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A multivariate Powell–Sabin interpolant

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Abstract We consider the problem of constructing a C^1 piecewise quadratic interpolant, Q, to positional and gradient data defined at the vertices of a tessellation of n-simplices in $\mathbb{R}^n$. The key to the interpolation scheme is to appropriately subdivide each simplex to ensure that certain necessary geometric constraints are satisfied by the subdivision points. We establish these constraints using the Bernstein–Bézier form for polynomials defined over simplices, and show how they can be satisfied. When constructed, the interpolant Q has full approximation power.

Keywords Piecewise quadratic interpolation · Bernstein–Bézier form · Powell–Sabin · n-simplices

Mathematics Subject Classifications (2000) Primary 41A05 · 41A10 · 65D05 · 65D07 · Secondary 41A15

1 Introduction

We consider the problem of constructing a C^1 piecewise quadratic interpolant, Q, to positional and gradient data defined at the vertices of a tessellation of n-simplices in $\mathbb{R}^n$, where by n-simplex we mean the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$. In practice, the tessellation of the data points *may* be prescribed as part

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of the problem. More commonly, however, only the location of the data points and the associated positional and gradient data are prescribed and the tessellation must be created so as to provide the domain over which the interpolant is defined.

Currently, there are two local polynomial spline schemes for interpolating positional and gradient data in $\mathbb{R}^n$: the Clough–Tocher interpolant constructed in [11], and tensor-product splines. The Clough–Tocher interpolant uses piecewise cubic polynomials defined over a tessellation of n-simplices in $\mathbb{R}^n$. In contrast to our scheme, the Clough–Tocher interpolant either requires sampling (estimating) additional gradient data at an arbitrary interior point of every edge of each simplex or imposing constraints on the cross-boundary derivative. Tensor-product cubic C^1 splines solve a Hermite problem on gridded data and in many practical applications including PDEs and medical imaging, the data in $\mathbb{R}^n$ are indeed prescribed on a regular grid. However, the total degree of piecewise polynomials used in cubic tensor-product splines is 3^n but the approximation order of the tensor-product scheme is only four. Although one of the main advantages of tensor-product splines is their computational simplicity, for visualization of isoparametric surfaces or for ray-casting in volume rendering, algebraic equations of high order have to be solved using numerical methods (see [7] and references therein). In the case of the Powell–Sabin interpolant, this can be accomplished by solving quadratic equations explicitly.

Despite being subject to some geometric constraints, the Powell–Sabin interpolant presented in this paper is a smooth polynomial spline of the lowest possible degree solving a Hermite interpolation problem in $\mathbb{R}^n$ which has full approximation power.

For $n = 2$, the well known Powell–Sabin element was constructed in [6]. The interpolation scheme subdivides each domain triangle into 6 subtriangles by joining an interior point to each vertex as well as to a suitably chosen point on each edge. To achieve C^1 continuity, the interior point chosen for an edge in the tessellation must lie on the line segment joining the interior points for the triangles that share that edge. This geometric constraint can always be satisfied. Higher degree schemes with C^r, $r > 1$, smoothness have been considered in [1] and [4].

Worsey and Piper in [12] extended this result to the case $n = 3$. Each tetrahedron, T, in the tessellation is subdivided into 24 subtetrahedra by choosing an interior point for each edge, and each face of T, and then joining a point on the interior of T to each point chosen on its boundary. Again, to achieve C^1 continuity, the points selected must satisfy a restrictive set of geometric constraints: certain sets of points must be collinear and other sets must be coplanar. It is shown that these constraints can be satisfied in many cases important for practical applications. However, it remains an open question whether these constraints can be satisfied for an arbitrary tessellation.

In this paper we generalize that result to $\mathbb{R}^n$. The key to solving the interpolation problem is to split the n-simplices in a tessellation appropriately. The geometric constraints on the splitting process are more involved than in the 3-dimensional case. It is an open and difficult problem whether or not these constraints can be satisfied for an arbitrary tessellation or even a Delaunay tessellation. We show how to satisfy the constraints for constructing the interpolant Q in cases where the data points form a regular lattice, which is often the case in practical applications. For non-gridded data, we show that the constraints can be satisfied by choosing the subdivision point of every face of each simplex to be its circumcenter, in cases where the circumcenters are interior to the face.

In Section 2 of the paper we describe the generic strategy for splitting an n-simplex S^n in order to admit the possibility of constructing the interpolant Q. Then in Section 3 the coefficients for the C^0 piecewise quadratic over the generic split so as to interpolate the given data are explicitly defined.

In Section 4 we establish the constraints that need to be satisfied by the split points in order for the interpolant Q to be C^1 over a single n-simplex S^n. This leads to a definition for the generic split to be Powell–Sabin.

In Section 5 we extend the results of Section 4 to consider the interpolation problem over a union of simplices in $\mathbb{R}^n$. In this case the solution is further constrained since the geometric constraints in the Powell–Sabin split of a single n-simplex impinge upon the geometry of the split for neighboring simplices. Again, we leave as an open question the matter of how they may be satisfied for an arbitrary prescribed tessellation of n-simplices, but we do consider, in Section 6, two important special cases of practical interest. Finally, in Section 7, we prove that the Powell–Sabin interpolant Q has full approximation power.

In our development, we use the Bernstein–Bézier form for polynomials defined over simplices. It is well-known that this form admits a geometric analysis of the smoothness of piecewise polynomials via the control nets (see [2]), and this is a key element in our analysis.

2 The generic split for an n-simplex

Before defining the generic split, some terminology and results regarding triangulations in $\mathbb{R}^n$ will be needed.

Definition 2.1 Let $v_1, \ldots, v_m$ be m points in $\mathbb{R}^n$. The linear combination

$$\lambda_1 v_1 + \cdots + \lambda_m v_m \quad \text{with} \quad \lambda_1 + \cdots + \lambda_m = 1$$

is called an affine combination of $v_1, \ldots, v_m$.

Definition 2.2 A set of points in $\mathbb{R}^n$ is called affinely independent if none of the points can be written as an affine combination of the others.

Let $V := \{v_0, \ldots, v_m\} \subset \mathbb{R}^n$ be affinely independent, and denote the convex hull of V by $[V]$ or, equivalently, by $[v_0, \ldots, v_m]$. Then $S^m := [v_0, \ldots, v_m]$ is an m-(dimensional) simplex, whose set of vertices is given by V. In this paper we will be primarily concerned with n-dimensional simplices in $\mathbb{R}^n$. In that context, the use of barycentric coordinates is important.

Definition 2.3 Let $v_0, \ldots, v_n$, be $n+1$ affinely independent points in $\mathbb{R}^n$, and let u be an arbitrary point in $\mathbb{R}^n$. The numbers $b_0, \ldots, b_n$, satisfying

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ v_0 & v_1 & \ldots & v_n \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = \begin{pmatrix} 1 \\ u \end{pmatrix} \tag{2.1}$$

are called barycentric coordinates of u relative to the simplex $[v_0, \ldots, v_n]$.

We note that the $(n+1) \times (n+1)$ matrix on the left side of (2.1) has a nonzero determinant since the points $v_0, \ldots, v_n$ are affinely independent, and it immediately follows from the definition that $b_0 + \cdots + b_n = 1$. The point with all equal barycentric coordinates is called the barycenter of the simplex $[v_0, \ldots, v_n]$.

Since our focus is interpolation over a union of simplices, it is first of all necessary to establish precisely the meaning of a triangulation in $\mathbb{R}^n$. Citing [5], we use the following definition in which facet refers to an $(n-1)$-dimensional face of an n-simplex.

Definition 2.4 Let $\mathcal{P}$ be a set of m distinct points in $\mathbb{R}^n$ $(n \geq 1,\ \ m \geq n+1)$. Assume that the points in $\mathcal{P}$ do not lie entirely in a hyperplane. A triangulation, $\triangle$, of $[\mathcal{P}]$ is a set of non-degenerate n-dimensional simplices, $\{T_i\}$, with the following properties:

(a) All vertices of each simplex are members of $\mathcal{P}$.
(b) The interiors of the simplices are pairwise disjoint.
(c) Each facet of a simplex is either on the boundary of $[\mathcal{P}]$, or else is a common facet of exactly two simplices.
(d) Each simplex contains no points of $\mathcal{P}$ other than its vertices.
(e) The union of $\{T_i\}$ is $[\mathcal{P}]$.

Now we are ready to describe the generic split of an n-simplex. It can be done recursively by systematically increasing the dimension by one, from 1 to n.

Definition 2.5 The generic split $\triangle(S^1)$ for a line segment S^1 is obtained by choosing any interior point of S^1. The split $\triangle(S^1)$ consists of the two subsegments. For a simplex S^n, $n \geq 2$, suppose each facet is subdivided into $n!$ subsimplices using the generic split. Then to form the generic split $\triangle(S^n)$ of S^n,

(1) Select an interior point in S^n,
(2) Join this interior point to all the points used in splitting every boundary face to form the edges of the subsimplices.

The generic split $\triangle(S^n)$ subdivides S^n into $n!(n+1) = (n+1)!$ subsimplices. This definition makes clear the recursive nature of the splitting process and its geometric construction. Figure 1 (left) shows the generic split $\triangle(S^2)$. In Fig. 1 (right) we show the edges of the twelve subtetrahedra built on the two visible faces of S^3.

However, Definition 2.5 has two drawbacks for constructing our interpolation scheme. First, it is not constructive. In order to analyze the piecewise quadratic interpolation scheme over the generic split in Section 3, we need to be able to identify each subsimplex in $\triangle(S^n)$. Second, it needs to be justified that the algorithm in Definition 2.5 indeed produces a unique n-dimensional triangulation, see Definition 2.4.

As shown in [5], n-dimensional triangulations are a quite delicate matter. In particular, in dimensions four and higher two possible triangulations may be indistinguishable from each other based on only information about connectivity of pairs of points. Although this is not the case for $\triangle(S^n)$, and Definition 2.5 does describe a unique n-dimensional triangulation of S^n, throughout the paper we work with a

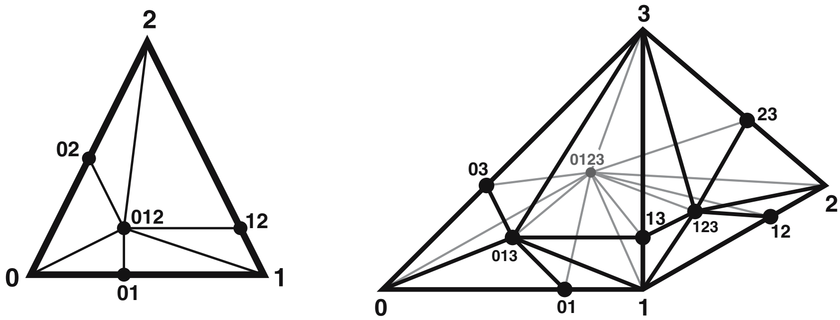

Fig. 1 The generic splits for a triangle and tetrahedron. On the *right*, the split for the two visible faces of the tetrahedron only is shown, namely $[v_0, v_1, v_3]$ and $[v_1, v_2, v_3]$. The interior points for the faces are v_{013} and v_{123}. The *interior* point for the tetrahedron v_{0123} is joined to all points on the boundary. All the points are labeled with their subscript indices only

different definition that makes use of multi-index notation for faces of a simplex and the points using for splitting them. The faces of S^n are formed by subsets of its $n+1$ vertices, and are simplices themselves, see [3].

Definition 2.6 Let $V := \{v_0, \ldots, v_n\}$ be affinely independent. A k-(dimensional) face $F_I := [v_{i_0}, \ldots, v_{i_k}]$ is the convex hull of $k+1$ distinct vertices $\{v_{i_j}\}_{j=0}^{k} \subset V$, where $I := (i_0, \ldots, i_k)$ is the standard multi-index notation and, by assumption, $i_0 < i_1 < \cdots < i_k$.

For brevity, we refer to k-dimensional faces as k-faces. In particular, a 0-face is a vertex of S^n, and the n-face is S^n itself. As usual, by $|I|$ we denote the number of entries in the multi-index I.

Given two multi-indices $I := (i_0, \ldots, i_k)$ and $J := (j_0, \ldots, j_m)$, we say that $I \subset J$ if $\{i_0, \ldots, i_k\} \subset \{j_0, \ldots, j_m\}$. Clearly, $I \subset J$ if and only if F_I is a k-face of the m-simplex F_J. Additionally, we introduce the complete set of multi-indices describing all faces of S^n:

$$\mathcal{I} := \{I, \text{ such that } F_I \text{ is a } k\text{-face of } S^n \text{ for all } k = 0, \ldots, n\}. \tag{2.2}$$

Definition 2.7 For each $I \in \mathcal{I}$, let v_I be an arbitrary point strictly interior to F_I (where F_I is as in Definition 2.6), with the assumption that for $|I| = 1$, v_I is the vertex F_I. The set of $(n+1)!$ simplices

$$\mathcal{V} := \Big\{[v_{I_0}, \ldots, v_{I_n}], \text{ for all } I_k \in \mathcal{I}, \text{ such that } |I_k| = k+1 \\ \text{for } k = 0, \ldots, n, \text{ and } I_k \subset I_{k+1} \text{ for } k = 0, \ldots, n-1\Big\}, \tag{2.3}$$

forms the generic split $\triangle(S^n)$ of S^n.

Clearly, Definition 2.7 needs to be justified. As a tool in proving that Definition 2.7 provides a triangulation of S^n we will use the concept of signature sets introduced by Lawson in [5], where the following theorem is proved.

Theorem 2.8 *Let $\mathcal{P}$ be a set of $n+2$ points, $v_1, \ldots, v_{n+2}$, in $\mathbb{R}^n$ not lying entirely in any hyperplane. There is a partitioning of the index set $\{1, 2, \ldots, n+2\}$ into three signature sets, P_0, P_1, and P_2, and a set of numbers, d_i, satisfying*

$$\begin{aligned} \sum_{i\in P_1} d_i v_i &= \sum_{i\in P_2} d_i v_i, \\ \sum_{i\in P_1} d_i &= \sum_{i\in P_2} d_i = 1, \\ d_i &= 0, \; i \in P_0, \\ d_i &> 0, \; i \in P_1 \cup P_2. \end{aligned} \tag{2.4}$$

The numbers d_i are uniquely determined by the set $\mathcal{P}$. The sets P_0, P_1, P_2 are also unique, with the understanding that the labeling of P_1 and P_2 could be arbitrarily interchanged.

Since P_1 and P_2 are not mutually distinguished we use P_1 to label the smaller of the two sets when they are not of the same size. For each $i = 1, \ldots, n+2$, let T_i denote the simplex formed using the points $\mathcal{P} \setminus v_i$ as vertices. The following result immediately follows from Theorem 2 of [5].

Theorem 2.9 *Let $\mathcal{P}$ be as in Theorem 2.8. If $|P_1| = 1$ then there exists a unique triangulation of $[\mathcal{P}]$, namely $\{T_i : i \in P_2\}$.*

Now we are ready to prove that the generic split $\triangle(S^n)$ is a triangulation.

Theorem 2.10 *The set of simplices $\mathcal{V}$ defined in (2.3) forms an n-dimensional triangulation of S^n for $n \geq 1$.*

Proof The proof proceeds by using a recursive argument. We first select a point strictly interior to S^n and use it to cone off to the vertices of S^n. This creates $n+1$ subsimplices. For each of them we then choose a point strictly interior to the facet which is also a facet of S^n, and use it to cone off to the vertices of that subsimplex, thereby creating n subsimplices. This process is now simply applied recursively. At each level we choose a point strictly interior to an original boundary k-face of S^n for $k = n-1, \ldots, 0$, and cone off to the vertices of the subsimplex obtained at the previous level. Whilst the process is the same at each level, we choose to describe it in three steps for clarity.

Step 1. Let $v_{0,\ldots,n}$ be an arbitrary point strictly interior to S^n. In terms of the definition of $\mathcal{V}$ in (2.3), since $|I_n| = n+1$, we set $v_{I_n} := v_{0,\ldots,n}$. Then

$$v_{I_n} = \sum_{j=0}^{n} \alpha_j^{I_n} v_j, \quad \sum_{j=0}^{n} \alpha_j^{I_n} = 1, \quad \alpha_j^{I_n} > 0, \; j = 0, \ldots, n, \tag{2.5}$$

where $\{\alpha_j^{I_n}, \; j = 0, \ldots, n\}$ are the barycentric coordinates of v_{I_n} relative to S^n. Now we apply Theorem 2.8 to the set $\mathcal{P}_0 := \{v_0, v_1, \ldots, v_n, v_{I_n}\}$ of $n+2$ points with the index set $\{0, \ldots, n, I_n\}$. Comparing (2.4) with (2.5), we conclude that the signature set P_0 is empty, the signature set P_1 consists of one element, I_n, and $P_2 = \{0, \ldots, n, \}$, with $d_j = \alpha_j^{I_n}$ for $j \in P_2$. Thus, by Theorem 2.9, there exists a unique triangulation of S^n into the $n+1$ simplices

$$\mathcal{B}_1 := \left\{ \left[v_{i_0}, \ldots, v_{i_{n-1}}, v_{I_n} \right], \text{ for all distinct } i_j \in P_2 \right\}.$$

We may denote $(i_0, \ldots, i_{n-1})$ by the multi-index I_{n-1}, since $(i_0, \ldots, i_{n-1}) \subset I_n$, and it contains n entries. Each of the simplices in $\mathcal{B}_1$ will be split in the same manner, so that without loss of generality we may choose one of them, namely

$$S_1^n := \left[v_0, \ldots, v_{n-1}, v_{I_n} \right], \text{ with } I_{n-1} := (0, \ldots, n-1),$$

to describe the next step in the process. We note that $F_{I_{n-1}} =: [v_0, \ldots, v_{n-1}]$ is the facet of S^n to be split in the next step.

Step 2. To proceed with the splitting, let $v_{0,\ldots,n-1}$ be an arbitrary point strictly interior to $F_{I_{n-1}}$. In terms of the definition of $\mathcal{V}$ in (2.3), since $|I_{n-1}| = n$ and $I_{n-1} \subset I_n$, we set $v_{I_{n-1}} := v_{0,\ldots,n-1}$. Then

$$v_{I_{n-1}} = \sum_{j=0}^{n-1} \alpha_j^{I_{n-1}} v_j, \quad \sum_{j=0}^{n-1} \alpha_j^{I_{n-1}} = 1, \quad \alpha_j^{I_{n-1}} > 0, \quad j = 0, \ldots, n-1, \tag{2.6}$$

where $\{\alpha_j^{I_{n-1}}, \ \ j = 0, \ldots, n-1\}$ are the barycentric coordinates of $v_{I_{n-1}}$ relative to $F_{I_{n-1}}$. Now we apply Theorem 2.8 to the set $\mathcal{P}_1 := \{v_0, \ldots, v_{n-1}, v_{I_{n-1}}, v_{I_n}\}$ of $n+2$ points with the index set $\{0, \ldots, n-1, I_{n-1}, I_n\}$. Comparing (2.4) with (2.6), we conclude that for $\mathcal{P}_1$ the signature set P_0 contains one element, I_n, the signature set P_1 consists of one element, I_{n-1}, and $P_2 = \{0, \ldots, n-1, \}$, with $d_j = \alpha_j^{I_{n-1}}$ for $j \in P_2$. Thus, by Theorem 2.9, there exists a unique triangulation of S_1^n into the n simplices

$$\mathcal{B}_2 := \left\{ \left[v_{j_0}, \ldots, v_{j_{n-2}}, v_{I_{n-1}}, v_{I_n} \right], \text{ for all distinct } j_i \in P_2 \right\}.$$

We may denote $(j_0, \ldots, j_{n-2})$ with $n-1$ entries by the multi-index I_{n-2}, since $(j_0, \ldots, j_{n-2}) \subset I_{n-1}$. For further consideration and without loss of generality we choose one simplex in $\mathcal{B}_2$, namely

$$S_2^n := \left[v_0, \ldots, v_{n-2}, v_{I_{n-1}}, v_{I_n} \right], \text{ with } I_{n-2} := (0, \ldots, n-2).$$

We note that $F_{I_{n-2}} =: [v_0, \ldots, v_{n-2}]$ is an $(n-2)$-dimensional boundary face of S^n, that will be split in the next step.

Step 3. Clearly, the procedure in *Step 2* can be carried out for as long as we can choose an arbitrary point strictly interior to the k-face obtained at the previous level of the construction. For $1 \le k \le n-1$, let

$$S_k^n := \left[v_0, \ldots, v_{n-k}, v_{I_{n-k+1}}, \ldots, v_{I_n} \right], \text{ with } I_{n-k} := (0, \ldots, n-k),$$

be the simplex chosen from the set $\mathcal{B}_k$ of $n-k+2$ simplices created at the previous level of recursion, and $F_{I_{n-k}} =: [v_0, \ldots, v_{n-k}]$ be the $(n-k)$-dimensional boundary face of S^n to be split. Now let $v_{0,\ldots,n-k}$ be a point strictly interior to $F_{I_{n-k}}$. In terms of the definition of $\mathcal{V}$ in (2.3), since $|I_{n-k}| = n-k+1$ and $I_{n-k} \subset I_{n-k+1}$, we set $v_{I_{n-k}} := v_{0,\ldots,n-k}$. Then

$$v_{I_{n-k}} = \sum_{j=0}^{n-k} \alpha_j^{I_{n-k}} v_l, \quad \sum_{j=0}^{n-k} \alpha_j^{I_{n-k}} = 1, \quad \alpha_j^{I_{n-k}} > 0, \quad j = 0, \ldots, n-k,$$

where $\{\alpha_j^{I_{n-k}}, \quad j = 0, \ldots, n-k\}$ are the barycentric coordinates of $v_{I_{n-k}}$ relative to $F_{I_{n-k}}$. Now applying Theorem 2.8 to the set $\mathcal{P}_k := \{v_0, \ldots, v_{n-k}, v_{I_{n-k}}, \ldots, v_{I_n}\}$ of $n+2$ points with the index set $\{0, \ldots, n-k, I_{n-k}, \ldots, I_n\}$, we conclude that $P_0 = \{I_{n-k+1}, \ldots, I_n\}$, the signature set P_1 consists of one element, I_{n-k}, and $P_2 = \{0, \ldots, n-k,\}$, with $d_j = \alpha_j^{I_{n-k}}$ for $j \in P_2$. Thus, by Theorem 2.9, there exists a unique triangulation of S_k^n into the $n-k+1$ simplices

$$\mathcal{B}_{k+1} := \left\{\left[v_{m_0}, \ldots, v_{m_{n-k-1}}, v_{I_{n-k}}, \ldots, v_{I_n}\right] \text{ for all distinct } m_i \in P_2\right\}.$$

We may denote the multi-index $(m_0, \ldots, m_{n-k-1})$ with $n-k$ entries by I_{n-k-1}, since $(m_0, \ldots, m_{n-k-1}) \subset I_{n-k}$.

The process terminates when we arrive at the simplex

$$S_n^n := \left[v_0, v_{I_1}, \ldots, v_{I_n}\right], \text{ with } I_0 := (0).$$

In terms of the definition (2.3), we set $v_{I_0} := v_0$, since $|I_0| = 1$ and $I_0 \subset I_1$, and a simple combinatorial argument shows that card $\mathcal{V} = (n+1)!$.The proof is complete. □

3 An interpolant over the generic split for an *n*-simplex

In constructing the interpolant Q over $\triangle(S_n)$, we will use the Bernstein–Bézier form for defining piecewise polynomial functions over a simplicial partition of a domain in $\mathbb{R}^n$. The underlying theory and details are discussed in [2] and [9]. For each subsimplex $T := [u_0, \ldots, u_n] \in \triangle(S^n)$, the quadratic polynomial piece p^T is given by

$$p^T = \sum_{i_0+\cdots+i_n=2} c^T_{i_0,\ldots,i_n} B^T_{i_0,\ldots,i_n}, \quad i_j \geq 0, \quad j = 0, \ldots, n, \tag{3.1}$$

where

$$B^T_{i_0,\ldots,i_n} = \frac{2}{i_0! \ldots i_n!} b_0^{i_0} \ldots b_n^{i_n}, \quad i_0 + \cdots + i_n = 2,$$

are the quadratic Bernstein polynomials associated with T. Here, $\{b_i\}_{i=0}^n$ are the barycentric coordinates relative to T. As usual, we associate the B-coefficients $c^T_{i_0,\ldots,i_n}$ of p with the domain points $\xi^T_{i_0,\ldots,i_n} := (i_0 u_0 + \cdots + i_n u_n)/2$ in T. The coefficient $c^T_{i_0,\ldots,i_n}$ associated with the domain point $\xi^T_{i_0,\ldots,i_n}$ will be also referred to as its ordinate, and the ordered pair in $\mathbb{R}^{n+1}$

$$\mathcal{C}^T_{i_0,\ldots,i_n} := \left(\xi^T_{i_0,\ldots,i_n}, c^T_{i_0,\ldots,i_n}\right), \quad i_0 + \cdots + i_n = 2, \tag{3.2}$$

will be called a control point.

For later use, we introduce some additional notation. Let $m \in \{0, 1, 2\}$ and let $0 \le k \le n$. The set of domain points

$$R_m^T(u_k) := \left\{ \xi^T_{i_0,\dots,i_{k-1},2-m,i_{k+1},\dots,i_n}, \quad \text{for all } i_j \text{ such that } \sum_{\substack{j=0\\ j\neq k}}^{n} i_j = m \right\}, \tag{3.3}$$

will be called the shell of radius m around the vertex u_k in T. The set of domain points

$$D_m^T(u_k) := \bigcup_{j=0}^{m} R_j(u_k), \tag{3.4}$$

will be referred to as the ball of radius m around the vertex u_k in T. Correspondingly, if u is a vertex of more than one simplex

$$R_m(u) := \{R_m^T(u), \quad \text{for all simplices } T \text{ having } u \text{ as a vertex}\},$$

$$D_m(u) := \{D_m^T(u), \quad \text{for all simplices } T \text{ having } u \text{ as a vertex}\},$$

will be referred to as the shell and the ball of radius m around u, respectively.

It is easy to derive conditions for a smooth join between two quadratic polynomials p and $\tilde{p}$ defined respectively on two simplices T and $\widetilde{T}$ with a common facet F. The smoothness conditions of order one involve only control points of p and $\tilde{p}$ associated with the domain points in balls of radius one around the vertices of F.

In order for p and $\tilde{p}$ to be C^1 across F, it is necessary and sufficient that it be C^1 at the vertices of F, and this admits the following geometric interpretation established in [2]:

Lemma 3.1 *Quadratic polynomials p and $\tilde{p}$, defined respectively on n-simplices T and $\widetilde{T}$ with a common facet F, join with C^1 continuity across F if and only if for every vertex u of F, the control points $\{(p, c_p),\ p \in D_1(u)\}$ lie in a (hyper)plane in $\mathbb{R}^{n+1}$.*

For the subsequent analysis of the interpolation scheme, it is useful to have explicit formulae for domain points in $\triangle(S^n)$ and their associated ordinates. A domain point is located either at a vertex of a subsimplex, that is at a split point v_I, or at the midpoint of a line segment connecting two vertices v_I and v_J, that is, at $(v_I + v_J)/2$. In fact, if we allow $I = J$, then in view of Definition 2.7, the set of all domain points in S^n is given by

$$\mathcal{D} = \{v_{I,J} := (v_I + v_J)/2, \quad \text{for all } I \subseteq J, \quad \text{where } \ I, J \in \mathcal{I}\}, \tag{3.5}$$

where $\mathcal{I}$ is as in (2.2). The following two lemmas provide obvious but useful formulae for the location of domain points.

Lemma 3.2 *Let v_I be the split point of the face $F_I := [v_{i_0}, \dots, v_{i_k}]$ as in Definition 2.7, and $\{\alpha_j^I,\ j = 0, \dots, k\}$ be the barycentric coordinates of v_I relative to F_I. Then, for any $J \supseteq I$*

$$v_{I,J} = \sum_{j=0}^{k} \alpha_j^I v_{i_j,J},$$

where $v_{i_j,J} = (v_{i_j} + v_J)/2$.

Proof Follows from (3.5) by substituting $v_I = \sum_{j=0}^{k} \alpha_j^I v_{i_j}$. □

Lemma 3.3 *Let v_I be the split point of the face F_I as in Definition* 2.7. *Then*

$$D_1(v_I) = \{v_{J,I},\ \ all\ J \subseteq I\} \cup \{v_{I,J},\ \ all\ J \supseteq I\}.$$

Proof The result immediately follows from Definition 2.7, and (3.5). □

Now we provide the algorithm for setting the ordinates of all the domain points in $\mathcal{D}$, thus uniquely defining the interpolant Q over $\triangle(S^n)$.

Definition 3.4 Let $f \in C^1(S^n)$, $\mathcal{D}$ be as in (3.5) and $\mathcal{V}$ be as in (2.3). Set

(1) $c_i := f(v_i),\ \ i = 0, \ldots, n;$
(2) $c_{i,J} := \frac{1}{2} D_{\langle v_i, v_J\rangle} f(v_i) + c_i,\ \ i = 0, \ldots, n,$ all $J \supset i$, *where* c_i is defined in (1), and $D_{\langle u,v\rangle} f$ is the derivative of f in the direction $\langle u, v\rangle$;
(3) $c_{I,J} := \sum_{j=0}^{k} \alpha_j^I c_{i_j,J}$, all $J \supseteq I$, all $|I| = k+1$, $k = 1 \ldots, n$, *where* $c_{i_j,J}$ is defined in (2), and α^I, and k are the same as in Lemma 3.2.

For each $T := [v_{I_0}, \ldots, v_{I_n}] \in \mathcal{V}$, the quadratic polynomial in (3.1) is

$$Qf(b_0, \ldots, b_n)|_T := 2 \sum_{k=0}^{n-1} \sum_{m=k+1}^{n} c_{I_k, I_m} b_k b_m + \sum_{k=0}^{n} c_{I_k, I_k} b_k^2,$$

where $b_0, \ldots, b_n$ are barycentric coordinates relative to T.

Since the ordinates of the domain points in the ball of radius one of every vertex of S^n are determined directly from the position and gradient data at that vertex, the next result follows immediately.

Lemma 3.5 *Let $f \in C^1(S^n)$. The interpolant Qf is continuous over $\triangle(S^n)$, and C^1 continuous at the vertices of S^n.*

In the remainder of the paper we will use Q and Qf interchangeably to represent the interpolation scheme and the result of the interpolation scheme. The difference is largely one of semantics.

4 Powell–Sabin split for an *n*-simplex

Let $\triangle(S^n)$ be the generic split of a simplex S^n as described in Definition 2.7. This definition puts no constraints on the choices for the interior points used in splitting S^n. However, in order to construct a piecewise quadratic C^1 interpolant over the generic split $\triangle(S^3)$ of S^3, certain geometric constraints must be satisfied by the interior points chosen selected for the split, see [12]. In this section we consider this issue for the n-dimensional case.

Definition 4.1 We call the generic split $\triangle(S^n)$, Powell–Sabin, if for $f \in C^1(S^n)$ the interpolant Qf of Definition 3.4 is C^1 continuous on $\triangle(S^n)$.

The following theorem provides sufficient conditions for $\triangle(S^n)$ to be Powell–Sabin.

Theorem 4.2 *The generic split $\triangle(S^n)$ is Powell–Sabin if for each $1 \le i \le n-1$, the interior point chosen for an i-dimensional face F is $(n-i)$-coplanar with the interior points chosen for all j-dimensional faces, $j = i+1, \ldots, n$, containing F as a face.*

Remark 4.3 The limits for i in Theorem 4.2 can in fact be set as $1 \le i \le n-2$, since for $i = n-1$, the only face containing the facet F is S^n itself, and two distinct points are always 1-coplanar (collinear).

Proof of Theorem 4.2 In order for the piecewise quadratic polynomial to be C^1 over $\triangle(S^n)$, it is necessary and sufficient that it be C^1 at the vertices of the $(n+1)!$ subsimplices, that is at every split point of $\triangle(S^n)$. Without loss of generality consider $v_I = v_{0,\ldots,i}$, which is the split point chosen for the i-face $F_I = [v_0, \ldots, v_i]$. For $i = 0$, the assertion follows from Lemma 3.5 since v_0 is a vertex of S^n.

We now consider $1 \le i \le n-1$. In order to apply Lemma 3.1, we need to show that the control points $\{(p, c_p),\ p \in D_1(v_I)\}$ are n-coplanar in $\mathbb{R}^{n+1}$. By Lemma 3.3

$$D_1(v_I) = W_I \cup U_I, \quad W_I := \{v_{J,I},\ \text{all } J \subseteq I\}, \quad U_I := \{v_{I,J},\ \text{all } J \supseteq I\}. \tag{4.1}$$

First we consider the domain points in W_I. For each $J = (j_0, \ldots, j_k) \subseteq I$, the face $F_J := [v_{j_0}, \ldots, v_{j_k}]$ has a split point v_J with the barycentric coordinates $\{\beta_m^J\}_{m=0}^k$. From Lemma 3.2 and Definition 3.4, the control point $(v_{J,I}, c_{J,I})$ can be written as an affine combination of $\{(v_{j_m,I}, c_{j_m,I})\}_{m=0}^k$

$$v_{J,I} = \sum_{m=0}^k \beta_m^J v_{j_m,I}, \quad c_{J,I} = \sum_{m=0}^k \beta_m^J c_{j_m,I}.$$

Since $J \subseteq I$, each control point (p, c_p), where $p \in W_I$, can be written as an affine combination of $\{(v_{j,I}, c_{j,I})\}_{j=0}^i$, and therefore,

$$\dim \langle (p, c_p),\ p \in W_I \rangle = i. \tag{4.2}$$

Next we consider the domain points in U_I. Under the hypothesis of the theorem, the domain points in U_I are $(n-i)$-coplanar, that is $\dim \langle U_I \rangle = n - i$. For each $0 \le m \le i$, we introduce the following shift of U_I:

$$U_I^m := U_I + \frac{v_m - v_I}{2} = \left\{ \frac{v_I + v_J}{2} + \frac{v_m - v_I}{2},\ \text{all } J \supseteq I \right\} = \{v_{m,J},\ \text{all } J \supseteq I\}.$$

The domain points in U_I^m are clearly $(n-i)$-coplanar as well. Moreover, since $U_I^m \subset D_1(v_m)$, and Q restricted to the convex hull $[U_I^m]$ is C^1 at v_m for $0 \le m \le i$, Lemma 3.1 implies that

$$\dim \langle (p, c_p),\ p \in U_I^m \rangle = n - i. \tag{4.3}$$

Also from Lemma 3.2 and Definition 3.4, for any $v_{I,J} \in U_I$, we have:

$$v_{I,J} = \sum_{m=0}^{i} \alpha_m^I v_{m,J}, \quad c_{I,J} = \sum_{m=0}^{i} \alpha_m^I c_{m,J}, \tag{4.4}$$

where $\{\alpha_m^I\}_{m=0}^i$ are the barycentric coordinates of the point v_I with respect to the face F_I. Since for each $0 \le m \le i$, the domain point $v_{m,J}$ is in U_I^m, the identities in (4.4) lead to

$$\langle (p, c_p),\ p \in U_I \rangle = \sum_{m=0}^{i} \alpha_m^I \ \langle (p, c_p),\ p \in U_I^m \rangle. \tag{4.5}$$

From (4.5) and (4.3) it follows that

$$\dim \langle (p, c_p),\ p \in U_I \rangle = n - i. \tag{4.6}$$

It remains to note that from (4.1), the intersection of the sets W_I and U_I contains the point v_I. Thus, (4.2) and (4.6) show that the control points in $\{(p, c_p),\ p \in D_1(v_I)\}$ are n-coplanar in $\mathbb{R}^{n+1}$ and, from Lemma 3.1, the proof is complete. □

Generic splits $\triangle(S^1)$ and $\triangle(S^2)$ are always Powell–Sabin, because the geometric constraints of Theorem 4.2 are automatically satisfied for $n = 1, 2$. The generic split $\triangle(S^3)$ is Powell–Sabin provided that in the splitting process the interior point chosen for each edge of S^3 lies on the plane determined by the interior points chosen for the two triangular faces sharing this edge and the interior point chosen for S^3 itself (cf. Theorem 3.2 in [12] and Fig. 2).

The geometric constraints of Theorem 4.2 can always be satisfied for a solitary n-simplex. If we arbitrarily select only the interior point $v_{0,1,\ldots,n} \in S^n$ as well as an interior point v_J for each of the $n+1$ facets F_J of S^n, then by recurring backwards, all other points needed to split lower dimensional faces of S^n are uniquely determined by the coplanarity conditions in the statement of Theorem 4.2. Consequently, the interpolation problem can always be solved on a single macro n-simplex S^n. In particular, we have the following result.

Theorem 4.4 *Let each v_I in Definition 2.7 be the barycenter of the associated face F_I. Then the split $\triangle(S^n)$ is Powell–Sabin.*

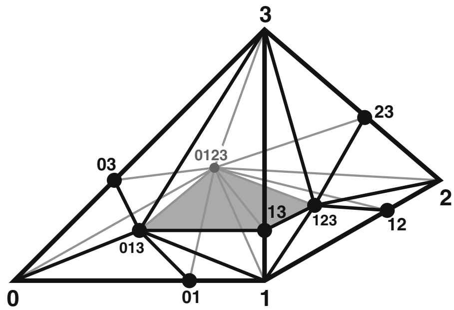

Fig. 2 The geometric constraint for a Powell–Sabin split of a tetrahedron T. For the two faces of T shown, the point v_{13} on the common edge must be coplanar with the points v_{013}, v_{123}, and v_{0123}. Similarly for all other edges of T. All the points are labeled with their subscript indices only

Proof Consider an i-face of S^n which, without loss of generality, we take to be $F_I := [v_0, \dots, v_i]$. Then $v_I := \frac{1}{i+1}\sum_{k=0}^{i} v_k$. For each $J \supset I$, the associated face F_J of S^n can be written as $[v_0, \dots, v_i, v_{l_1}, \dots, v_{l_m}]$, where $l_j \notin \{0, \dots, i\}$, and $1 \le m \le n-i$. The corresponding barycenter v_J can be written as

$$\frac{1}{i+m+1}\left[\sum_{k=0}^{i} v_k + \sum_{j=1}^{m} v_{l_j}\right] = \frac{b_0}{i+1}\sum_{k=0}^{i} v_k + \sum_{j=1}^{m}\frac{b_j}{i+2}\left[\sum_{k=0}^{i} v_k + v_{l_j}\right],$$

$$\text{where} \quad b_0 = \frac{(1-m)(i+1)}{i+m+1}, \quad b_j = \frac{i+2}{i+m+1}, \quad j = 0, \dots, m.$$

Since $b_0 + \cdots + b_m = 1$ each v_J can be written as an affine combination of at most $n-i+1$ points, namely v_I, $\{v_{0,\dots,i,l_j}\}$, $j = i+1, \dots, n$, and Theorem 4.2 applies. □

We conclude this section with a result on the cross-boundary derivative of the interpolant.

Theorem 4.5 *Given a Powell–Sabin split $\triangle(S^n)$ of S^n, let F_I be a facet of S^n, v_I be the split point in F_I, and $v := v_{0,\dots,n}$ be the split point in S^n as in Definition 2.7. If $f \in C^1(S^n)$, then on F_I the derivative of Qf in the direction $\langle v_I - v\rangle$ is a global linear polynomial in $n-1$ variables. This polynomial is uniquely determined by interpolation to the position and gradient data prescribed at the vertices of F_I.*

Proof The proof is straightforward and similar to those of Theorem 3.4 and Theorem 3.3 in [11] and [12], respectively. Without loss of generality, we let $F_I := [v_0, \dots, v_{n-1}]$. For simplicity, we denote $[v_0, \dots, v_{n-1}, v] \cap \triangle(S^n)$ by $\triangle_T$. To evaluate the derivative of Qf in the direction $\langle v_I - v\rangle$ on F_I one subtracts two piecewise linear Bézier subnets (of dimension $n-1$) from each other (see [2]). These come respectively from the ordinates on the interior of F_I and those in $R_1(v)|_{\triangle_T}$. The fact that the ordinates associated with the domain points on the boundary of F_I are not involved is a consequence of the particular directional derivative being considered. If we evaluate the barycentric coordinates of $\langle v_I - v\rangle$ relative to the subsimplices in $\triangle_T$, the coordinates corresponding to v are -1, the coordinates corresponding to v_I are 1, while all others are zeros. The two piecewise linear polynomials defined by these subnets are C^1, since Qf is C^1 on S^n. It is clear that a C^1 piecewise linear polynomial is a global linear polynomial. Therefore, the derivative of Qf in the direction $\langle v_I - v\rangle$ on F_I being the difference of those polynomials, is also a global linear polynomial in $n-1$ variables.

The last assertion in the statement of the theorem now follows immediately from Definition 3.4. □

5 The global interpolation scheme

The results of Section 4 mean that the interpolation problem posed in the introduction will only have a solution if the Powell–Sabin splitting strategy is carefully and systematically applied to all n-simplices in the triangulation so that the necessary geometric constraints from Theorem 4.2 are satisfied for each simplex. Moreover, we have to address the issue of C^1 continuity across the common boundary between macro n-simplices, and this will lead to additional geometric constraints when selecting the splitting points.

To begin, we consider the interpolation problem for two macro n-simplices S_1 and S_2 with a common facet B_{12}. When both S_1 and S_2 are subdivided using the generic splitting procedure of Definition 2.7, constrained so that the geometric conditions of Theorem 4.2 are satisfied, we can construct a unique piecewise quadratic interpolant Q which is C^1 on each macro-simplex separately. However, we need Q to be globally C^1. That is it must be continuously differentiable across the common facet B_{12}. We now consider this issue.

Theorem 5.1 *Let S_1, S_2 be two n-simplices with a common facet B_{12}. Let $\triangle(S_1)$ and $\triangle(S_2)$ be their respective Powell–Sabin splits such that $\triangle(S_1)|_{B_{12}} = \triangle(S_2)|_{B_{12}}$. Then for any $f \in C^1(S_1 \cup S_2)$ the interpolant Qf, and its first order derivative in any direction within B_{12}, are continuous functions over B_{12}.*

Proof On B_{12}, the given vertex data and the Powell–Sabin split are precisely the configuration for the $(n-1)$-dimensional interpolation problem. Therefore, from the analysis in Section 4, all the Bézier ordinates on B_{12} are uniquely determined by C^1 interpolation to the data. Since the data are common to both S_1 and S_2, and Qf is C^1 on each, the result follows. □

This result now places a rather significant restriction on how the points used in the Powell–Sabin split of S_1 and S_2 may be chosen. Theorem 4.2 imposes restrictions on the split of B_{12}. Moreover, these restrictions cannot now be viewed for S_1 and S_2 in isolation, since the points used in splitting the common boundary B_{12} impact those coplanarity conditions for both S_1 and S_2. The next theorem immediately follows from Theorem 4.2.

Theorem 5.2 *Given two macro n-simplices S_1 and S_2 with common facet B_{12}, and their respective Powell–Sabin splits, where the conditions of Theorem* 5.1 *are met, let the interior point chosen for each i-face F of B_{12} be $(n-i)$-coplanar with the interior points selected for all j-faces, $j = i+1, \ldots, n$, which have F as a face in both S_1 and S_2. Then for any $f \in C^1(S_1 \cup S_2)$ the interpolant Qf is C^1 across B_{12}.*

We will return to this point in Theorem 5.4, as well as Section 6, since there are choices of points used in the Powell–Sabin split which will satisfy the constraints, but before considering the details, we need to examine the cross-boundary derivative of Qf across the boundary between S_1 and S_2.

Theorem 5.3 *Given Powell–Sabin splits of two macro n-simplices S_1 and S_2 with common facet B_{12}, where the conditions of Theorem* 5.2 *are met, let u_1, u_2, and u_{12} be the interior points used for each respectively. Then for any $f \in C^1(S_1 \cup S_2)$, the interpolant Qf is C^1 across B_{12} if u_1, u_2, and u_{12} are collinear.*

Proof Given that the split satisfies the constraints of Theorem 5.2, it follows that we have only to prove that a particular cross boundary derivative of Qf is continuous across B_{12}. From Theorem 4.5 it follows that considered as a limiting value on S_1, the normalized derivative of Qf in the direction $\langle u_1 - u_{12}\rangle$ restricted to B_{12} is a global linear polynomial that is uniquely determined by the interpolation to the derivative data given at the vertices of B_{12} only. The same holds for the normalized derivative of Qf in the direction $\langle u_{12} - u_2\rangle$, taken as a limiting value from S_2 and restricted to B_{12}. Since u_1, u_2, and u_{12} are collinear, the result follows. □

From these results, together with those of Section 4, we may now conclude the following:

Theorem 5.4 *Given a triangulation of points in* $\mathbb{R}^n$ *into the simplices* $S_1, \ldots, S_m$, *where* $\Omega := S_1 \cup \cdots \cup S_m$, *let the positional and gradient data at each point be prescribed from* $f \in C^1(\Omega)$. *In the triangulation, select a unique interior point for each i-face,* $i = 1, \ldots, n$, *as in Definition* 2.7 *to split each* S_j, $j = 1, \ldots, m$. *This yields a simplicial partition* $\triangle$ *of* Ω *consisting of* $m(n+1)!$ *subsimplices. Let* $\mathcal{S}_2^1(\triangle)$ *be the space of* C^1 *piecewise quadratic polynomials defined on* $\triangle$. *Then the interpolant* Qf, *defined on each* S_j, $j = 1, \ldots, m$, *using Definition* 3.4, *belongs to* $\mathcal{S}_2^1(\triangle)$ *provided that:*

(1) $\triangle(S_j)$ *is Powell–Sabin for all* $j = 1, \ldots, m$;
(2) *When splitting any two neighboring simplices in the triangulation the interior points chosen in both is collinear with the interior point chosen for the common facet.*

Proof We only have to prove that the conditions of Theorem 5.2 are satisfied. Consider any two neighboring macro n-simplices $S_1 := [v_0, \ldots, v_{n-1}, v_n]$ and $S_2 := [v_0, \ldots, v_{n-1}, v_{n+1}]$ sharing the facet $B_{12} := [v_0, \ldots, v_{n-1}]$. Let $v_{0,\ldots,n-1,n}$, $v_{0,\ldots,n-1,n+1}$, and $v_{0,\ldots,n-1}$ be the interior points used for each respectively. By assumption (2) these points are collinear. Without loss of generality, assume $F := [v_0, \ldots, v_i]$ be an i-face of B_{12}. Since the split on S_1 is Powell–Sabin, the interior point chosen for F and the interior points selected for all j-faces, $j = i+1, \ldots, n$, which have F as a face in S_1, lie in a $(n-i)$-dimensional affine subspace

$$A_1 = \langle v_{0,\ldots,i}, \ldots, v_{0,\ldots,n-1}, v_{0,\ldots,n-1,n}\rangle.$$

Similarly, the interior point chosen for F and the interior points selected for all j-faces, $j = i+1, \ldots, n$, which have F as a face in S_2, lie in a $(n-i)$-dimensional affine subspace

$$A_2 = \langle v_{0,\ldots,i}, \ldots, v_{0,\ldots,n-1}, v_{0,\ldots,n-1,n+1}\rangle.$$

Assumption (2) clearly implies that A_1 must coincide with A_2. □

6 A splitting algorithm for a Powell–Sabin Interpolant

We now consider an important special case where the constraints of Theorem 5.4 are met and the interpolation problem can always be solved. Namely, when the positional and gradient data are prescribed at the vertices of a uniform regular lattice. This situation is of practical interest since in many applications data are prescribed on a uniform grid. In this case, we first of all construct a specific triangulation of the data sites into n-simplices, so that a Powell–Sabin split can be generated.

Definition 6.1 The matrix representation of a k-simplex S is the $(k+1) \times n$ matrix $M_S := \{a_{ij}\}$, whose rows are the Cartesian coordinates of the vertices of S.

Definition 6.2 Let $B := [0, 1]^n$, and $\hat{S}$ be an n-simplex whose $(n+1) \times n$ representation matrix $M_{\hat{S}} = \{a_{ij}\}$ is given by

$$a_{ij} = \begin{cases} 1, & if \quad i > j, \\ 0, & otherwise. \end{cases} \tag{6.1}$$

The collection of permutations of the columns of $M_{\hat{S}}$ provides $n!$ congruent simplices. This splits B into subsimplices and creates the triangulation $\triangle_B$ of B. Integer shifts of $\triangle_B$ form a triangulation $\triangle$ of $\mathbb{R}^n$ into simplices.

This triangulation is the underlying domain for the problem of interpolating positional and gradient data at the points in $\mathbb{R}^n$ with the Cartesian coordinates

$$(i_1, \ldots, i_n), \quad i_j \in \{0, \ldots, N_j\}, \text{ where } N_j \in \mathbb{Z}^+.$$

Clearly, this lattice of points can be dilated or shifted. We now provide an algorithm for creating a Powell–Sabin split of the triangulation $\triangle$, so that the interpolation problem can be solved.

Algorithm 6.3 Let $\triangle$ be as in Definition 6.2. Then,

(a) Select centroids in all n-simplices in $\triangle$. This provides interior points for all n-faces;
(b) As an interior point for a i-face F, choose the point of intersection of F with the $(n-i)$-dimensional affine subspace passing through the centroids of all n-faces sharing F.

Figure 3 illustrates the algorithm defined in Algorithm 6.3 for a two-dimensional domain. It is shown in [9] by directly checking the conditions in Theorem 5.4 that the triangulation presented in Algorithm 6.3 is indeed Powell–Sabin, and thus we can always solve the interpolation problem in the case of uniformly gridded data.

It is an open and, we believe, difficult problem whether or not the interpolation problem can be solved for an arbitrary triangulation in $\mathbb{R}^n$. However, we make the following observations, which will lead to a solution in certain cases.

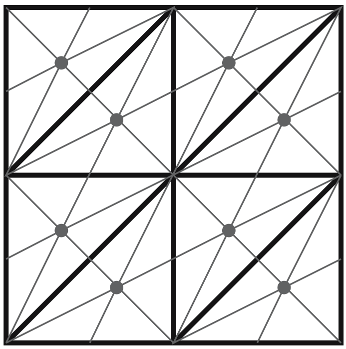

Fig. 3 The Powell–Sabin split created by Algorithm 6.3 on a uniform grid in $\mathbb{R}^2$. The centroids of the triangles are marked as *gray dots*

Remark 6.4 The constraints of coplanarity imposed by Theorem 5.4 can be met by choosing, as the interior points in the splitting process, the circumcenters for each simplex, as in [12]. However, the circumcenters need not be interior points and for our purposes they must be if they are to be used in the splitting algorithm. This leads to a definition of an acute simplex.

Definition 6.5 A simplex S^n is called acute if for each $2 \leq k \leq n$, and each k-face F of S^n, the circumcenter of F is strictly interior to F.

It is easy to see that if every simplex in a triangulation of a domain in $\mathbb{R}^n$ is acute, then the circumcenters of each face can be chosen as the interior points for a Powell–Sabin split. An example for the 3-dimensional case is given in [12].

7 Approximation power

Let Ω be as in Theorem 5.4. The interpolation scheme described in Section 5 defines a linear interpolation operator Q mapping $C^1(\Omega)$ into $\mathcal{S}_2^1(\triangle)$. We note that $Qp = p$ for any polynomial p of degree two. The following results show that this operator provides optimal order approximation. Given a simplex S^n, let $|S^n|$ be its diameter and let $\|\cdot\|_{S^n}$ be the ∞-norm on S^n. Let D^β be the derivative operator in its standard multi-index notation. Given a function $f \in C^k(S^n)$, let $\|D^k f\|_{S^n} := \max_{|\beta|=k} \|D^\beta f\|_{S^n}$.

Theorem 7.1 *Let $f \in C^1(S^n)$. Then*

$$\|f - Qf\|_{S^n} \leq 1.5\,|S^n|\,\|D^1 f\|_{S^n}. \tag{7.1}$$

Proof Let T be one of the subsimplices in $\triangle(S^n)$. Since Bernstein polynomials form a partition of unity on T, it follows from (3.1) that

$$f(x) - Qf(x) = \sum_{i_0+\cdots+i_n=2} \left(f(x) - c^T_{i_0,\ldots,i_n}\right)\, B^T_{i_0,\ldots,i_n}(x), \quad \text{for any } x \in T, \tag{7.2}$$

where $c^T_{i_0,\ldots,i_n}$ are the B-coefficients of $Qf|_T$. The next result immediately follows from Definition 3.4:

$$\left| f(x) - c^T_{i_0,\ldots,i_n} \right| \leq 1.5\,|S^n|\,\|D^1 f\|_{S^n}, \quad \text{for any } x \in T, \quad \text{for any } i_0 + \cdots + i_n = 2.$$

Inserting these estimates into (7.2), and taking the maximum over all $T \in \triangle(S^n)$ leads to (7.1). □

Our next result includes an error bound for second derivatives. Since the second derivatives of Qf are not continuous on S^n in general, we provide local estimates. Given a simplex $T \in \triangle(S^n)$, let $|T|$ be its diameter, and $\|\cdot\|_T$ be the ∞-norm on T. Similarly to the notation above, for $f \in C^k(T)$, let $\|D^k f\|_T := \max_{|\beta|=k} \|D^\beta f\|_T$. Additionally, we need to define a constant related to the geometry of $\triangle(S^n)$:

$$\gamma_T := \frac{|T|}{\rho_T}, \quad \text{where } \rho_T \text{ is the inradius of } T.$$

Theorem 7.2 *There exists a constant K depending only on n, γ_T and $|S^n|/|T|$ such that for every $f \in C^{m+1}(S^n)$, $0 \leq m \leq 2$,*

$$\left\| D^\beta(f - Qf) \right\|_T \leq K|S^n|^{m+1-|\beta|} \left\| D^{m+1} f \right\|_{S^n}, \tag{7.3}$$

for all $0 \leq |\beta| \leq m$.

Proof The idea of the proof is similar to that of the proof of Theorem 6.2 in [8]. Let p be the Taylor polynomial of degree m generated by $f \in C^{m+1}(S^n)$ about the incenter of S^n. Then

$$\left\| D^\beta(f - p) \right\|_{S^n} \leq K_1|S^n|^{m+1-|\beta|} \left\| D^{m+1} f \right\|_{S^n}, \tag{7.4}$$

where K_1 depends on n. Since $Qp = p$, we have

$$\left\| D^\beta(f - Qf) \right\|_T \leq \left\| D^\beta(f - p) \right\|_T + \left\| D^\beta Q(f - p) \right\|_T. \tag{7.5}$$

In view of (7.4), it suffices to estimate the second term in (7.5). By the Markov inequality [10], we obtain

$$\left\| D^\beta Q(f - p) \right\|_T \leq \frac{K_2}{\rho_T^{|\beta|}} \; \| Q(f - p) \|_T, \tag{7.6}$$

and since Bernstein polynomials form a partition of unity on T, from Definition 3.4 it follows that

$$\| Q(f - p) \|_T \leq \max_{i_0 + \cdots + i_n = 2} \left| c^T_{i_0,\ldots,i_n} \right| \leq \| f - p \|_{S^n} + \frac{1}{2} |S^n| \, \| D^1(f - p) \|_{S^n}.$$

Using (7.4) we obtain the further estimate of $\| Q(f - p) \|_T$

$$\| Q(f - p) \|_T \leq K_3|S^n|^{m+1} \, \| D^{m+1} f \|_{S^n}, \quad \text{where} \quad K_3 = 1.5 \, K_1.$$

Inserting the last inequality into (7.6), and combining it with (7.5) leads to

$$\begin{aligned} \left\| D^\beta(f - Qf) \right\|_T \leq & K_1|S^n|^{m+1-|\beta|} \, \| D^{m+1} f \|_{S^n} + \frac{K_2 \, K_3}{\rho_T^{|\beta|}} |S^n|^{m+1} \, \| D^{m+1} f \|_{S^n} \\ \leq & |S^n|^{m+1-|\beta|} \, \| D^{m+1} f \|_{S^n} \left\{ K_1 + K_2 \, K_3 \, \gamma_T^{|\beta|} \, \frac{|S^n|^{|\beta|}}{|T|^{|\beta|}} \right\}. \end{aligned}$$

The proof is complete. □

Remark 7.3 In Theorem 7.2, when $|\beta| = 0$ the Markov inequality (7.6) does not need to be used, and the constant K does not depend on the geometry of $\triangle(S^n)$.

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