A C^1 Cross Polytope Macro-element in Four Variables

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Abstract. A C^1 macro-element in four variables is constructed based on a split of a cross polytope into sixteen simplices. The element uses supersplines of degree nine, and provides optimal order approximation of smooth functions.

§1. Introduction

In comparison with a variety of bivariate macro-elements, and several trivariate ones (see [9]), there are only three macro-elements which work in an arbitrary number of variables: the Clough-Tocher macro-element developed in [13]), a macro-element on a general simplicial partition considered in [7], and the Powell-Sabin macro-element constructed in [11].

A multivariate spline is usually understood as a possibly smooth, piecewise polynomial function of several arguments. More precisely, let $\triangle$ be a partition of a domain $\Omega \in \mathbb{R}^n$ into polytopes (usually simplices). We recall that in n-dimensional space, any $(n+1)$ points in general position are the vertices of an n-dimensional simplex. The faces of an n-simplex are formed by subsets of its $n+1$ vertices. In a single formula, a k-dimensional face F_k formed by $k+1$ vertices $\{v_{i_j}\}_{j=o}^{k}$ is given by

$$F_k := \{\sum_{j=0}^{k} \gamma_j v_{i_j}, \quad \sum_{j=0}^{k} \gamma_j = 1, \quad \text{for all } j, \quad \gamma_j \geq 0\}. \tag{1}$$

The collection of polytopes $\mathcal{B}$ in $\mathbb{R}^n$ is constructed so that any two polytopes can only intersect at a common k-dimensional face, $0 \leq k < n$,

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see (1). Then the space of polynomial splines in n variables of degree d and smoothness r associated with $\triangle$ is

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \ s|_T \in \mathcal{P}_d(\mathbb{R}^n), \text{ all polytopes } T \in \triangle\}, \qquad (2)$$

where as usual, $\mathcal{P}_d(\mathbb{R}^n)$ denotes the space of polynomials of total degree d in n variables. Following [4] we define a polytope as a bounded convex region of n-dimensional space enclosed by a finite number of hyperplanes.

Definition 1. *Suppose $\mathcal{B}$ is a collection of polytopes $\{B_j\}_1^N$ whose union is some set $\Omega \in \mathbb{R}^n$. Let $\triangle_{B_j}$ be a partition of the polytope B_j into simplices, and let $\triangle := \bigcup_{j=1}^N \triangle_{B_j}$ be the corresponding partition of Ω. Suppose for each j, $\mathcal{S}(\triangle_{B_j})$ is a subspace of $\mathcal{S}_d^1(\triangle_{B_j})$, $\mathcal{S}(\triangle)$ is a subspace of $\mathcal{S}_d^1(\triangle)$, and that Λ_{B_j} is a set of functionals on B_j (consisting usually of point evaluation of s or its derivatives at points on the $(n-1)$-dimensional faces of B_j) such that*

1) *if two polytopes B_i and B_j share a $(n-1)$-dimensional face F, then $\triangle_{B_i}|_F = \triangle_{B_j}|_F$, $\Lambda_{B_i}|_F = \Lambda_{B_j}|_F$, and $\mathcal{S}(\triangle_{B_i})|_F = \mathcal{S}(\triangle_{B_j})|_F$;*
2) *each spline $s_{B_j} \in \mathcal{S}(\triangle_{B_j})$ is uniquely determined by Λ_{B_j};*
3) *if a piecewise polynomial s is defined on Ω by $s|_{B_j} := s_{B_j}$, i.e. the spline constructed in 2) for each polytope B_j, then $s \in \mathcal{S}(\triangle)$.*

In this case we call $(\mathcal{S}(\triangle_{B_j}), \Lambda_{B_j})$, a C^1 macro-element *of degree d associated with B_j. We also call $(\mathcal{S}(\triangle), \Lambda)$, where $\Lambda := \{\Lambda_{B_i}\}_1^N$, a C^1* macro-element space *of degree d.*

Remark 2. In the collection $\mathcal{B}$ it is desirable to have polytopes of a similar geometric nature, so that 2) in Definition 1 can be carried out by the same algorithm for every polytope.

Definition 3. *Given $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}^n$, let $\mathcal{S}_{d_n}^{\alpha}(\triangle)$ be the space of all splines in $\mathcal{S}_{d_n}^1(\triangle)$ satisfying the following additional smoothness conditions: for each k-dimensional face F_k of $\mathcal{B}$, $s \in C^{\alpha_k}(F_k)$, $0 \le k \le n-1$.*

As we mentioned in the beginning of this section, there are only three n-variate macro-elements. The first one uses an unsplit simplex as a building block. The following result is due to A. Le Méhauté, and can be found in [7].

Theorem 4. *With the notation of Definition 1, suppose $\mathcal{B}$ is a collection of not split simplices in $\mathbb{R}^n$, such that any two simplices only intersect at a common k-dimensional face, $0 \le k \le n$, see (1). Then for any C^1 macro-element space $(\mathcal{S}(\triangle), \Lambda)$, where $\triangle$ is formed by not split simplices, the* *associated spline space $\mathcal{S}(\triangle)$ must be of degree d_n not less than 2^n+1, and supersmoothness α not less than $(2^{n-1}, 2^{n-2}, \ldots, 2, 1)$, see Definition 3.*

The Clough-Tocher macro-element (see [13]) also uses a simplex as a building block. However, each simplex is subdivided into $(n+1)!/2$ subsimplices. The resulting C^1 macro-element space has a fixed degree three that does not depend on the number of variables. However, certain geometric constraints on the locations of split points have to be satisfied.

For the Powell-Sabin macro-element (see [11]) each simplex is split into $(n+1)!$ subsimplices, and the C^1 macro-element space has a fixed low degree two. The locations of split points in this case are subject to severe geometric constraints.

The main purpose of this paper is to construct a new macro-element in four variables, show how it can be used to solve certain Hermite interpolation problems, and prove that it has full approximation power. The macro-element is based on a split of a four-dimensional cross polytope into sixteen simplices. The construction is a generalization of an octahedral macro-element in three variables presented in [6], see also the description in Section 2.

The paper is organized as follows. In Section 2 we collect useful facts about the Bernstein-Bézier representation of multivariate polynomial splines, and provide a detailed description of a polynomial and octahedral C^1 macro-elements in three variables (see [14] and [6], respectively). Both macro-elements are needed for the construction of the four-dimensional cross polytope macro-element in Section 3. In Section 4 a Hermite interpolation scheme is constructed. Section 5 contains some results on higher-dimensional constructions. In Section 6 we present an algorithm for tesselation a collection of boxes in $\mathbb{R}^n$ into cross polytopes, and Section 7 provides a result on approximation power.

§2. Preliminaries

The Bernstein-Bézier representation, referred to as the BB-form, is based on barycentric coordinates with respect to a simplex. Most of the facts presented in this section can be found in [2].

A multi-index is any vector with nonnegative integer entries. The length of a multi-index α is the sum of all its entries, i.e., $|\alpha| := \sum_i \alpha_i$. Let D_i be differentiation with respect to the i-th coordinate. Then $D^\alpha := D_1^{\alpha_1} \ldots D_n^{\alpha_n}$. Let T be an ordered $(n+1)$-point set $T = \{v_j\}_{j=0}^n \subset \mathbb{R}^n$ in general position. Then the Bernstein basis polynomials of degree d relative to T are

$$B_i^d(v) = B_{i_0,\ldots,i_n}^d(v) := \frac{d!}{i_0! \cdots i_n!} b_0^{i_0} \cdots b_n^{i_n}, \quad |i| = d, \tag{3}$$

where $b_0, \ldots, b_n$ are the barycentric coordinates of v relative to T.

Every polynomial p of degree d on a simplex T can be written uniquely in the form

$$p = \sum_{|i|=d} c_i B_i^d, \tag{4}$$

where B_i^d are the Bernstein basis polynomials associated with T. We call $c_i = c_{i_0,\dots,i_n}$ the B-coefficients of p, and define the associated set of domain points to be

$$\mathcal{D}_{d,T} := \{\xi_i = \xi_{i_0,\dots,i_n} := \frac{1}{d}\sum_{j=0}^{n} v_j i_j\}, \quad \mathcal{D}_{d,\triangle} := \bigcup_{T\in\triangle} \mathcal{D}_{d,T}.$$

Let $F_k = \langle v_{i_0}, \dots, v_{i_k}\rangle$ be a k-dimensional face in $\triangle$ as in (1), and $\xi_i \in \mathcal{D}_{d,\triangle}$ be a domain point in a simplex $T = \langle v_0, \dots, v_n\rangle$ containing F_k as a face. We say that ξ_i lies at a distance m of F_k if $\sum_{j=0}^{k} i_j = d-m$. The set of all domain points at a distance m of F_k is called the ring of radius m around the face F_k, and denoted by $R_m(F_k)$. The set of domain points within a distance m of F_k is referred to as the disk of radius k around the face F_k, and is denoted by $D_m(F_k)$. We will need two additional pieces of notation: $R_m^T(F_k) := R_m(F_k) \cap T$, and $D_m^T(F_k) := D_m(F_k) \cap T$.

We now recall a formula for directional derivatives of p. Suppose $\{u^k\}_{k=1}^m$ is a set of m vectors in $\mathbb{R}^n$. Let $a^{(k)} = (a_0^{(k)}, \dots, a_n^{(k)})$ be the barycentric coordinates of the vector u^k, $1 \le k \le m$. Then

$$D_{u_m} \dots D_{u_1} p(v) = \frac{d!}{(d-m)!} \sum_{|i|=d-m} c_i^{(m)}(a^{(1)}, \dots, a^{(m)}) B_i^{d-m}(v), \tag{5}$$

where $c_i^{(m)}(a^{(1)}, \dots, a^{(m)})$ is obtained by applying the de Casteljau transformation (see [2]) m times using $a^{(1)}, \dots, a^{(m)}$ in order.

With the help of (5) it is easy to derive conditions for a smooth join between two polynomials on adjoining simplices $T = \langle v_0, \dots, v_n\rangle$ and $\widetilde{T} := \langle v_{n+1}, v_n, \dots, v_1\rangle$. These two simplices share a $(n-1)$-dimensional face $F = \langle v_1, \dots, v_n\rangle$. Let

$$p(v) = \sum_{|i|=d} c_i B_i^d(v),$$

$$\tilde{p}(v) = \sum_{|i|=d} \tilde{c}_i \tilde{B}_i^d(v),$$

where $\{B_i^d\}$, and $\{\tilde{B}_i^d\}$ are the Bernstein basis functions associated with T and $\widetilde{T}$, respectively. Suppose u is any direction not coplanar with F. Then $D_u^k p(v) = D_u^k \tilde{p}(v)$, $0 \le k \le d$, if and only if

$$\tilde{c}_{k,j_1,\dots,j_n} = \sum_{|i|=k} c_{i_0, j_n+i_1, j_{n-1}+i_2, \dots, j_1+i_n} B_i^k(v_{n+1}), \quad j_1 + \dots + j_n = d-k.$$

Finally, we recall that $\mathcal{M} \subseteq \mathcal{D}_{d,\triangle}$ is called a determining set for $\mathcal{S} \subseteq \mathcal{S}_d^r(\triangle)$ provided that setting the B-coefficients corresponding to every $\xi \in \mathcal{M}$ to zero implies that $s \equiv 0$. The set $\mathcal{M}$ is called a minimal determining set (MDS) for $\mathcal{S}$ provided that it is the smallest determining set, or equivalently, if prescribing the B-coefficients corresponding to $\xi \in \mathcal{M}$ *uniquely* determines a spline $s \in \mathcal{S}$.

For later use we now consider in more detail the trivariate polynomial and octahedral C^1 macro-elements.

Let $\mathcal{B}$ be a collection of N tetrahedra in $\mathbb{R}^3$ such that any two tetrahedra can only intersect at a single vertex, along a common edge, or along a common face. The union of such tetrahedra is the set Ω. Suppose n_v, n_e, n_f, N are the number of vertices, edges, faces, and tetrahedra in $\mathcal{B}$, respectively. By D^α we denote the derivative operator in standard multi-index notation. For each edge e in $\mathcal{B}$, let $\{D_e^\alpha\}_{|\alpha|=k}$ be a basis for the space of directional derivatives of order k orthogonal to e. For each triangular face f in $\mathcal{B}$, let D_f^k be a k-th order directional derivative orthogonal to f.

The following results are due to A. Ženišek, and can be found in [14].

Theorem 5. *Let $\mathcal{S}(\triangle) := \mathcal{S}_9^{4,2,1}(\triangle)$. The dimension of the space $\mathcal{S}(\triangle)$ is given by*

$$\dim \mathcal{S}(\triangle) = 35n_v + 8n_e + 7n_f + 4N.$$

Moreover, any spline $s \in \mathcal{S}(\triangle)$ is uniquely determined by the following set of nodal data*:*

1) *for every vertex v of $\mathcal{B}$, $\{D^\alpha s(v)\}_{|\alpha|\le 4}$;*
2) *for every edge e of $\mathcal{B}$, $\{D_e^\alpha s(v_e)\}_{|\alpha|=1}$, where v_e is the midpoint of e;*
3) *for every edge $e := \langle u, v\rangle$ of $\mathcal{B}$, $\{D_e^\alpha s(v_u)\}_{|\alpha|=2}$, $\{D_e^\alpha s(u_v)\}_{|\alpha|=2}$, where $v_u := (3v+u)/4$, $u_v := (3u+v)/4$;*
4) *for every face f of $\mathcal{B}$, $s(u_f)$, $\{D^\alpha(D_f s)|_f(u_f)\}_{|\alpha|\le 2}$, where u_f is the centroid of f;*
5) *for every tetrahedron t of $\mathcal{B}$, $\{D^\alpha s(u_t)\}_{|\alpha|\le 1}$, where u_t is the centroid of t,*

and the pair $(\mathcal{S}(\triangle), \Lambda)$, where Λ is given by the nodal data in 1)–5), forms a C^1 macro-element space as in Definition 1.

The trivariate C^1 octahedral macro-element uses an octahedron as a building block. The following results are due to Ming-Jun Lai and A. Le Méhauté, and can be found in [6].

Definition 6. *Let B be an octahedron in $\mathbb{R}^3$ such that the three diagonals intersect at a common point O, and let $\triangle_B$ be the partition of B into eight tetrahedra obtained by connecting O with each of the vertices of B.*

Note that an octahedron is a three-dimensional cross polytope as defined in Section 3, see also Fig. 1 on the right.

Let $\mathcal{B}$ be a collection of N octahedra in $\mathbb{R}^3$ such that any two octahedra can only intersect at a common vertex, edge or triangular face. The union of those octahedra forms a set Ω. Let $\triangle$ be a simplicial partition Ω obtained by splitting each octahedron $B \in \mathcal{B}$ as in Definition 6.

Theorem 7. *Let $\mathcal{S}(\triangle) := \mathcal{S}_3^{(2,1,1)}(\triangle)$. The dimension of the space $\mathcal{S}(\triangle)$ is given by*

$$\dim \mathcal{S}(\triangle) = 10n_v + 2n_e + 3n_f + 8N.$$

Moreover, any spline $s \in \mathcal{S}(\triangle)$ is uniquely determined by the following set of nodal data:

1) *for every vertex v of $\mathcal{B}$, $\{D^\alpha s(v)\}_{|\alpha|\le 2}$;*
2) *for every edge e of $\mathcal{B}$, $\{D_e^\alpha s(v_e)\}_{|\alpha|=1}$, where v_e is the midpoint of e;*
3) *for every face f of $\mathcal{B}$, $\{D_f^1 s(f_i)\}_{1\le i\le 3}$, where f_i, $1 \le i \le 3$, are three locations on f whose barycentric coordinates with respect to f are $(2/5, 2/5, 1/5)$, $(1/5, 2/5, 2/5)$, $(2/5, 1/5, 2/5)$;*
4) *for every face f of $\mathcal{B}$, $D_f^2 s(u_f)$, where u_f is the centroid of f,*

and the pair $(\mathcal{S}(\triangle), \Lambda)$, where Λ is given by the nodal data in 1)–4), forms a C^1 macro-element space as in Definition 1.

Remark 8. In [6] the second derivatives at the centroids of the faces are not listed as nodal data associated with the term $8n$ in the dimension count. The authors only note that "the domain points can also be associated with nodal values."

Remark 9. In [8] it is shown that the dimension of the octahedral macro-element can be reduced to $10n_v + 2n_e + 3n_f + 5N$, by imposing additional supersmoothness C^2 at the centers of each octahedron.

§3. A cross polytope macro-element

By drawing n mutually perpendicular lines through any point O in n-space we obtain a Cartesian frame or cross. Two points on each line equidistant from O in both directions are the vertices of a cross polytope B^n. The set of the vertices of B^n contains $2n$ points. Each k-dimensional face of B^n is a k-simplex, $0 \le k \le n-1$, see (1). For a n-dimensional cross polytope B^n, the number N_k of k-dimensional faces F_k is given by:

$$N_k = 2^{k+1}\binom{n}{k+1}, \quad 0 \le k \le n-1,$$

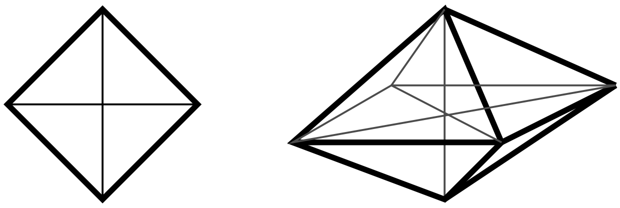

Fig. 1. 2-dim and 3-dim splits for cross polytopes.

see e.g. [4]. In particular, B^n has $2n$ vertices and 2^n cells, or n-dimensional faces. Each of the cells is a n-simplex (see (1)).

Analytically, the permutations of $(\pm 1, 0, 0, \ldots, 0)$ are the coordinates for the vertices of a n-dimensional cross polytope. For example, a 2-dimensional cross polytope is a rhombus, that is a quadrilateral with all sides equal in length; a 3-dimensional cross polytope is an octahedron (see Fig. 1).

We now define the partition of a four-dimensional cross polytope B to be used throughout the rest of this section.

Definition 10. *Let a cross polytope B in $\mathbb{R}^4$ have vertices $v_1, \ldots, v_8$, and v_0 be the point of intersection of the diagonals of B. Then we connect each of the vertices to v_0. This produces a simplicial decomposition of B, which we denote by $\triangle_B$.*

It is easy to see that the partition $\triangle_B$ consists of 16 simplices, and that it has 32 edges, 56 triangular faces, and 48 tetrahedral faces. Each simplex T has one vertex at v_0, and four vertices on the surface of B. We call the tetrahedral face t of T lying on the surface of B the outer face of T. A tetrahedral face t of T which contains v_0 will be called the interior face of B.

The set of domain points $\mathcal{D}_{9,\triangle_B}$ consists of $n_\triangle = 9 + 8 \cdot 32 + 28 \cdot 56 + 56 \cdot 48 + 70 \cdot 16 = 5641$ points. For each $0 \le m \le 9$, the domain points on the ring $R_m(v_0)$ can be regarded as lying on the surface of a cross polytope B_m that is split into simplices in the same way as B. We will always consider v_0 as the first vertex for barycentric coordinates.

Definition 11. *As in Definition 1, suppose B is a four-dimensional cross polytope, where $\triangle_B$ is as in Definition 10. Define $\mathcal{S}(\triangle_B) = \mathcal{S}_9^{(4,2,1,1)}(\triangle_B)$, as described in Definition 3.*

Definition 12. *Associated with the space $\mathcal{S}(\triangle_B)$ described in Definition 11, let $\mathcal{M}$ be the following set of domain points:*

1) *for each of the eight vertices v of the cross polytope B, the domain points in $\mathcal{M}(v) := D_4^T(v)$, where T is some simplex attached to v;*
2) *for each of the twenty-four edges $e := \langle v, u \rangle$ of the cross polytope B, the domain points in $\mathcal{M}(e) := D_2^T(e) \setminus \Big(\mathcal{M}(v) \cup \mathcal{M}(u)\Big)$ for some*

simplex T attached to e;

3) *for each of the thirty-two triangular faces f of the cross polytope B, the domain points in $\mathcal{M}(f) := D_1^T(f) \setminus \Big(\bigcup_{v \in \mathcal{V}} \mathcal{M}(v) \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}(e) \Big)$ for some simplex T attached to f, where $\mathcal{E}$ is the set of three edges of f, and $\mathcal{V}$ is the set of three vertices of f;*

4) *for each of the sixteen tetrahedral faces t of the cross polytope B, the domain points in $\mathcal{M}(t) := D_1^T(t) \setminus \Big(\bigcup_{v \in \mathcal{V}} \mathcal{M}(v) \cup \bigcup_{e \in \mathcal{E}} \mathcal{M}(e) \cup \bigcup_{f \in \mathcal{F}} \mathcal{M}(f) \Big)$ for some simplex T attached to t, where $\mathcal{F}$ is the set of four triangular faces of t, $\mathcal{E}$ is the set of six edges of t, and $\mathcal{V}$ is the set of four vertices of t;*

5) *for each of the sixteen simplices T, the domain points in $\mathcal{M}(T) := \bigcup_{2 \le i \le 5, jklm \ne 0} \xi_{ijklm}^T$.*

Theorem 13. *The dimension of the space $\mathcal{S}(\triangle_B)$ is 2456, and the set $\mathcal{M}$ forms a minimal determining set for $\mathcal{S}(\triangle_B)$.*

Proof: First we show that $\mathcal{M}$ is a minimal determining set. To this end, suppose we set the coefficients of $s \in \mathcal{S}(\triangle_B)$ corresponding to the domain points in $\mathcal{M}$.

Remark. We immediately observe that for each of the eight vertices v, and each of the twenty-four edges e of the cross polytope B, the coefficients corresponding to the domain points in $D_4(v)$, and $D_2(e)$ are uniquely defined by $\mathcal{M}(v)$, $\mathcal{M}(e)$, and the C^4, C^2 continuity at v and e respectively.

We now show that for each $0 \le i \le 9$ and each tetrahedral face t of B_i, the coefficients associated with the domain points lying on t are *uniquely* determined by smoothness conditions.

The cross polytopes B_9, B_8. The coefficients corresponding to the domain points in B_9, B_8 are either set, because the points are in $\mathcal{M}(v)$, $\mathcal{M}(e)$, $\mathcal{M}(f)$, $\mathcal{M}(t)$ for some v, e, f or t, or uniquely defined by the remark above.

The cross polytope B_7. Let t be an arbitrary tetrahedral face of B_7, and T_1 be the simplex containing t. By the remark above, the coefficients in $D_2(v)$, and $D_0(e)$ are uniquely defined for each vertex v, and each edge e of B_7. Since we agreed to consider v_0 as the first vertex for barycentric coordinates, $i = 2$ for all the domain points $\xi_{ijklm}^{T_1}$ on B_7, and the domain points that are strictly interior to t are in $\mathcal{M}(T_1)$. Thus, the yet undetermined coefficients correspond only to the domain points that are strictly interior to the triangular faces of t. For each such point, exactly one of the barycentric coordinates is equal to zero. Without loss of generality, assume $j = 0$, and $klm \neq 0$. Then the C^1 smoothness condition associated with $\xi^{T_1}_{20klm}$ has a very simple form, which we will call a single cross:

$$c^{T_1}_{20klm} = (c^{T_1}_{11klm} + c^{T_2}_{11klm})/2,$$

where T_1 and T_2 are the two simplices sharing t. The coefficients $c^{T_i}_{11klm}$ for $i = 1, 2, 3$, correspond to the domain points on B_8, and therefore have already been determined. This completes B_7.

The cross polytope B_6. Let t be an arbitrary tetrahedral face of B_6, and T_1 be the simplex containing t. By the remark above, the coefficients in $D_1(v)$ are uniquely defined for each vertex v. For all the domain points $\xi^{T_1}_{ijklm}$ on B_6, $i = 3$, and the domain points that are strictly interior to t are in $\mathcal{M}(T_1)$. Thus, the yet undetermined coefficients correspond to the domain points that are strictly interior to the triangular faces of t, and edges of t. Each coefficient corresponding to a point that is strictly interior to a triangular face can be computed again by the single cross smoothness condition from the points on B_7. For each point on the edge e of t, exactly two the barycentric coordinates are equal to zero. Without loss of generality, assume $j = k = 0$, and $lm \neq 0$. Then the double cross smoothness condition has to be satisfied:

$$c^{T_1}_{300lm} = (c^{T_1}_{210lm} + c^{T_2}_{210lm})/2 = (c^{T_3}_{201lm} + c^{T_4}_{201lm})/2,$$

where $T_i, i = 1, 2, 3, 4$, are the four simplices sharing e. However, this does not cause an overdetermination, because each of the $c^{T_i}_{210lm}, i = 1, 2$, $c^{T_i}_{201lm}, i = 3, 4$, was in turn determined by the single cross condition on B_7, and therefore:

$$c^{T_1}_{300lm} = \frac{1}{4}\sum_{i=1}^{4} c^{T_i}_{111lm}.$$

This completes B_6.

The cross polytope B_5. Let t be an arbitrary tetrahedral face of B_5, and T_1 be the simplex containing t. By the remark above, the coefficients in $D_0(v)$ are uniquely defined for each vertex v. For all the domain points $\xi^{T_1}_{ijklm}$ on B_5, $i = 4$, and the domain points that are strictly interior to t are in $\mathcal{M}(T_1)$. Thus, the yet undetermined coefficients correspond to the domain points that are strictly interior to the triangular faces of t and the edges of t. Each coefficient corresponding to a point that is strictly interior to a triangular face can be computed again by the single cross smoothness condition from the points on B_6. Each coefficient corresponding to a point that is strictly interior to an edge can be computed by the double cross smoothness condition from the points on B_7.

The cross polytope B_4. Let t be an arbitrary tetrahedral face of B_4, and T_1 be the simplex containing t. For all the domain points $\xi^{T_1}_{ijklm}$ on B_4, $i = 5$, and the only domain point $\xi^{T_1}_{51111}$ strictly interior to t is in $\mathcal{M}(T_1)$. Each coefficient corresponding to a point that is strictly interior to a triangular face can be computed again by the single cross smoothness condition from the points on B_5. Each coefficient corresponding to a point that is strictly interior to an edge can be computed by the double cross smoothness condition from the points on B_6. Similarly, each coefficient corresponding to a vertex v of t can be computed by the triple cross smoothness condition without an overdetermination, e.g. for $m \neq 0$:

$$c^{T_1}_{5000m} = \frac{1}{8} \sum_{i=1}^{4} c^{T_i}_{211lm},$$

where $T_i,\ i = 1, \ldots, 6$, are the eight simplices sharing v. This completes B_4.

The cross polytopes B_3, B_2, B_1. The tetrahedral faces t of B_3, B_2, B_1 have no strictly interior points. The coefficients strictly interior to triangular faces, edges and vertices of each t can be therefore uniquely computed using the single, double and triple cross continuity conditions respectively.

The cross polytopes B_0. The coefficient corresponding to the point ξ_{90000}, which belongs to all sixteen simplices $T_i,\ i = 1, \ldots, 16$, can be computed from the quadruple smoothness condition:

$$c_{90000} = \frac{1}{16} \sum_{i=1}^{16} c^{T_i}_{51111}.$$

To establish the dimension statement, we now count the number of points in $\mathcal{M}$. It is easy to see that the number of points described in 1)–5) for each v, e, f, t, and T respectively, equals 70, 15, 13, 35 and 35. □

§4. A Hermite interpolation scheme in $\mathbb{R}^4$

Next we describe a natural way to use the macro-element constructed above to build a C^1 spline space defined on a set Ω which has been partitioned into cross polytopes. We also show how this spline space can be used to solve associated Hermite interpolation problems. Let $\mathcal{B}$ be a collection of N cross polytopes in $\mathbb{R}^4$ such that any two cross polytopes can only intersect at a single vertex, along a common edge, or along a common triangular or tetrahedral face, and the union of such cross polytopes is the set Ω. Let n_v, n_e, n_f, n_t, and n_T be the number of vertices, edges, triangular, tetrahedral faces, and cells of the cross polytopes of $\mathcal{B}$, respectively.

Let $\triangle$ be the tetrahedral partition of Ω which is obtained by partitioning each cross polytope $B \in \mathcal{B}$ into 16 simplices as described in Definition 10.

For each edge e of a cross polytope in $\mathcal{B}$, let $\{D_e^\alpha\}_{|\alpha|=k}$ be a basis for the space of directional derivatives of order k orthogonal to e. For each triangular face f of a cross polytope in $\mathcal{B}$, let $\{D_f^\alpha\}_{|\alpha|=k}$ be a basis for the space of directional derivatives of order k orthogonal to f. Finally, for each tetrahedral face t of a cross polytope in $\mathcal{B}$, let D_t^l be the l-th directional derivative corresponding to a vector orthogonal to t.

Theorem 14. *Let $\mathcal{S}(\triangle) := \mathcal{S}_9^{(4,2,1,1)}(\triangle)$. The dimension of the space $\mathcal{S}(\triangle)$ is given by*

$$\dim \mathcal{S}(\triangle) = 70n_v + 15n_e + 13n_f + 35n_t + 35n_T.$$

Moreover, any spline $s \in \mathcal{S}(\triangle)$ is uniquely determined by the following set of nodal data*:*

1) *for every vertex v of $\mathcal{B}$, $\{D^\alpha s(v)\}_{|\alpha|\le 4}$;*
2) *for every edge e of $\mathcal{B}$, $\{D_e^\alpha s(v_e)\}_{|\alpha|=1}$, where v_e is the midpoint of e;*
3) *for every edge $e := \langle u, v\rangle$ of $\mathcal{B}$, $\{D_e^\alpha s(v_u)\}_{|\alpha|=2}$, $\{D_e^\alpha s(u_v)\}_{|\alpha|=2}$, where $v_u := (3v+u)/4$, $u_v := (3u+v)/4$;*
4) *for every triangular face f of $\mathcal{B}$, $s(u_f)$, $\{D^\alpha(D_f^\beta s)|_f(u_f)\}_{|\beta|=1,|\alpha|\le 2}$, where u_f is the centroid of f;*
5) *For every tetrahedral face t of $\mathcal{B}$, $\{D^\alpha s(u_t)\}_{|\alpha|\le 1}$, where u_t is the centroid of t;*
6) *for every tetrahedral face $t := \langle v_1, v_2, v_3, v_4\rangle$ of $\mathcal{B}$, $\{D_t^1 s(\xi_{ijkl})\}_{(ijkl)\in\mathcal{I}}$, where $\xi_{ijkl} = (iv_1 + jv_2 + kv_3 + lv_4)/8$, and $\mathcal{I}$ is the set of multiindices $(ijkl)$ with the properties $i+j+k+l = 8$, $ijkl \ne 0$, and $ijkl \ne 5$;*
7) *for every tetrahedral face t of $\mathcal{B}$, $\{D^\alpha D_t^i s(u_t)\}_{2\le i\le 5,|\alpha|\le 5-i}$, where u_t is the centroid of t,*

and the pair $(\mathcal{S}(\triangle), \Lambda)$, where Λ is given by the nodal data in 1)–7), forms a C^1 macro-element space as in Definition 1.

Proof: First we show that for each cross polytope $B \in \mathcal{B}$, the nodal data listed above uniquely determines a spline $s_B \in \mathcal{S}(\triangle_B)$. Let $\mathcal{M}$ be the minimal determining set for $\mathcal{S}(\triangle_B)$ as in Definition 12. Clearly, the data in 1) determine the coefficients corresponding to domain points in the disks $D_4(v)$ for each corner of B, and the data in 2), 3) complete the coefficients corresponding to the remaining domain points in $D_2(e)$ for every edge of B. Moreover, the data in 1)–5) restricted to a tetrahedral face t of B is precisely the Hermite interpolation data for Ženišek's macro-element, see

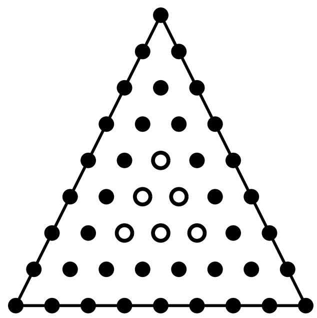

Fig. 2. Domain points on a face of B_8.

Theorem 5. Thus that data determine the coefficients corresponding to the remaining domain points in $\mathcal{M}$ in B_9.

We now consider the remaining domain points in $\mathcal{M} \cap B_8$. Let $\tilde{t}$ be one of the tetrahedral faces of B_8. On each of the triangular faces of $\tilde{t}$ there are 6 domain points left. They are shown as open dots in Fig. 2 on the right. The unknown coefficients corresponding to these points can be computed using the remaining six second derivatives in 4). The 6×6 matrix for this evaluation is nonsingular because the locations of the domain points match the locations of the entire set of the domain points for a bivariate quadratic polynomial.

There are $\binom{3+4}{3} = 35$ interior domain points in $\tilde{t}$. Four of them, located in the corners, have already been computed using the data in 1). The 31 remaining unknown coefficients are:

$c_{11124}, c_{11133}, c_{11142}, c_{11214}, c_{11223}, c_{11232}, c_{11241}, c_{11313}, c_{11322}, c_{11331},$
$c_{11412}, c_{11421}, c_{12114}, c_{12123}, c_{12132}, c_{12141}, c_{12213}, c_{12222}, c_{12231}, c_{12312},$
$c_{12321}, c_{12411}, c_{13113}, c_{13122}, c_{13131}, c_{13212}, c_{13221}, c_{13311}, c_{14112}, c_{14121},$
and c_{14211},

where the first index is associated with v_0, and the remaining four with the vertices of $\tilde{t}$. These coefficients can be determined using 30 pieces of data in 6) together with one remaining derivative in 5). The 31×31 matrix M for this evaluation is nonsingular, see Remark 5.54 in [10]. This fact supports the conjecture on matrices formed from Bernstein basis polynomials made by L.L. Schumaker in [1]. Nonsingularity of M can be shown explicitly. It also follows from the results in [5].

Next we claim that the data in 7) determine the MDS points in 5) of Theorem 12. First we note that for each $2 \leq i \leq 5$ and for each simplex T in $\mathcal{B}$ the MDS points $\{\xi_{ijklm}, jklm \neq 0\}$ form a stencil for a BB-form of a polynomial p of degree $5 - i$ in three variables. The data in 7) for a fixed i is equivalent to setting a value and all possible derivatives of p at one point. Therefore, the 20×20, 10×10, and 4×4 determinants for

$i = 2, 3, 4$ respectively, are nonzero. For $i = 5$ we only have one equation with one unknown, c_{51111}, and one piece of data, namely $D_t^5 s(u_t)$.

This completes the proof that the data in 1)–7) uniquely defines the coefficients corresponding to MDS domain points.

Let B and $\widetilde{W}$ be two cross polytopes sharing a tetrahedral face face t, and let $s = s_B$ and $\tilde{s} = s_{\widetilde{W}}$. It is clear that s and $\tilde{s}$ join with C^1 smoothness across t, since the coefficients corresponding to domain points on $D_1(t)$ of both splines are computed from the same data. □

§5. Remarks on a cross polytope macro-element in $\mathbb{R}^n$

Similarly to Definition 10, we define the partition of a cross polytope B in n variables.

Definition 15. *Let a cross polytope B in $\mathbb{R}^n$ have vertices $v_1, \ldots, v_{2n}$, and v_0 be the point of intersection of the diagonals of B. Then we connect each of the vertices to v_0. This produces a simplicial decomposition of B, which we denote by $\triangle_B$.*

Definition 16. *As in Definition 1, suppose $\mathcal{B}$ is a collection of cross polytopes in $\mathbb{R}^n$, such that any two cross polytopes only intersect at a common k-dimensional face, $0 \le k \le n$, see (1). Let the union of such cross polytopes be the set Ω. Suppose each cross polytope is split as in Definition 15. This produces a simplicial decomposition of Ω, which we denote by $\triangle$.*

Our next theorem shows that any macro-element space $(\mathcal{S}(\triangle), \Lambda)$ in n variables has a very large number of degrees of freedom for any Λ, and thus may not be useful in practice.

Theorem 17. *With the notation of Definition 1, for any C^1 macro-element space $(\mathcal{S}(\triangle), \Lambda)$, the associated spline space $\mathcal{S}(\triangle)$ must be of degree not less than $2^{n-1} + 1$.*

Proof: First we observe that an arbitrary triangulation in $\mathbb{R}^{n-1}$ induces a partition of a set in $\mathbb{R}^n$ into cross polytopes. Indeed, each $(n-1)$-simplex of the partition can be considered as the face of an n-simplex, which in turn is a cell of a cross polytope. Fig. 3 illustrates the idea of the construction when lifting from a 1-dimensional partition to a 2-dimensional Sibson split. Such constructions are possible because we can keep the projection of each cross polytope onto $(n-1)$-space entirely inside the generating $(n-1)$-simplex. This observation implies supersmoothness $C^{2^{n-1}}$ in all vertices of cross polytopes, and thus $d_n \ge 2d_{n-1} - 1 = 2^{n-1} + 1$. For example, if several octahedra are put together in such a way that some of their faces form a planar triangulation, then we must have supersmoothness C^2 in the corners of each triangle in the triangulation, and, hence, degree 5. For

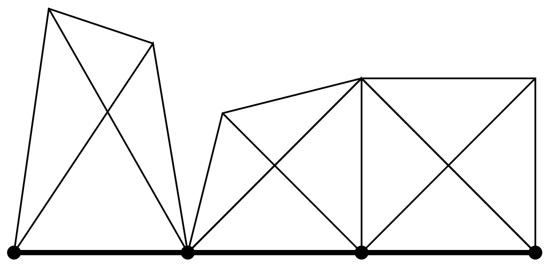

Fig. 3. Lifting 1-dim partition to 2-dim Sibson split.

4-dimensional space, the outer faces will form a tetrahedralization of a 3-dimensional region, which leads to Ženišek's macro-element [14], and, hence, degree 9. □

§6. A splitting algorithm in $\mathbb{R}^n$

In this section we show how to create a tiling of $\mathbb{R}^n$ into cross polytopes. The construction also provides a tesselation of a collection of boxes into cross polytopes.

An n-dimensional honeycomb is an infinite set of n-dimensional polytopes fitting together to fill $\mathbb{R}^n$ just once, so that every $(n-1)$-dimensional face of each polytope belongs to just one other polytope. Consider a honeycomb H formed by shifts of the n-dimensional parallelotope or box $P := [-1, 1]^n$. Since P has an interior point, namely the origin, which can be called the center of P, we can choose this same center for the inscribed sphere. The points of contact of this sphere with the $(n-1)$-dimensional faces of P are the vertices of the reciprocal polytope.

The following algorithm will lead to a tiling of $\mathbb{R}^n$ with three families of cross polytopes.

1) *Reciprocal cross polytopes.* For each box of H, the vertices are the centers of its $(n-1)$-dimensional faces. For example, the vertices of the reciprocal cross polytope for P are

$$\{(\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\}. \tag{6}$$

2) *"Corners".* Consider an arbitrary vertex v of H. There are 2^n boxes coming together at v. Therefore, we can "cut" the corners from each of 2^n boxes, and the collection of these corners forms a cross polytope centered at v. The vertices are the centers of the edges forming the cross at v. For example, for the corner $v := (1, 1, \ldots, 1)$ of P the vertices of the cross polytope that has v as the center are

$$\begin{gathered}\{(0, 1, \ldots, 1), (1, 0, 1, \ldots, 1), \ldots, (1, \ldots, 1, 0), \\ (0, 2, \ldots, 2), (2, 0, 2, \ldots, 2), \ldots, (2, \ldots, 2, 0)\}.\end{gathered} \tag{7}$$

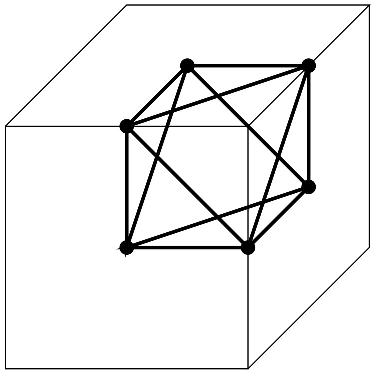

Fig. 4. Buffer octahedron in the split of a box.

3) *"Buffers"*. What is left between the reciprocal cross polytopes and the corners actually forms a collection of nonorthogonal cross polytopes as well. For example, the vertices of such a buffer cross polytope left in P between the reciprocal one described in (6) and the corner one described in (7) are

$$\begin{aligned}\{&(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1),\\ &(0,1,\ldots,1),(1,0,1,\ldots,1),\ldots,(1,\ldots,1,0)\}.\end{aligned} \tag{8}$$

The diagonals of this cross polytope are the segments connecting the two vertices whose coordinates add up to $(1,\ldots,1)$. They intersect at the point $(1/2,\ldots,1/2)$, which gives the center of the cross polytope. Note that this point is not the midpoint of the diagonals.

Remark 18. The buffers defined in 3) are not strictly speaking cross polytopes, since their diagonals are not mutually orthogonal, and the vertices are not equidistant from the center. However, as long as the diagonals intersect at one point all the results in this chapter hold true. See Fig. 4 for a buffer octahedron in the split of a box.

For the rest of this paper we relax the definition of a cross polytope according to Remark 18. In the most general setting the following algorithm provides a construction of a cross polytope:

a) start with a simplex in $\mathbb{R}^n$ with vertices $v_0,\ldots,v_n$;
b) choose one of the vertices, say v_0, to be the center of the cross polytope;
c) for every $i=1,\ldots,n$, extend the segment $\langle v_i, v_0\rangle$ from v_i to a point $v_{\bar{i}}$, which is a new vertex of the cross polytope. Connect $v_{\bar{i}}$ with all v_j, where $j\neq i$. Each of the segments $\langle v_j, v_{\bar{i}}\rangle$ is an edge of the cross polytope.

Remark 19. The tiling of $\mathbb{R}^3$ with octahedra proposed in [6] unfortunately does not generalize to a tiling of $\mathbb{R}^n$.

§7. Approximation power

Theorem 14 defines a linear interpolation operator $\mathcal{I}$ mapping $C^4(\Omega)$ into $\mathcal{S}(\triangle)$. We note that $\mathcal{I}p = p$ for any polynomial p of degree nine. Our next theorem shows that this operator provides optimal order approximation. Given a cross polytope $B \in \mathcal{B}$, let $|B|$ be its diameter. Given $1 \le m$, let $W^m_\infty(B)$ be the usual Sobolev space with seminorm

$$|g|_{m,B} := \sum_{|\alpha|=m} \|D^\alpha g\|_B, \tag{9}$$

where D^α is the derivative operator in standard multi-index notation, and $\|\cdot\|_B$ is the ∞-norm on B. Let δ_B be the ratio of the length of the longest diagonal of B to the length of the radius of the largest ball contained in B and centered at the point of intersection of the diagonals of B.

Theorem20. *There exists a constant C depending only on δ_B such that for every $g \in W^{m+1}_\infty(B)$ with $4 \le m \le 9$,*

$$\|D^\alpha(g - \mathcal{I}g)\|_B \le C|B|^{m+1-|\alpha|}|g|_{m+1,B}, \tag{10}$$

for all $0 \le |\alpha| \le m$.

Proof: The proof is similar to the proof of Theorem 6.2 in [8]. Fix $4 \le m \le 9$, and let $g \in W^{m+1}_\infty(B)$. By Lemma 4.3.8 of [3], there exists a polynomial $q := q_{g,B} \in \mathcal{P}_9$ such that for all $0 \le |\beta| \le m$

$$\|D^\beta(g - q)\|_B \le |(g-q)|_{|\beta|,B} \le K_1|B|^{m+1-|\beta|}|g|_{m+1,B}. \tag{11}$$

Using $\mathcal{I}q = q$, it is clear that

$$\|D^\alpha(g - \mathcal{I}g)\|_B \le \|D^\alpha(g-q)\|_B + \|D^\alpha \mathcal{I}(g-q)\|_B.$$

In view of (11), it suffices to estimate the second term. By the Markov inequality [12], applied to each of the polynomials $\mathcal{I}(g-q)|_{T_n}$, where $T_1, \ldots, T_{16}$ are the subsimplices in $\triangle_B$, see Definition 10, we have

$$\|D^\alpha \mathcal{I}(g-q)\|_{T_n} \le \left(\frac{64}{w_{T_n}}\right)^{|\alpha|} \|\mathcal{I}(g-q)\|_{T_n}, \tag{12}$$

where w_{T_n} is the shortest height of the simplex T_n, which implies

$$\|D^\alpha \mathcal{I}(g-q)\|_{T_n} \le K_2|B|^{-|\alpha|}\|\mathcal{I}(g-q)\|_{T_n},$$

where K_2 depends on δ_B.

Since the Bernstein basis polynomials $B_{ijklm}^{T_n}$ of degree nine form a partition of unity, it is easy to see that

$$\|\mathcal{I}(g-q)\|_{T_n} \le \max_{\xi \in \mathcal{D}_{9,T_n}} |c_\xi|,$$

where c_ξ are the associated B-coefficients of $\mathcal{I}(g-q)|_{T_n}$. We show below that there exists a constant K_3 that depends on δ_B such that

$$|c_\xi| \le K_3 \sum_{i=0}^{4} |B|^i |g-q|_{i,B}, \tag{13}$$

for all $\xi \in \mathcal{D}_{9,B}$. Inserting this in the above inequalities, taking the maximum over n, and using (11) leads immediately to (10).

To complete the proof, we now justify (13). First we consider c_ξ for ξ in the minimal determining set $\mathcal{M}$ described in Definition 12. As shown in the proof of Theorem 14, each of these coefficients can be determined from the nodal data (function values and derivatives at points on the faces of B). For most coefficients, this is a standard computation that leads to the bound (13). For example, see Theorem 8.2 of [8] for coefficients corresponding to domain points in disks of the form $D_4(v)$. As shown in the proof of Theorem 14, some of the coefficients are computed by solving linear systems, corresponding to fixed nonsingular matrices, and having a right-hand side which involves coefficients which have already been shown to satisfy (13). Thus, these coefficients also satisfy (13). Now the remaining coefficients of s are computed from the $\{c_\xi\}_{\xi \in \mathcal{M}}$ by smoothness conditions: single, double, triple or quadruple crosses, see Theorem 13, where a weighted average may need to be used instead of a simple one. Clearly, coefficients computed this way also satisfy (13). □

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