

An Application of the Riemann-Roch Formula in the Blow-up of a Nonsingular Scheme

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Joint work with Claudia Miller

Homological Conjecture in Commutative Algebra

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If $Y = \{\text{pt.}\}$, then

one obtains the Riemann-Roch Formula.

Basic Setting

$X = \text{Proj } R$, nonsingular, $\dim X = d$

R : a Noetherian graded regular ring (finite over $k = \bar{k}$)

$$\left\{ \begin{array}{l} \text{f.g. graded} \\ \text{modules / } R \end{array} \right\} \quad \left\{ \begin{array}{l} \text{coherent sheaves} \\ \text{on } X \end{array} \right\}$$

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 \downarrow \cong & & \parallel \\
 N_n & \longrightarrow & \widetilde{N} \\
 \forall n \gg 0 & &
 \end{array}$$

The Hilbert Polynomial

of M or \widetilde{M}

$$M = \bigoplus M_n$$

$$\mathcal{P}_{\widetilde{M}}(n) = \mathcal{P}_M(n) = \dim_k M_n$$

$$\forall n \gg 0.$$

The Chern Classes of \widetilde{M}

$c_i := c_i(\widetilde{M})$: the i -th Chern class of \widetilde{M}

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$$A_j(X) \longrightarrow A_{j-i}(X)$$

$$C_{\widetilde{M}}(h) := 1 + c_1 h + c_2 h^2 + \cdots + c_d h^d$$

An Example

If $X = \mathbb{P}_k^d$, then

$$\begin{aligned} A_*(X) &= \mathbb{Z}^{d+1} \\ &= \mathbb{Z}[\mathbb{P}_k^d] \oplus \mathbb{Z}[\mathbb{P}_k^{d-1}] \oplus \cdots \oplus \mathbb{Z}[\mathbb{P}_k^0] \end{aligned}$$

$$\mathcal{O}_X = \widetilde{R} = k[\overbrace{x_0, \dots, x_d}]$$

$$C_{\mathcal{O}_X}(h) = C_{\widetilde{R}}(h) = 1$$

$$C_{\mathcal{O}_{X^{(\ell)}}}(h) = C_{\widetilde{R^{(\ell)}}}(h) = 1 + \ell h$$

Whitney Sum Formula

$$0 \longrightarrow \widetilde{M}_1 \longrightarrow \widetilde{M}_2 \longrightarrow \widetilde{M}_3 \longrightarrow 0$$

$$\Rightarrow C_{\widetilde{M}_2}(h) = C_{\widetilde{M}_1}(h) \cdot C_{\widetilde{M}_3}(h)$$

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Thus, for any M : a f.g. graded module over R

$$0 \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

$$\begin{aligned} C_{\widetilde{M}}(h) &= \prod_{i=0}^d (C_{\widetilde{F}_i}(h))^{(-1)^i} \\ &= 1 + c_1 h + c_2 h^2 + \cdots + c_d h^d \end{aligned}$$

$$\text{ch}(\widetilde{M}) = r + c_1 h + \frac{1}{2!}(c_1^2 - 2c_2)h^2 + \cdots$$

The Riemann-Roch Formula

for Sheaves over $\mathbb{P}_k^d = E, k = \bar{k}$

$$\chi(E, \widetilde{M}(n)) = \int e^{nh} \text{ch}(\widetilde{M}) \text{td}(\Omega_X^\vee)$$

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$$\mathcal{P}_{\widetilde{M}}(t) = \text{the coeff. of } h^t \text{ in}$$

$$e^{th} \text{ch}(\widetilde{M}) \left(\frac{h}{1-e^{-h}}\right)^{d+1}$$

Riemann-Roch formula over

$$\begin{array}{ccc} & \mathbb{P}_k^{d_1} \times \mathbb{P}_k^{d_2} & \\ & \swarrow p_1 & \searrow p_2 \\ \mathbb{P}_k^{d_1} & & \mathbb{P}_k^{d_1} \end{array}$$

$$\mathcal{F}(m, n) = \mathcal{F} \otimes p_1^* \mathcal{O}(m) \otimes p_2^* \mathcal{O}(n)$$

\rightsquigarrow a multigraded module

$$\forall m, n \gg 0,$$

$$\mathcal{P}_{\mathcal{F}}(m, n) = \text{the coefficient of } x^{m+1} y^{n+1} \text{ in}$$
$$e^{mx+ny} \text{ch}(\mathcal{F}) \left(\frac{x}{1-e^{-x}} \right)^{m+1} \left(\frac{y}{1-e^{-y}} \right)^{n+1}$$

– due to Serre's vanishing theorem and Künneth formula.

A Variation

$$\mathbb{P}_k^d = \text{Proj } k[x_0, \dots, x_d] \rightsquigarrow A[x_0, \dots, x_d] = \mathbb{P}_A^d$$

$k[x_0, \dots, x_d] \rightsquigarrow A$: a regular local ring.

$\text{Spec } A$

Main Question

A : a regular local ring

M : a f.g. module over A

The Hilbert-Samuel poly. of M

$$HS_M(i) = \ell(M/\mathfrak{m}^i M) \quad \forall i \gg 0$$

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$$HS_M(i) = \ell(M/\mathfrak{m}^i M) = \sum_{n=0}^i \ell(\mathfrak{m}^n M / \mathfrak{m}^{n+1} M)$$

$$\mathcal{P}_{\text{gr}(M)}(n) = \ell(\mathfrak{m}^n M / \mathfrak{m}^{n+1} M)$$

where $\text{gr}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$.

Idea

$$\mathcal{R}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M$$

$$\left[\widetilde{\mathcal{R}(M)} \right] \quad K_0(\text{Proj } A[mt]) \xrightarrow{\tau_X} A_*(\text{Proj } A[mt])$$

$$\text{Spec } A \longleftarrow \text{Proj } A[mt]$$

Idea

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 \left[\widetilde{\mathcal{R}(M)} \right] & K_0(\text{Proj } A[mt]) & \xrightarrow{\tau_X} & A_*(\text{Proj } A[mt]) \\
 \downarrow \text{Thm.} & \begin{array}{c} f^* \downarrow \\ K_0(E) \end{array} & \xrightarrow{\tau_E} & \begin{array}{c} \downarrow \cap \\ A_*(E) \end{array} \\
 \left[\widetilde{\mathfrak{gr}(M)} \right] & & \longrightarrow & \left(\begin{array}{l} \text{leading to Hilbert} \\ \text{poly. of } \mathfrak{gr}(M) \end{array} \right)
 \end{array}$$

Spec $A \longleftarrow$

Proj $A[mt]$

$f \uparrow$

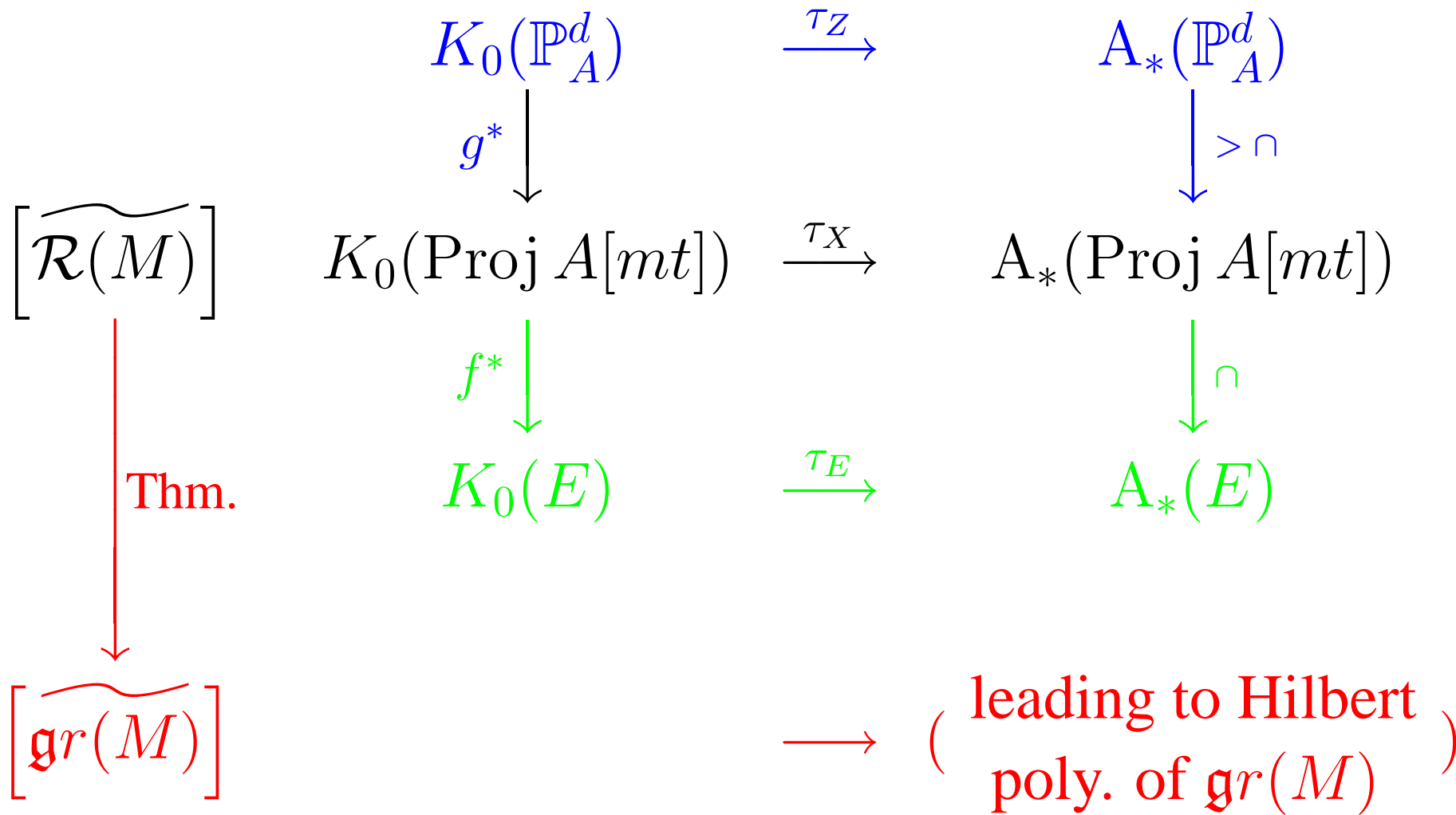
$$E = \mathbb{P}_k^d = \text{Proj}(\mathfrak{gr}(A))$$

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$$\begin{array}{ccc}
 \text{Spec } A & \longleftarrow & \text{Proj } A[mt] & \hookrightarrow & \mathbb{P}_A^d \\
 & & \begin{array}{c} f \uparrow \\ E = \mathbb{P}_k^d = \text{Proj}(\mathfrak{gr}(A)) \end{array} & &
 \end{array}$$



Theorem 1 (C.–Miller) *Let $R = A[\mathfrak{m}t]$.*

$$f^* : K_0(\text{Proj } R) \longrightarrow K_0(E)$$

$$[\widetilde{\mathcal{R}(M)}] \longrightarrow [\widetilde{\mathfrak{g}r(M)}].$$

Theorem 2 (C.–Miller) *Let $R = A[\mathfrak{m}t]$.*

$$f^* : K_0(\text{Proj } R) \longrightarrow K_0(E)$$

$$[\widetilde{\mathcal{R}(M)}] \longrightarrow [\widetilde{\text{gr}(M)}].$$

Define

$$f^*([\widetilde{\mathcal{R}(M)}]) = \sum (-1)^i [\widetilde{\text{Tor}_i^R(\mathcal{R}(M), R/\mathfrak{m}R)}]$$

$$E = \text{Proj}(R/\mathfrak{m}R)$$

Proof of Theorem 1

By the definition of f^* ,

$$\begin{aligned} f^*([\widetilde{\mathcal{R}(M)}]) &= \sum (-1)^i [\widetilde{\mathrm{Tor}_i^R(M, R/\mathfrak{m}R)}] \\ &\stackrel{\text{claim}}{=} [\widetilde{\mathrm{Tor}_0^R(R_{\mathfrak{m}}(M), R/\mathfrak{m}R)}] \\ &= [\widetilde{\mathcal{R}(M) \otimes_R R/\mathfrak{m}R}] \\ &= [\widetilde{\mathfrak{g}r(M)}] \end{aligned}$$

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Note: Tor_i^R is not necessarily zero but $\widetilde{\mathrm{Tor}_i^R} = 0$.

The Claim

Proof. $\dim A = d + 1$, $\mathfrak{m} = (a_0, \dots, a_d)$.

$R := A[\mathfrak{m}t] = A[t_0, \dots, t_d]/I$ ($I = I_2\left(\begin{array}{ccc} a_0, & \dots, & a_d \\ t_0, & \dots, & t_d \end{array}\right)$).

Locally in an open cover $R_{(t_i)}$,

$$\mathfrak{m}_{(t_i)} = (a_i)R_{(t_i)}.$$

$$\implies 0 \longrightarrow R_{(t_i)} \xrightarrow{a_i} R_{(t_i)} \longrightarrow R_{(t_i)}/\mathfrak{m}R_{(t_i)} \longrightarrow 0.$$

$$\implies \mathrm{Tor}_i^R = 0 \quad \text{for } i \geq 2$$

Tor_1^R

$$\begin{aligned} \text{Tor}_1^R(\mathcal{R}(M), R/\mathfrak{m}R)_{(t_i)} &\cong (0 :_{\mathcal{R}(M)_{(t_i)}} a_i) \cong (0 : \mathfrak{m}R_{(t_i)}) \\ &\cong (0 :_{\mathcal{R}(M)} \mathfrak{m})_{(t_i)} \end{aligned}$$

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\implies

$$\widetilde{\text{Tor}}_1^R = (0 :_{\mathcal{R}(M)} \mathfrak{m})$$

$$(0 : \mathfrak{m}) \subset \mathcal{R}(M) = \bigoplus \mathfrak{m}^n M$$

$(0 : \mathfrak{m})_n \subset \mathfrak{m}^n M$ as A -modules.

$(0 : \mathfrak{m})_n = \text{Hom}_A(A/\mathfrak{m}, \mathfrak{m}^n M) = 0 \ \forall n \gg 0$ since
depth $M > 0$.

Thus,

$$\widetilde{\text{Tor}}_1^R = 0.$$

$$\begin{array}{ccccc}
\left[\widetilde{\mathcal{R}(M)} \right] & K_0(\text{Proj } A[mt]) & \xrightarrow{\tau_X} & A_*(\text{Proj } A[mt]) & \\
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\left[\widetilde{\text{gr}(M)} \right] & & \longrightarrow & \left(\begin{array}{c} \text{leading to Hilbert} \\ \text{poly. of } \text{gr}(M) \end{array} \right) &
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\left[\widetilde{\mathfrak{g}r(M)} \right] & K_0(E) & \xrightarrow{\tau_E} & A_*(E) & \\
& & \longrightarrow & \tau_E(\left[\widetilde{\mathfrak{g}r(M)} \right]) &
\end{array}$$

$$\tau_E(\left[\widetilde{\mathfrak{g}r(M)} \right]) = a_n [\mathbb{P}_k^n] + a_{n-1} [\mathbb{P}_k^{n-1}] + \cdots + a_1 [\mathbb{P}_k^1] + a_0 [\mathbb{P}_k^0].$$

$n = \dim \text{Supp}(\mathfrak{g}r(M)).$

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$$\begin{aligned}
P_{\mathfrak{gr}(M)}(t) &= e_n \binom{t+n}{n} - e_{n-1} \binom{t+n-1}{n-1} + \cdots \\
&= b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0
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\end{aligned}$$

$$b_\ell = \frac{a_\ell}{\ell!}$$

$$\begin{array}{ccccc}
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$$g^*([\widetilde{A[t_0, \dots, t_d]}]) = [\widetilde{A[\mathfrak{m}t]}] = [\widetilde{\mathcal{R}(A)}]$$

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$$g^*([\widetilde{A[t_0, \dots, t_d]}]) = [\widetilde{A[\mathfrak{m}t]}] = [\widetilde{\mathcal{R}(A)}]$$

$$g^*([\widetilde{M[t_0, \dots, t_d]}]) \stackrel{?}{=} [\widetilde{\mathcal{R}(M)}]$$

$$g^* : \begin{array}{ccc} K_0(\mathbb{P}_A^d) & \longrightarrow & K_0(\text{Proj } A[mt]) \\ \widetilde{[M[t]]} & \xrightarrow{\text{?}} & \widetilde{[\mathcal{R}(M)]} \end{array}$$

Take $M = A/(x)$ where x is a nonzerodivisor in A .
 Then, $\mathcal{R}(M) \neq 0$ and $\widetilde{[\mathcal{R}(M)]} \neq 0$ in $K_0(\text{Proj } A[mt])$.

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Then, $\mathcal{R}(M) \neq 0$ and $\widetilde{[\mathcal{R}(M)]} \neq 0$ in $K_0(\text{Proj } A[mt])$.

But,

$$M[\underline{t}] \cong A[t_0, \dots, t_d]/(x) \text{ and}$$

$$[A[t_0, \dots, t_d]/(x)] = 0$$

in $K_0(\mathbb{P}_A^d)$ since $\deg x = 0$. So

$$g^*(M[t_0, \dots, t_d]) = 0$$