. \otimes \mathbf{H}.)\,(\tau_R(R))$$
$$= \mathrm{ch}(\mathbf{G}. \otimes \mathbf{H}.)\,(q_d) + \sum_{i=0}^{d-1} \mathrm{ch}(\mathbf{G}. \otimes \mathbf{H}.)\,(q_i)$$

Therefore, $\chi_R(M,N) = 0$ if R is a complete intersection.

Question 1

Let $\mathbf{F}. \in C^m(R)$ and $\alpha \in A_i(R)$ for $i < d$. When $\mathrm{ch}(\mathbf{F}.)\,(\alpha) = 0$?

Remark Let $\mathbf{F}. \in C^m(R)$ and $d > k \geq 0$.

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(1) $\chi_{\mathbf{F}.}(M) = 0 \quad \forall M$ with $\dim M \leq k$.

(2) $\mathrm{ch}(\mathbf{F}.)\,(\alpha) = 0 \quad \forall i \leq k, \forall \alpha \in A_i(R)$.

(3) $\overline{A_i(R)} = 0 \quad \forall i \leq k$.

$\underline{R : \text{a RLR}} \Longrightarrow \chi_{\mathbf{F}.}(M) = 0 \quad \forall \mathbf{F}. \in C^m(R)$, $\forall M$ with $\dim M < d$.

$\underline{d > \dim M = 0} \Longrightarrow \chi_{\mathbf{F}.}(M) = 0 \quad \forall \mathbf{F}. \in C^m(R)$.

Dutta-Hochster-MacLaughlin (1985)

$\exists R$ of $d = 3$, $\exists \mathbf{F}. \in C^m(R)$, $\exists \alpha \in A_2(R)$ such that $\mathrm{ch}(\mathbf{F}.)\,(\alpha) \neq 0$.

Miller-Singh (2000)

$\exists R$ of $d = 5$, $\exists \mathbf{F}. \in C^m(R)$, $\exists \alpha \in A_3(R)$ such that $\mathrm{ch}(\mathbf{F}.)\,(\alpha) \neq 0$.

Foxby, Roberts

If $\dim R/P \geq 2 \quad \forall P \in \mathrm{Min}(R)$, then $\mathrm{ch}(\mathbf{F}.)\,(\alpha) = 0 \quad \forall \mathbf{F}. \in C^m(R), \forall \alpha \in A_1(R)$.

Example Let A be a standard graded ring over $\mathbf{C}$. Assume that $\mathrm{Proj}(A)$ is a curve of $g > 0$. Set $R = A_{(A_1)}$.

Then $d = 2$ and $\dim A_1(R)_{\mathbf{Q}} = \infty$.

But, $\mathrm{ch}(\mathbf{F}.)\,(\alpha) = 0 \quad \forall \mathbf{F}. \in C^m(R)$, $\forall \alpha \in A_1(R)$.

Definition

(1) $\alpha \in G_0(R)$ is numerically equivalent to 0 $\overset{\text{def}}{\Longleftrightarrow} \chi_{\mathbf{F}.}(\alpha) = 0 \quad \forall \mathbf{F}. \in C^m(R)$.

$NG_0(R) := \{\alpha \in G_0(R) \mid \alpha \sim_{\text{num}} 0\}$.

$\overline{G_0(R)} := G_0(R)/NG_0(R)$.

(2) $\alpha \in A_*(R)$ is numerically equivalent to 0 $\overset{\text{def}}{\Longleftrightarrow} \mathrm{ch}(\mathbf{F}.)(\alpha) = 0 \quad \forall \mathbf{F}. \in C^m(R)$.

$NA_*(R) := \{\alpha \in A_*(R) \mid \alpha \sim_{\text{num}} 0\}$.

$\overline{A_*(R)} := A_*(R)/NA_*(R)$.

Remark (1) Easy to see

$$\tau_R(NG_0(R)_{\mathbf{Q}}) = NA_*(R)_{\mathbf{Q}}.$$

(2) $NA_*(R) = \oplus_{i=0}^{d} NA_i(R)$,
where $NA_i(R) = NA_*(R) \cap A_i(R)$. (Not easy. Use Adams operation due to Gillet-Soulé.)

$$\begin{array}{lccc} & G_0(R)_{\mathbf{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbf{Q}} \\ (3) & \downarrow & & \downarrow \\ & \overline{G_0(R)}_{\mathbf{Q}} & \xrightarrow{\overline{\tau_R}} & \overline{A_*(R)}_{\mathbf{Q}} = \oplus_{i=0}^{d} \overline{A_i(R)}_{\mathbf{Q}} \end{array}$$

K (2004) Let R be an excellent local ring such that

- $R \supset \mathbf{Q}$

or

- R is essentially of finite type / $\mathbf{Z}$, a field or a complete DVR.

Then, $\overline{G_0(R)}$ and $\overline{A_*(R)}$ are finitely generated free abelian group (of the same rank).

Outline of the proof

Reduce to the case where R is a domain.

Let $\pi : Z \longrightarrow \mathrm{Spec}R$ be a projective generically finite morphism such that Z is regular and $\pi^{-1}(m)_{\mathrm{red}} = \cup_\ell E_\ell$ is a normal crossing divisor of Z.

Prove that the kernel of the composite map

$$A_*(Z)_{\mathbf{Q}} \to \oplus_\ell \mathrm{CH}^{\cdot}(E_\ell)_{\mathbf{Q}} \to \oplus_\ell \mathrm{CH}^{\cdot}_{\mathrm{num}}(E_\ell)_{\mathbf{Q}}$$

is contained in the kernel of the surjection

$$A_*(Z)_{\mathbf{Q}} \stackrel{\pi_*}{\to} A_*(R)_{\mathbf{Q}} \to \overline{A_*(R)}_{\mathbf{Q}}.$$

Then $\overline{A_*(R)}_{\mathbf{Q}}$ is the subquotient of the finite dim. $\mathbf{Q}$-vect. space $\oplus_\ell \mathrm{CH}^{\cdot}_{\mathrm{num}}(E_\ell)_{\mathbf{Q}}$.

<u>Example</u> Let A be a standard graded ring over $\mathbf{C}$ such that $X := \mathrm{Proj}(A)$ is smooth/$\mathbf{C}$. Set $R := A_{(A_1)}$ and $d = \dim R$.
(1) rank$\overline{A_{d-1}(R)} \leq \rho(X) - 1$, where $\rho(X)$ is the Picard number of X.
(If $\mathrm{CH}^{\cdot}(X)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbf{Q}}$, then we have $A_*(R)_{\mathbf{Q}} \xrightarrow{\sim} \overline{A_*(R)}_{\mathbf{Q}}$ and rank$\overline{A_{d-1}(R)} = \rho(X) - 1$ by Roberts-Srinivas 2003.)
(2) It is conjectured that $\mathrm{CH}^{\cdot}_{\mathrm{hom}}(X)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbf{Q}}$ (the standard conjecture). (It is known to be true when $\dim X \leq 3$ or X is an abelian variety.)
If $\mathrm{CH}^{\cdot}_{\mathrm{hom}}(X)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{CH}^{\cdot}_{\mathrm{num}}(X)_{\mathbf{Q}}$, then

$$\overline{A_i(R)} = 0$$

for $i \leq d/2$.
(It is true for $d \leq 4$.)

Question 2 For any R and $i \leq d/2$, $\overline{A_i(R)} = 0$?

(It is equivalent to $\chi_{\mathbf{F}.}(M) = 0 \quad \forall \mathbf{F}. \in C^m(R)$ and $\forall M$ with $\dim M \leq d/2$.)

The next question immediately follows from Question 2.

Question 3 Let M and N be finitely generated R-modules such that $\mathrm{pd}_R N < \infty$ and $\ell_R(M \otimes_R N) < \infty$.

If $2 \cdot \dim M + \dim N \leq \dim R$ and $\dim M \neq 0$, then $\chi_R(M, N) = 0$?

Proposition Let R be a Noetherian local domain, and $\pi : Z \longrightarrow \mathrm{Spec}(R)$ be a proper surjective map such that Z is a regular scheme.

Then $\overline{A_i(R)} = 0$ if $i < \dim Z - \dim \pi^{-1}(m)$.

Example Set $R := k[x_{ij} \mid i = 1, \ldots, m \ ; \ i = 1, \ldots, n]/I_2(x_{ij})$ and $m \leq n$.

Let $\pi : Z \longrightarrow \mathrm{Spec}(R)$ be the blow-up of $\mathrm{Spec}(R)$ along $(x_{11}, x_{21}, \ldots, x_{m1})$.

Then $\dim Z = \dim R = n+m-1$, $\pi^{-1}(m) = \mathbf{P}^{m-1}$. Therefore, $\overline{A_i(R)} = 0$ if $i < \dim Z - \dim \pi^{-1}(m) = n$. In this case, it is known that

$$A_i(R) \xrightarrow{\sim} \overline{A_i(R)} = \begin{cases} \mathbf{Z} & (n \leq i \leq n+m-1) \\ 0 & (\text{otherwise}) \end{cases}$$