On the n-dimensional Clough-Tocher Interpolant

T. Sorokina [1]

Abstract. We show that the n-dimensional Clough-Tocher interpolant constructed in [6] is not smooth for $n \geq 4$ unless additional conditions are imposed.

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§1. Introduction

We shall remind the reader the main results of [6]. The authors construct a C^1 piecewise cubic interpolant to data defined on a tessellation in $\mathbb{R}^n$ of n-simplices. The given data are:

(a) Position and gradient at every vertex;
(b) $(n-1)$ directional derivatives at the mid-edge points of the tessellation. The directions being such that, together with the direction determined by the associated edge, they provide a basis for $\mathbb{R}^n$.

The generalized Clough-Tocher splitting algorithm of a n-simplex P_n is introduced by induction. For P_2 it is a standard bivariate Clough-Tocher split of a triangle into three subtriangles coning off an arbitrary interior point. To obtain the split a m-simplex, first each of its boundary $(m-1)$-dimensional faces is split according to the lower-dimensional scheme. Then every split point on the boundary of P_m is connected to an arbitrary interior point in P_m. The main result is given in Theorem 3.3 of [6]:

Result 1.1. *If P_n is split according to the inductive buildup described above, then there can be defined a cubic polynomial over each subsimplex to obtain, over P_n, a unique C^1 interpolant, Q, to the given data.*

The goal of our paper is to show that the interpolant Q fails to be smooth for $n \geq 4$. We shall use is the following theorem from [2]:

Theorem 1.2. *A piecewise continuous polynomial over a tessellation of $\mathbb{R}^n$ into simplices is C^1 at a vertex v of the tessellation if and only if the control points of the domain points in the disk of radius one around v lie in a (hyper)plane in $\mathbb{R}^{n+1}$.*

The paper is organized as follows. In the remainder of the introduction, we provide basic definitions. In Section 2 we construct an interpolant Q over the

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[1] Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA, `sorokina@math.utah.edu`

generalized Clough-Tocher split of a single four-dimensional simplex as described in [6], and show that it fails to be smooth. In Section 3 we state how to "fix" the construction of [6] in order to enforce C^1 smoothness. The proof of the new result is rather lengthy and technical, and will be presented in a different paper, see [4].

In the remainder of this section we review basic Bernstein-Bézier techniques. For a detailed exposition see [2] and [3].

Definition 1.3. *Let* $v_1, \ldots, v_m$ *be* m *points in* $\mathbb{R}^n$*. The linear combination*

$$\lambda_0 v_0 + \ldots + \lambda_m v_m, \quad \text{where} \;\; \lambda_0 + \ldots + \lambda_m = 1$$

is called an affine combination of $v_1, \ldots, v_m$*.*

Definition 1.4. *A set of points in* $\mathbb{R}^n$ *is called affinely independent if none of the points can be written as an affine combination of the others.*

Let $V := \{v_0, \ldots, v_m\} \subset \mathbb{R}^n$ be affinely independent, and denote the convex hull of V by $[V]$ or, equivalently, by $[v_0, \ldots, v_m]$. Then $P_m := [v_0, \ldots, v_m]$ is an m-(dimensional) simplex, whose set of vertices is given by V.

Definition 1.5. *Let* $v_0, \ldots, v_n$*, be* $n+1$ *affinely independent points in* $\mathbb{R}^n$*, and let* u *be an arbitrary point in* $\mathbb{R}^n$*. The numbers* $b_0, \ldots, b_n$*, satisfying*

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ v_0 & v_1 & \ldots & v_n \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = \begin{pmatrix} 1 \\ u \end{pmatrix} \tag{1.1}$$

are called the barycentric coordinates of u *relative to the simplex* $[v_0, \ldots, v_n]$*.*

The Bernstein-Bézier (BB) form of the cubic polynomial g relative to the simplex $P_n := [v_0, \ldots, v_n]$ is given by

$$g = \sum_{i_0+\ldots+i_n=3} c_{i_0,\ldots,i_n} \; B_{i_0,\ldots,i_n}, \quad i_j \geq 0, \quad j = 0, \ldots, n, \tag{1.2}$$

where

$$B_{i_0,\ldots,i_n} = \frac{3!}{i_0! \ldots i_n!} \, b_0^{i_0} \ldots b_n^{i_n}, \quad i_0 + \ldots + i_n = 3,$$

are the cubic Bernstein polynomials associated with P_n. Here, $\{b_i\}_{i=0}^n$ are the barycentric coordinates relative to P_n. As usual, we associate the BB-coefficients $c_{i_0,\ldots,i_n}$ of p with the domain points $\xi_{i_0,\ldots,i_n} := (i_0 v_0 + \cdots + i_n v_n)/3$ in P_n. The domain point for a cubic polynomial can be located either at a vertex of P_n, or on an edge of P_n, or at the centroid of a triangular face of P_n. The coefficient $c_{i_0,\ldots,i_n}$ associated with the domain point $\xi_{i_0,\ldots,i_n}$ will be also referred to as its ordinate, and the ordered pair in $\mathbb{R}^{n+1}$

$$\mathcal{C}_{i_0,\ldots,i_n} := (\xi_{i_0,\ldots,i_n}, c_{i_0,\ldots,i_n}), \quad i_0 + \cdots + i_n = 3, \tag{1.3}$$

will be called a control point.

Suppose v is a vertex of a tessellation $\triangle$ of $\mathbb{R}^n$ into simplices. Then the sets of domain points

$$R_1(v) := \big\{(2\ v+u)/3, \quad \text{for all edges } \ [v,u] \in \triangle\big\}, \quad \text{and} \quad D_1(v) := v \cup R_1(v),$$

will be referred to as the ring and disk of radius one around the vertex v in $\triangle$.

Suppose $e := [v,u]$ is an edge of $\triangle$. Then the set of domain points

$$R_1(e) := R_1(v) \cup R_1(u) \cup \big\{(v+u+w)/3, \quad \text{for all faces } \ [v,u,w] \in \triangle\big\},$$

will be referred to as the ring of radius one around the edge e in $\triangle$.

It is easy to derive conditions for a smooth join between two cubic polynomials g and $\tilde{g}$ defined respectively on two simplices P_n and $\widetilde{P}_n$ with a common face F. To describe these in more detail, suppose that $P_n := [v_0, \ldots v_n]$ and $\widetilde{P}_n := [v_{n+1}, v_2, \ldots, v_n]$ are two adjoining simplices sharing the face $F := [v_2, \ldots, v_n]$. Suppose

$$g = \sum_{i_0+\ldots+i_n=3} c_{i_0,\ldots,i_n}\ B_{i_0,\ldots,i_n}, \quad \tilde{g} = \sum_{i_0+\ldots+i_n=3} \tilde{c}_{i_0,\ldots,i_n}\ \widetilde{B}_{i_0,\ldots,i_n},$$

where $\{\widetilde{B}_{i_0,\ldots,i_n}, \ \ i_0+\ldots+i_n=3\}$ are the cubic Bernstein polynomials associated with $\widetilde{P}_n$. It is well known that g and $\tilde{g}$ join smoothly across the face F if and only if for all admissible $i_1, \ldots, i_n$

$$\begin{aligned} &c_{0,i_1,\ldots,i_n} = \tilde{c}_{0,i_1,\ldots,i_n}, \quad i_1+\ldots+i_n = 3, \quad \text{and} \\ &c_{1,i_1,\ldots,i_n} = \tilde{c}_{1,i_1,\ldots,i_n} \tilde{b}_{i_0}(v_0) + \sum_{j=0}^{n} \tilde{c}_{i_0,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_n} \tilde{b}_{i_j}(v_0), \ i_1+\ldots+i_n = 2, \end{aligned}$$

where $\{\tilde{b}_{i_j}(v_0)\}_{j=0}^n$ are the barycentric coordinates of v_0 relative to $\widetilde{P}_n$.

§2. A counterexample

Let $S := [v_0, v_1, v_2, v_3, v_4]$ be a simplex in $\mathbb{R}^4$. By v_{ijk} we denote the split point in the interior of the triangular face $[v_i, v_j, v_k]$ of S. For all admissible $(ijk) \neq (012)$, we define $v_{ijk} := (v_i + v_j + v_k)/3$. Let $v_{012} = (v_0 + v_1)/4 + v_2/2$. Similarly, let v_{ijkl} be the split point in the interior of the tetrahedral face $[v_i, v_j, v_k, v_l]$ of S. For all admissible $(ijkl)$, we define $v_{ikjl} := (v_i + v_j + v_k + v_l)/4$. Finally, by $v_{01234} := (v_0 + v_1 + v_2 + v_3 + v_4)/5$ we denote the split point in the interior of S, thus, introducing the generalized Clough-Tocher split $\triangle(S)$ as described in [6]. To summarize the construction, we note that the split points for all two- and three-dimensional faces, except $[v_0, v_1, v_2]$, are chosen to be the centroids.

Throughout the paper, the notation $\langle \phi, \psi \rangle$ shall be reserved for the vector from the point ϕ to the point ψ as opposed to the line segment $[\phi, \psi]$. In order to define the interpolant, we set the following data:

a. For each vertex v_i of S, the value and the gradient at v_i are set to be zeros.

b1. For each edge $[v_i, v_j]$ of S, except $[v_0, v_1]$, the three required directional derivatives of order one at the mid-edge points are set to zero as well.

b2. The following directional derivatives of order one are set at the midpoint m of $[v_0, v_1]$:

$$D_{\langle m, v_{01234}\rangle} f(m) = 0, \quad D_{\langle m, v_{0123}\rangle} f(m) = 0, \quad D_{\langle m, v_{0124}\rangle} f(m) = 3.$$

Our goal is to show that the Clough-Tocher interpolant defined over $\triangle(S)$ by the data in a, b1 and b2, is not C^1 at the point v_{012}.

Below are six domain points in the disk of radius one around v_{012} along with their respective BB-coefficients (the computation of each coefficient follows later on in this section). The points are located on true one-dimensional edges emanating from v_{012}.

$$\begin{aligned} p_0 &:= 2/3 v_{012} + 1/3 v_0, & c_{p_0} &= 1, & p_1 &:= 2/3 v_{012} + 1/3 v_1, & c_{p_1} &= 1, \\ p_2 &:= 2/3 v_{012} + 1/3 v_2, & c_{p_2} &= 0, & p_3 &:= 2/3 v_{012} + 1/3 v_{0123}, & c_{p_3} &= 0, \\ p_4 &:= 2/3 v_{012} + 1/3 v_{0124}, & c_{p_4} &= 1/4, & p &:= 2/3 v_{012} + 1/3 v_{01234}, & c_p &= 0. \end{aligned} \tag{2.1}$$

The six control points $\{(p, c_p), (p_i, c_{p_i})\}$ are not 4-coplanar in $\mathbb{R}^5$ since

$$\begin{aligned} p &= -1/5(p_0 + p_1 + p_2) + 4/5(p_3 + p_4), \\ c_p &= 0 \neq -1/5(c_{p_0} + c_{p_1} + c_{p_2}) + 4/5(c_{p_3} + c_{p_4}) = -1/5. \end{aligned}$$

Theorem 1.2 shows that the interpolant is not C^1 at v_{012}.

In the remainder of this section we show how to compute the BB-coefficients associated with the six domain points of (2.1). Since it is rather difficult to visualize four-dimensional geometry, we shall place every domain point of interest in a suitable two- or three-dimensional object. This is possible because all domain points for a cubic spline lie in two-dimensional simplices – triangular faces.

All domain points for a cubic spline over $\triangle(S)$ are located either on one-dimensional edges or at the centroids of the triangular faces. The data in a. shows that the BB-coefficients associated with all the domain points in the disks of radius one around the vertices of S vanish. Moreover, the data in b1. implies that the domain points located at the centroids of all triangular faces, except those sharing both v_0 and v_1, have coefficients equal to zero. In Table 1, we summarize information about the BB-coefficients associated with the domain points w_J located at the centroids of the triangular faces $[v_0, v_1, v_J]$ having both v_0 and v_1 as vertices. J is a multi-index in its standard notation, m is the mid-point of $[v_0, v_1]$.

The values for the derivatives in the last three rows are given by b2. The derivative in the first row is computed as follows:

$$\begin{aligned} &\langle m, v_{012}\rangle = -5/2\langle m, v_{01234}\rangle + 2\langle m, v_{0123}\rangle + 2\langle m, v_{0124}\rangle, \\ &D_{\langle m, v_{012}\rangle} f(m) = -5/2 D_{\langle m, v\rangle} f(m) + 2 D_{\langle m, v_{0123}\rangle} f(m) + 2 D_{\langle m, v_{0124}\rangle} f(m) = 6. \end{aligned} \tag{2.2}$$

face $[v_0, v_1, v_J]$	$w_J =: (v_0 + v_1 + v_J)/3$	$D_{\langle m, v_J\rangle} f(m)$	$c(w_J)$
$[v_0, v_1, v_{012}]$	$(5v_0 + 5v_1 + 2v_2)/12$	6	4
$[v_0, v_1, v_{0123}]$	$(5v_0 + 5v_1 + v_2 + v_3)/12$	0	0
$[v_0, v_1, v_{0124}]$	$(5v_0 + 5v_1 + v_2 + v_4)/12$	3	2
$[v_0, v_1, v_{01234}]$	$(6v_0 + 6v_1 + v_2 + v_3 + v_4)/15$	0	0

Tab. 1. Domain points w_J at the centroids of the faces $[v_0, v_1, v_J]$, and their BB-coefficients $c(w_J)$. The midpoint of $[v_0, v_1]$ is denoted by m.

The barycentric coordinates of m relative to the triangle $[v_0, v_1, v_J]$ are $(1/2, 1/2, 0)$. For all J, the barycentric coordinates of the vector $\langle m, u\rangle$ relative to the triangle $[v_0, v_1, v_J]$ are $(-1/2, -1/2, 1)$, see Fig. 1 (left). Hence, the coefficient $c(w_J) =: c_{111}$ in each row can be computed from the formula, see [3]:

$$D_{\langle m,u\rangle} f(m) = \frac{3!}{2!}\Big(\big[-\frac{1}{2}(c_{300} + c_{210}) + c_{201}\big]\frac{1}{4} + \big[-\frac{1}{2}(c_{210} + c_{120}) + c_{111}\big]\frac{1}{2} + \big[-\frac{1}{2}(c_{120} + c_{030}) + c_{021}\big]\frac{1}{4}\Big), \tag{2.3}$$

where c_{ijk} is the BB-coefficient associated with the domain point ξ_{ijk} relative to the triangle $[v_0, v_1, v_J]$. All the coefficients in (2.3), except c_{111}, vanish since their respective domain points are located in disks of radius one around either v_0 or v_1. In Fig. 1 (left) the domain points, whose coefficients vanish, are marked with black dots. Thus, formula (2.3) yields $c_{111} = c(w_J) = 2\ D_{\langle m,v_J\rangle} f(m)/3$.

The first three domain points – p_0, p_1, p_2 – lie on the boundary triangular face $T := [v_0, v_1, v_2]$ of S. This face is split according to the bivariate Clough-Tocher scheme, see Fig. 1 (right), The data in a. and b1. implies that the coefficients associated with the domain points shown as black dots in Fig. 1 (right) vanish, including the coefficient associated with p_2. From the first row of Table 1 we know that $c(w_{012}) = 4$. Using the C^1 smoothness conditions across the edges $[v_0, v_{012}]$, $[v_0, v_{012}]$, we conclude that $c_{p_0} = c_{p_1} = c(w_{012})/4 = 1$.

The boundary triangular face T is shared by two boundary tetrahedral faces $F := [v_0, v_1, v_2, v_3]$ and $G := [v_0, v_1, v_2, v_4]$ of S. The domain point p_3 is located in in F, while p_4 is in G. Both F and G are split according to the three-dimensional Worsey-Farin scheme with v_{0123} and v_{0124} being the split points strictly interior to F and G, respectively.

We shall start by placing p_3 in the triangle $\mathcal{T} := [t_0, t_1, t_2]$, where

$$t_0 := 2/3v_0 + 1/3v_{0123},\ t_1 := 2/3v_1 + 1/3v_{0123},\ t_2 := 2/3v_2 + 1/3v_{0123}.$$

$\mathcal{T}$ is similar to T and split in the same way, see Fig. 2 (left). The domain points in $\mathcal{T}$ can be associated with a quadratic spline over the Clough-Tocher split. The domain points on the edges of $\mathcal{T}$ are located in the ring of radius one around the

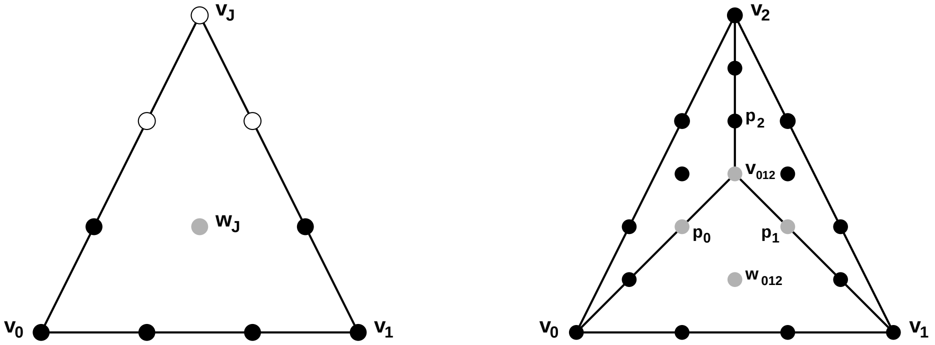

Fig. 1. Left: domain points in the triangle $[v_0, v_1, v_J]$, compare with Table 1. Right: domain points in $T =: [v_0, v_1, v_2]$. The BB-coefficients associated with the points shown as black dots vanish.

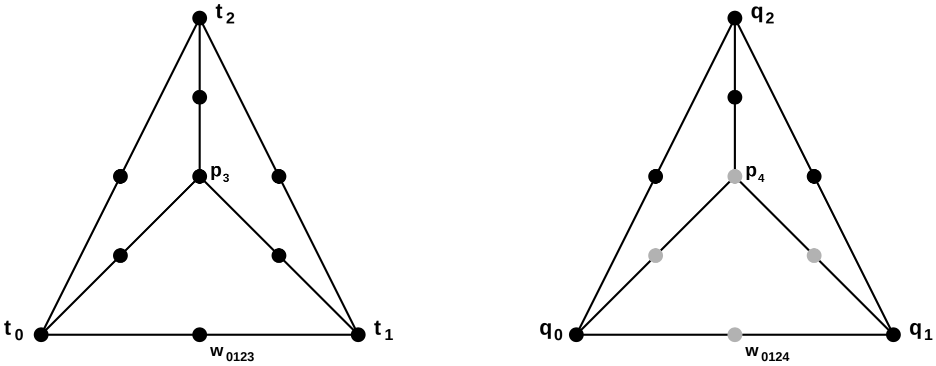

Fig. 2. Left: domain points in $\mathcal{T} =: [t_0, t_1, t_2]$. Right: domain points in $\mathcal{Q} =: [q_0, q_1, q_2]$. The BB-coefficients associated with the points shown as black dots vanish.

edges of T. Thus, the data in a., and b1. immediately implies that the coefficients associated with all the domain points on the edges, except w_{0123}, are zeros. The location of w_{0123} can be written as

$$w_{0123} := (t_0 + t_1)/2 = (v_0 + v_1 + v_{0123})/3 \in [v_0, v_1, v_{0123}].$$

From Table 1 we know that $c(w_{0123}) = 0$. Applying C^1 conditions within $\mathcal{T}$, we conclude that $c_{p_3} = 0$.

Next we consider p_4 by placing it the triangle $\mathcal{Q} := [q_0, q_1, q_2]$, where

$$q_0 := 2/3v_0 + 1/3v_{0124}, \; q_1 := 2/3v_1 + 1/3v_{0124}, \; q_2 := 2/3v_2 + 1/3v_{0124}.$$

$\mathcal{Q}$ is similar to T and split in the same way, see Fig. 2 (right). The domain points in $\mathcal{Q}$ can be also associated with a quadratic spline over the Clough-Tocher split. The domain points on the edges of $\mathcal{Q}$ are located in the ring of radius one around the edges of T. Thus, the data in a., and b1. immediately implies that the coefficients associated with all the domain points on the edges, except w_{0124}, are zeros. The location of w_{0124} can be written as

$$w_{0124} := (q_0 + q_1)/2 = (v_0 + v_1 + v_{0124})/3 \in [v_0, v_1, v_{0124}].$$

From Table 1 we know that $c(w_{0124}) = 2$. Applying C^1 conditions within $\mathcal{Q}$, we conclude that $c_{p_4} = c(w_{0124})/8 = 1/4$.

Finally, we consider the domain point p. It is neither in F nor in G. However, we can place p in the tetrahedron $[v_0, v_1, v_2, v_{01234}]$ that is split into three subtetrahedra coning off v_{012} in $\triangle(S)$. Moreover, we shall place p in the triangle $\mathcal{R} := [r_0, r_1, r_2]$

$$r_0 := 2/3v_0 + 1/3v_{01234}, \;\; r_1 := 2/3v_1 + 1/3v_{01234}, \;\; r_2 := 2/3v_2 + 1/3v_{01234},$$

see Fig. 2 (left) again with r_i in place of t_i, $i = 0, 1, 2$ and p in place of p_3. The exact same argument as the one for the triangle $\mathcal{T}$ and the point p_3 applies, and we conclude that $c_p = 0$. $\square$

§3. Conclusion

The counterexample of Section 2 suggests that some geometric constraints must be imposed in order for the multivariate Clough-Tocher interpolant to be smooth. To introduce the constraints we need additional notation.

Definition 3.1. *Let* $V := \{v_0, \ldots, v_n\}$ *be the set of vertices of a simplex* P_n *in* $\mathbb{R}^n$. *A* k*-dimensional face* $F_I := [v_{i_0}, \ldots, v_{i_k}]$ *is the convex hull of* $k+1$ *distinct vertices* $\{v_{i_j}\}_{j=0}^{k} \subset V$, *where* $I := (i_0, \ldots, i_k)$ *is the standard multi-index notation.*

Our next definition describes rather severe geometric constraints similar to those for the multivariate Powell-Sabin interpolant of [5]. As usual, points in $\mathbb{R}^n$ are m-coplanar if they lie in a m-dimensional plane, $m \le n$.

Definition 3.2. *The generalized Clough-Tocher split* $\triangle$ *of a simplex* P_n *in* $\mathbb{R}^n$ *is constrained if for each* $2 \le k \le n-2$, *the interior point chosen for an* k*-dimensional face* F *is* $(n-k)$*-coplanar with the interior points chosen for all* j*-dimensional faces,* $j = k+1, \ldots, n$, *containing* F *as a face.*

Now we are ready to state the main result using the terminology of Result 1.1.

Theorem 3.3. *If the generalized split* $\triangle$ *of* P_n *is constrained, then there can be defined a cubic polynomial over each subsimplex to obtain, over* P_n, *a unique* C^1 *interpolant,* Q, *to the data in (a), (b) of Section 1.*

The proof of this theorem is the subject of another paper, see [4].

Remark 3.4. *The construction of [6] yields a smooth interpolant for $n = 3$ because* **only** *in this case the condition of Theorem 1.2 is automatically satisfied. Indeed, consider the generalized Clough-Tocher split of a tetrahedron $S := [v_0, v_1, v_2, v_3]$ in $\mathbb{R}^3$. Let v_{012} be the split point on the triangular face $T := [v_0, v_1, v_2]$. There are five domain points in $D_1(v_{012})$:*

$$D_1(v_{012}) = \{v_{012}, (2v_{012}+v_0)/3, (2v_{012}+v_1)/3, (2v_{012}+v_2)/3, (2v_{012}+v)/3\}, \quad (3.1)$$

where v is the split point in the interior of S. The first four points of (3.1) lie in the two-dimensional face T. According to the bivariate Clough-Toucher construction, their control points lie in a two-dimensional plane in $\mathbb{R}^3$. Therefore, the five control points associated with the domain points in (3.1) must lie in a hyperplane in $\mathbb{R}^4$.

Remark 3.5. *For $n = 4$, Theorem 3.3 yields the weakest constraint – the coplanarity conditions must be imposed only on triangular faces. A triangular face T of a four-dimensional simplex S is shared by two tetrahedral faces F and G, cf. our counterexample of Section 2. The split point chosen in the interior of T must be coplanar (in the usual two-dimensional sense) with the split points chosen in the interiors of F, G, and S itself. As our next remark shows, violation of this condition on one or more faces yeilds unrecoverable loss of smoothness.*

Remark 3.6. *Let $WF(\triangle)$ be the space of C^1 smooth piecewise cubic polynomials over the generalized Clough-Tocher split (not necessarily constrained) of a single simplex S in $\mathbb{R}^4$. Then, according to [6], the dimension of $WF(\triangle)$ must be equal to the cardinality of the data set in (a), (b). That is dim $WF(\triangle) = 55$. In Table 2 we summarize the results on analyzing the dimension of $WF(\triangle)$ obtained using a software package designed by Peter Alfeld, see [1]. The variable k in the first row shows the number of triangular faces for which the constraint of Theorem 3.3 is satisfied. The number in the second row is the corresponding dimension of $WF(\triangle)$. A four-dimensional simplex has 10 triangular faces. Thus, the first column gives the dimension for the constrained split. It is interesting to note, that in the most general case, that is when $k = 0$, the space $WF(\triangle)$ coincides with the space of cubic polynomials in four variables.*

k	10	9	8	7	6	5	4	3	2	1	0
dim	55	53	51	49	47	45	43	41	39	37	35

Tab. 2. Dimension of $WF(\triangle)$ over splits with decreasing number of the geometric constraints satisfied. Courtesy of P. Alfeld.

Remark 3.7. *The geometric constraints of Theorem 3.3 can be always satisfied on a single simplex. In particular, this can be accomplished by choosing centroids as the split points for each face. However, this does not resolve the issue of smoothness* *between two neighboring macro-simplices. The constraint of Theorem 3.3 on each k-dimensional face F, for $2 \leq k \leq n-1$, must be satisfied* **simultaneously** *for all simplices sharing F. This observation imposes a severe restriction on the geometry of the tessellation, and thus, makes the Clough-Tocher interpolant in its explicit form non-applicable to scattered data interpolation. We address this issue in more detail in [4].*

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