Our usual schedule

Monday though Friday:

10:00am – 1:00pm Presentations. Room 304.

2:00pm — 5:00pm Problem solving. Room 304.

Our schedule for the first four days

Mon, May 16

10:30am – 1:00pm Introduction. Room 304.

2:00pm — 5:00pm Problem solving. Room 304.

Tue, May 17

3:00pm — 6:00pm Presentations. Room 304.

Wed, May 18

3:00pm — 6:00pm Problem solving. Room 304.

Thur, May 19

8:15 am – 9:00 am Attending a conference talk. Student Learning Center. Room 171.

Important! Please, get all together before entering the room – there will be an earlier talk there. Do not enter until the first talk is over (usually people applaud in the end of the talk, you will hear it). Leave quietly after the talk.

12:30pm — 2:00pm Presentations. Room 304.

3:00pm — 6:00pm Problem solving. Room 304.

On Fri. we resume our usual schedule.

Problem 1. Given the quadratic polynomial $p(x) = x^2 - 2x + 7$, the functions $x^2, x, 1$ are called basis functions. The numbers $1, -2, 7$ are called the coefficients associated with the basis functions. Find out what meaning the coefficients have.

Hint: 7 is the value of $p(x)$ at the point $x = 0$.

Problem 2. Given the polynomial $p(x) = (x-1)^2 - 2(x-1) + 7$, the functions $(x-1)^2, x-1, 1$ are the basis functions. The numbers $1, -2, 7$ are the coefficients associated with the basis functions. Find out what meaning the coefficients have.

Problem 3. The polynomial $p(x) = (x-1)^2 - 2(x-1) + 7$ can be also written as $p(x) = x^2 - 4x + 10$ using a different basis: $x^2, x, 1$. Thus, the basis is not unique. Write down the same polynomial $p(x)$ in tems of some third basis. What do all three sets of basis functions have in common?

Problem 4. Given the polynomial

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x + a_0,$$

of degree d, the functions

$$x^d, x^{d-1}, \ldots, x, 1$$

are called basis functions. The numbers

$$a_d, a_{d-1}, \ldots, a_1, a_0$$

are called the coefficients associated with the basis functions. Find out what meaning the coefficients have. How many derivatives does $p(x)$ have?

Problem 5. The derivative from the left is:

$$f'_- := \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}.$$

The derivative from the right is:

$$f' + - := \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}.$$

For a continuous function f, $f'(x)$ exists if and only if both derivatives from the right and from the left exist, and $f'_- = f'_+$. Moreover, then $f'(x) = f'_- = f'_+$. The piecewise polynomial function is defined as

$$s(x) := \left\{ \begin{matrix} x, & x \in [0,1] \\ 2-x, & x \in (1,2] \end{matrix} \right\}.$$

Find $f'_-(1)$, $f'_+(1)$, $f'(1)$ using the definitions above.

Problem 6. A point x on the coordinate line has one Cartesian coordinate, which can be geometrically interpreted as the distance from the origin with the plus sign if the point is to the right of the origin, and with the minus sign if the point is from the left of the origin. That is, the Cartesian coordinate is defined with respect to the origin. If one moves the origin, the Cartesian coordinate of a fixed point will change.

The barycentric coordinates with respect to a segment $[x_1, x_2]$ of a point x are defined as follows:

$$\begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}.$$

The solution (b_1, b_2) of this system of equations gives two barycentric coordinates b_1, and b_2 of the point x.

The definition above is "very complicated" but not entirely useless. We learn that the point x has two barycentric coordinates (cf. only one Cartesian coordinate). The barycentric coordinates are associated with a segment (cf. the Cartesian coordinate is associated with the origin – one point).

1). Prove that the system of two equations given above always have a solution, i.e. that barycentric coordinates are well defined.

2). Find the barycentric coordinates of the end points of $[x_1, x_2]$, and of the midpoint of the segment.

3). Consider a specific segment $[1, 4]$. Find the barycentric coordinates of the following points: $x = 1, 4, 2, 3, 0, 5$.

4). Find a geometric interpretation of barycentric coordinates similar to the one for Cartesian coordinates described in the beginning of the problem.

5). Prove this interpretation using the definition.

Problem 7. A function f is in the class $C^r[a,b]$ if all the derivatives of f of order $0 \le i \le r$ exist at every point $x \in [a,b]$ (the one-sided derivatives at the end points).

1) What is r for the function $s(x)$ in Problem 5? $s(x)$ is a polynomial spline of degree one and smoothness r (insert the correct r) defined on [0,2] with the knot $x = 1$.

2) Consider

$$s(x) := \left\{ \begin{matrix} x^2, & x \in [0,1] \\ -x^2 - x + 3, & x \in (1,2] \end{matrix} \right\}.$$

This is a quadratic (polynomial of degree 2) spline of smoothness r. Find the correct value for r.

Problem 8. Construct the polynomial spline $s(x)$ of degree two and smoothness one on the interval [2,6] with the knot at $x = 4$ satisfying the following properties:

$$s(2) = 0, \quad s(6) = 1, \quad s(4) = 2 \quad , s'(2) = 1.$$

Problem 9. Assume that we have a point with the Cartesian coordinate x and barycentric coordinates b_1, b_2 with respect to a segment $[x_1, x_2]$.

1). Find a formula for x in terms of b_1, b_2

2). Find a formula for b_1, b_2 in terms of x

3). Write down the polynomial $p(x) = x^2 + 3x - 1$ in terms of b_1, b_2

4). Write down the polynomial $p(b_1, b_2) = 2b_1^2 + b_1 b_2 - b_2^2$ in terms of x.

Problem 10. As we already know from Problem 4, a polynomial p of degree d can be written as

$$p(x) = \sum_{i=0}^{d} a_i x^i,$$

where $\{x^i\}_{i=0}^d$ are basis functions, and the numbers $\{a_i\}_{i=0}^d$ are the coefficients associated with the basis functions.

As we learn from Problem 8, p can be rewritten in terms of barycentric coordinates (how?). Thus, we will have different basis functions, and they will not have to unique (which problem shows it?). Similar to the case of Cartesian coordinates, where we usually choose one basis $\{x^i\}_{i=0}^d$ to work with, there is the most commonly used basis for barycentric coordinates – Bernstein basis:

$$B_{ij}^d(b_1, b_2) := \frac{d!}{i!j!} b_1^i b_2^j, \quad i + j = d,$$

where d is the degree of p. That is, p can be written as

$$p(b_1, b_2) = \sum_{i+j=d} c_{ij} B_{ij}^d(b_1, b_2),$$

where $\{B_{ij}^d\}_{i+j=d}$ are basis functions, and the numbers $\{c_{ij}\}_{i+j=d}$ are the coefficients associated with the basis functions.

1). The basis $\{B_{ij}^d\}_{i+j=d}$ should have the same number of functions as the basis $\{x^i\}_{i=0}^d$ (why?). Show it.

2). Write down the coefficients c_{ij} for the polynomial in Problem 10 (part 4).

3). Write down the coefficients c_{ij} for the polynomial in Problem 10 (part 3).

4). Graph B_{ij}^1 for all $i + j = 1$ with respect to the interval [1,2]. How would they look if we change the interval to [6,10]?

5). Prove that every polynomial $p(x) = \sum_{i=0}^{d} a_i x^i$ can be written as $p(b_1, b_2) = \sum_{i+j=d} c_{ij} B_{ij}^d(b_1, b_2)$.

6). Prove that every polynomial $p(b_1, b_2) = \sum_{i+j=d} c_{ij} B_{ij}^d(b_1, b_2)$ can be written as $p(x) = \sum_{i=0}^{d} a_i x^i$

Problem 11. Using the BB(Bernstein-Bezier) form of the polynomial p with respect to the segment $[x_1, x_2]$:

$$p(b_1, b_2) = \sum_{i+j=d} c_{ij} B^d_{ij}(b_1, b_2),$$

find the values of $p(x)$ at $x = x_1$ and $x = x_2$.

Problem 12. Find the derivative of $B^2_{ij}(b_1, b_2)$ with respect to x for all possible choices of i and j (three choices). Hint: use chain-rule.

Problem 13.
(a) Using the BB(Bernstein-Bezier) – form of the quadratic polynomial p with respect to the segment $[x_1, x_2]$:

$$p(b_1, b_2) = \sum_{i+j=2} c_{ij} B^2_{ij}(b_1, b_2),$$

find the derivative of $p(b_1, b_2)$ with respect to x.

(b) Using the BB(Bernstein-Bezier) form of the polynomial p with respect to the segment $[x_1, x_2]$:

$$p(b_1, b_2) = \sum_{i+j=d} c_{ij} B^d_{ij}(b_1, b_2),$$

find the derivative of $p(b_1, b_2)$ with respect to x.

(c) What is the value of $p'(x_1)$, $p'(x_2)$? Give explicit formulae for the coeff. $c_{d0}, c_{0d}, c_{d-1,1}, c_{1,d-1}$ in terms of the values and derivatives of p at the endpoints.

Problem 14.
(a) Find the maximum of $B^2_{ij}(b_1, b_2)$ on $[x_1, x_2]$.

(b) Find the the maximum of $B^d_{ij}(b_1, b_2)$ on $[x_1, x_2]$.

Problem 15.
(a) Show that $0 \leq B^d_{ij}(b_1, b_2) \leq 1$ on $[x_1, x_2]$ for all possible choices of i and j.

(b) Show that $\sum_{i+j=d} B^d_{ij}(b_1, b_2) = 1$. Hint: use the binomial expansion.

Problem 15*. Associated with the BB-form of p we define the set of domain points to be

$$\mathcal{D}_{d,[x_1,x_2]} := \{\xi_{ij} = (ix_1 + jx_2)/d\}_{i+j} = d.$$

We can now think of the coefficients of a polynomial written in B-form as of the domain points on the segment $[x_1, x_2]$.

(a) Depict the domain points for $d = 1, 2, 3, 4$ on the segment $[x_1, x_2]$.

(b) Consider the segment $[0, 6]$ and three pieces of polynomials on the subsegments $[0, 2], [4, 2], [4, 6]$. Depict the domain points for each one.

Problem 16. Consider the partition of the segment $[0, 6]$ described in Problem 15 (b), and a spline $s \in \mathcal{S}_3^r$ on it. Let $\{c_{ij}\}$ be the coeff. of the first polynomial on $[0, 2]$, $\{\tilde{c}_{ij}\}$ be the coeff. of the second polynomial on $[4, 2]$, and $\{\mathbf{c}_{ij}\}$ be the coeff. of the third polynomial on $[4, 6]$.

(a) Find the conditions on the coeff. of all three polynomials which guarantee that $r = 0$.

(b) Find the conditions on the coeff. of all three polynomials which guarantee that $r = 1$. Hint: the conditions are **very** simple.

Problem 17. Describe a step-by-step algorithm for setting up a qubic C^1 spline s on the segment $[a, b]$ with n equally spaced knots $x_0 = a, \dots, x_{n+1} = b$. Your input should be a, b, n, $n + 2$ values $f(x_i)$ at $x_0 = a, \dots, x_{n+1} = b$, and $n + 2$ values $f'(x_i)$ at $x_0 = a, \dots, x_{n+1} = b$. The output should be the value of $s(x)$ at any point $x \in [a, b]$. If you program this algorithm, skip the next problem.

Problem 18. Consider the segment $[0, 6]$ and two pieces of polynomials on the subsegments $[0, 2], [6, 2]$. Depict the domain points for each one. Answer the questions in Problem 16 for the new partition of the segment.

Problem 19. Given a bivariate polynomial

$$p(x, y) = \sum_{i+j=d} a_{ij} x^i y^j,$$

the set $\{x^i y^j\}_{i+j=d}$ forms a basis. How many basis functions are there? What about a trivariate polynomial? Polynomials of n variables? What is the meaning of the coefficients a_{ij}? (use Taylor expansion).

Theoretical Exporations: Vector Spaces and Dimensions

Read pp. 150–153

Problem 20. Without looking into your notes prove:

1) Proposition 1.2

2) Theorem 1.3

3) Corollary 1.4

Problem 21. Prove that the spaces of univariate and bivariate polynomials of degree d ($\mathcal{P}_d^1$, and $\mathcal{P}_d^2$) form vector spaces over the field of real numbers $\mathbb{R}$. Prove that polynomials of degree $n \leq d$ form subspaces of those vector spaces.

Problem 22. Let $\triangle_{[a,b]}$ be the the uniform partition of $[a, b]$ into $n + 1$ subintervals as in Problem 17. Prove that the space $\mathcal{S}_d^r(\triangle_{[a,b]})$ form vector spaces over the field of real numbers $\mathbb{R}$. Give obvious lower and upper bounds for the dimension of $\mathcal{S}_d^r(\triangle_{[a,b]})$. (Hint: use subspaces)

Problem 23. Prove that the monomials form bases for $\mathcal{P}_d^1$ and $\mathcal{P}_d^2$.

Problem 24. Prove that Bernstein polynomials B_{ij}^d form a basis for $\mathcal{P}_d^1$. (Hint: use the Problems solved earlier).

Problem 25. Let $\triangle$ be a partition of $[a, b]$ into subintervals, and $\mathcal{S}$ a space of polynomial splines defined on $\triangle$. Suppose $\Gamma \subset \mathcal{D}_{d,\triangle}$ a subset of the set of all domain points is such that if $s \in \mathcal{S}$ and $c_\xi = 0$ for all $\xi \in \Gamma$, then $s \equiv 0$. Then we say that Γ is a minimal determining set (MDS) for $\mathcal{S}$. If $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$ and $\mathcal{M}$ has the smallest cardinality, then we call $\mathcal{M}$ a minimal determining set.

1). Prove that the cardinalty of $\mathcal{M} < \infty$.

2). Assume that $\triangle$ is a uniform partition of $[a, b]$ as in Problem 17. Find MDS for $\mathcal{S}_d^0(\triangle)$ and $\mathcal{S}_3^1(\triangle_)$ Hint: for the latter, use Problem 17.

Problem 26. Without looking into your notes prove Theorem 6.8.

Problem 27. Find the dimensions of $\mathcal{S}_d^0(\triangle)$ and $\mathcal{S}_3^1(\triangle)$.

Problem 28. Find the dimension of $\mathcal{S}_2^1(\triangle)$. Hint: find an MDS.

Problem 29. Using the same reasoning as in Problem 28, derive the formula for the dimension of $\mathcal{S}_d^1(\triangle)$ for an arbitrary d.

Problem 30. Using the same reasoning as in Problem 29, derive the formula for the dimension of $\mathcal{S}_d^r(\triangle)$ for an arbitrary d and r. It is a highly nontrivial problem if you consider Cartesian coordinates. However, very easy for barycentric ones. Hint: set the coeff. corresponding to the domain points in the first interval. Move on to the second one using C^r, and so on. This will give you an MDS.

Problem 31. Suppose T is a nondegenerate triangle in $\mathbb{R}^2$ with vertices

$$v_i := (x_i, y_i), \quad i = 1, 2, 3.$$

We will write $T = \langle v_1, v_2, v_3 \rangle$. Let $v = (x, y)$, and

The barycentric coordinates with respect to a triangle T of a point $v = (x, y)$ are defined as follows:

$$\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

The solution (b_1, b_2, b_3) of this system of equations gives three barycentric coordinates b_1, b_2 and b_3 of the point v.

1). Prove that the system of three equations given above always have a solution, i.e. that barycentric coordinates are well defined. Hint: the 3×3 matrix is nonsingular due to its geometric interpretation.

2). Find the barycentric coordinates of the vertices v_1, v_2, v_3 of the triangle T, and of the midpoint of the segment $[v_2, v_3]$.

4). Find a geometric interpretation of barycentric coordinates. Hint: solve the system using Cramer's rule

Problem 32. The point with barycentric coordinates (1/3,1/3,1/3) is called the barycenter of T. Where is it located?

Problem 33. Prove that for each $i = 1, 2, 3$, the function b_i is a linear polynomial in x and y which assumes the value 1 at the vertex v_i and vanishes at all points on the edge of T opposite to v_i.

Problem 34. Prove that a point v lies in the interior of T if and only if all three of its barycentric coordinates are positive. Explain when each one is negative or zero.

Problem 35. Bernstein basis polynomials of degree d relative to T are:

$$B^d_{ijk} := \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k, \quad i + j + k = d.$$

Trinomial expansion formula:

$$1 = (b_1 + b_2 + b_3)^d = \sum_{i+j+k=d} \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k.$$

Show that $0 \le B^d_{ijk}(b_1, b_2, b_3) \le 1$ for all v in the triangle T. Sketch B^1_{ijk} for all possible i, j, k.

Problem 36. Prove that the set $\{B^d_{ijk}\}_{i+j+k=d}$ is a basis for $\mathcal{P}^2_d$. Hint: use the dimension argument and induction on d.

Problem 36*. Given an arbitrary triangle T with the vertices A, B, C, let O be the midpoint of AC. Let D be the point on AB such that $AD = 3DB$, and E be the point on CB such that $CE = 3EB$. Find the barycentric coordinates of D with respect to the triangle BOE.

Problem 37. Given a vector $\langle u_x, u_y\rangle$, the associated directional derivative of a function f at the point (x_0, y_0) is defined by

$$D_{\langle u_x,u_y\rangle} f(x_0, y_0) := \frac{d}{dt} f(x_0 + tu_x, y_0 + tu_y)|_{t=0}.$$

In particular, $\partial f/\partial x = D_{\langle 1,0\rangle} f = D_x f$, and $\partial f/\partial y = D_{\langle 0,1\rangle} f = D_y f$. Let

$$f := \begin{Bmatrix} x^2 + y, & x \geq 0 \\ -x^2 + y, & x \leq 0 \end{Bmatrix}.$$

(a) Using the definition above find $D_{\langle 1,0\rangle} f(0, 0)$,$D_{\langle -1,0\rangle} f(0, 0)$, $D_{\langle 0,1\rangle} f(0, 0)$, $D_{\langle 0,-1\rangle} f(0, 0)$, $D_{\langle 2,4\rangle} f(0, 0)$.

(b) Is f C^1 at $(0, 0)$?

Problem 38. Assume f is C^1 at the point (x_0, y_0). Then, it is well-known from calculus that

$$D_{\langle u_x,u_y\rangle} f(x_0, y_0) = u_x D_x f(x_0, y_0) + u_y D_y f(x_0, y_0).$$

(a) Let $D_x f(x_0, y_0) = 2$, $D_y f(x_0, y_0) = -3$. Find $D_{\langle -1,2\rangle} f(x_0, y_0)$.

(b) Let $D_{\langle -1,2\rangle} f(x_0, y_0) = 3$, $D_{\langle 0,3\rangle} f(x_0, y_0) = 2$. Find $D_x f(x_0, y_0)$ and $D_y f(x_0, y_0)$.

Problem 39. Construct an example of a function that has both partial derivatives at the origin, but is not continuous at the origin. Notice, that it would not be possible for univariate functions.

Problem 40. Every bivariate polynomial of degree d can be written in its BB-form with respect to a triangle $T := \langle u, v, w\rangle$:

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}.$$

The set of domain points is defined by

$$\mathcal{D}_{d,T} := \{\xi_{ijk} = \frac{iu + jv + kw}{d}, \quad i + j + k = d\}.$$

Notice that each domain point ξ_{ijk} corresponds to a coefficient c_{ijk}. Draw the sets of domain points for polynomials of degree 1,2,3,4. Show that $p(u) = p(\xi_{d00}) = c_{d00}$. How about $p(v), p(w)$?

Problem 41. Prove that if p is a polynomial of degree d, then

$$D_{\langle u_x,u_y\rangle} p = d(\sum_{i+j+k=d-1} ac_{i+1,j,k} B^{d-1}_{ijk} + bc_{i,j+1,k} B^{d-1}_{ijk} + ec_{i,j,k+1} B^{d-1}_{ijk}),$$

where (a, b, e) are the barycentric coordinates of the vector $\langle u_x, u_y\rangle$ with respect to T.

Problem .

Suppose you are given the coeff. associated with the domain points 0,1,2,3,4, and 6. You are also given the Cartesian coordinates of all three vertices, and m_y – the directional derivative at m – the midpoint of $\langle v_1, v_2\rangle$ in the unit direction perpendicular to $\langle v_1, v_2\rangle$. 1) Evaluate $D_{\langle v_1,v_2\rangle}p(m)$ from the univariate polynomial on $\langle v_1, v_2\rangle$. Divide it by the length $\langle v_1, v_2\rangle$. This will give you $D_x p(m)$, if you assume that $\langle v_1, v_2\rangle$ points in the direction of x–axis. Then $m_y = D_y p(m)$

2) Using $D_x p(m)$ and m_y evaluate $D_{\langle m,v_3\rangle}p(m)$.

3) Reevaluate $D_{\langle m,v_3\rangle}p(m)$ using the formula for the derivative of p in B-form. This should give you the coeff. associated with domain point 5.

Problem .

Suppose you are given the coeff. associated with the domain points 0,1,2,3,4,5 and 8. You are also given the Cartesian coordinates of all three vertices, m_y^1 – the directional derivative at domain point 3 in the unit direction perpendicular to $\langle v_1, v_2\rangle$, and m_y^2 – the directional derivative at domain point 1 in the unit direction perpendicular to $\langle v_1, v_2\rangle$, Show how to evaluate the coeff. associated with domain points 6 and 7. You may skip the conversion from D_x, D_y to directional derivatives, i.e., assume that the directional derivatives that you need are given.

Problem . The partition $\triangle$ of the plane shown below is known as type-2 triangulation. Suppose the values at the points where the diagonals intersect are given (black dots). Your goal is to construct a spline $s \in S_2^1(\triangle)$. The "mask" below shows how to evaluate the coeff. corresponding to the domain point in the middle of each edge of the rectangle. Build masks for evaluating the remaining 5 coeff. of each polynomial piece, i.e. two corners of the rectangle, the point of the intersection of the diagonals, and two more midedges from the corners of the rectangle to the the intersection of the diagonals. Hint: The coeff. at the intersection of the diagonals is not equal to the given value, i.e., s is not an interpolating spline. Show that all C^1 conditions are satisfied.