§1. Basic definitions

Let $\mathbb{R}$ be the set of all real numbers, while $\overline{\mathbb{R}} := \mathbb{R} \bigcup \{-\infty; \infty\}$ be an extended set of real numbers, and $\mathbb{N} := \{1, 2 \ldots, n, \ldots\}$ be the set of all natural numbers.

Definition 1.1. *Let X be a set of elements of arbitrary nature. For any $n \in \mathbb{N}$, by x_n we denote an element of X corresponding to n. This defines a set of numbered elements $\{x_1, x_2, \ldots, x_n, \ldots\}$ (or, for brevity, $\{x_n\}$) called a sequence defined on X.*

If $X = \mathbb{R}$ then $\{x_n\}$ is called a sequence of numbers. If X is a set of functions: $x_n = x_n(t)$, then $\{x_n(t)\}$ is a sequence of functions.

Definition 1.2. *A sequence $\{x_n\}$ is called stationary if $x_n = x_{n+1} \quad \forall n \in \mathbb{N}$.*

Definition 1.3. *Two sequences $\{x_n\}$ and $\{y_n\}$ are called equal if $x_n = y_n \ \ \forall n \in \mathbb{N}$.*

Sequences can also be defined as mappings from $\mathbb{N}$ to X. More precisely,

Definition 1.4. *A sequence is a single-valued mapping from $\mathbb{N}$ to X.*

In this study, we mainly consider sequences of numbers, and for simplicity refer to them as to sequences.

Definition 1.5. *The sum (difference, product, quotient) of two sequences $\{x_n\}$ and $\{y_n\}$ is the sequence $\{z_n\}$ such that*

$$z_n = x_n + y_n \quad (\ x_n - y_n, \quad x_n \cdot y_n, \quad x_n/y_n\ ) \quad \forall\ n \in \mathbb{N},$$

where the quotient makes sense only if $y_n \neq 0 \ \ \forall\ n \in \mathbb{N}$.

Definition 1.6. *A sequence $\{x_n\}$ is called bounded below (above) if $\exists K \in \mathbb{R}$ such that*

$$x_n \geq K \quad (\ x_n \leq K\ ) \quad \forall n \in \mathbb{N}.$$

Definition 1.7. *A sequence that is bounded below and above is called bounded.*

Or, equivalently

Definition 1.8. *A sequence $\{x_n\}$ is called bounded if $\exists K \in \mathbb{R}$ such that*

$$|x_n| \leq K \quad \forall n \in \mathbb{N}.$$

Counterdefinitions are extremely important for better understanding of new concepts. For example, from Definition 1.8 we can define an unbounded sequence.

Definition 1.9. *A sequence $\{x_n\}$ is called unbounded if $\forall K \in \mathbb{R} \ \ \exists n_K \in \mathbb{N}$ such that*

$$|x_{n_K}| > K.$$

Remark 1.10. *Clearly, for any number $K \in \mathbb{R}$ there exist infinitely many terms of the sequence satisfying the inequality in Definition 1.9. If it were not true the* *sequence would be bounded. Analogous statements hold for not bounded below and not bounded above sequences.*

Examples:

1. *the sequence* $\{n^2\}$ *is bounded below:* $n^2 > 0\ \forall n \in \mathbb{N}$ *but not bounded above;*
2. *the sequence* $\{-n\}$ *is bounded above:* $-n < 0\ \forall n \in \mathbb{N}$ *but not bounded below;*
3. *the sequence* $\{(-1)^n + 1\}$ *is bounded :* $|(-1)^n + 1| \leq 2\ \forall n \in \mathbb{N}$.

Infinitely large sequences represent an important subset of unbounded sequences.

Definition 1.11. *A sequence* $\{x_n\}$ *is called infinitely large if* $\forall K \in \mathbb{R}\ \ \exists n_K \in \mathbb{N}$ *such that*

$$|x_n| > K \quad \forall n \geq n_K.$$

As an example, we show that the sequence $\{(-1)^n\ n^3\}$ is infinitely large. Indeed, for any number K, we can find n_k such that $|(-1)^n\ n^3| > K \quad \forall n \geq n_K$. To this end, we solve the inequality $n^3 > K$, and $n > \sqrt[3]{K}$. t Let $n_k = \lfloor \sqrt[3]{K} \rfloor + 1$, where $\lfloor c \rfloor$ is the integer part of c. Then for $n \geq n_K$ we obtain

$$n \geq n_k > \sqrt[3]{K} \quad \Rightarrow \quad n^3 > K \quad \Rightarrow \quad |(-1)^n\ n^3| > K.$$

From Definitions 1.9 and 1.11 it follows that any infinitely large sequence is unbounded. However, the converse is not true: there exist unbounded sequences that are not infinitely large. For example, such is the sequence $\{(1 - (-1)^n)n\}$.

The concept of a limit plays a very important role in mathematics.

Definition 1.12. *A number* $a \in \mathbb{R}$ *is called a limit of a sequence* $\{x_n\}$ *if* $\forall \varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ *such that*

$$|x_n - a| < \varepsilon \qquad \forall n \geq n_K.$$

Remark 1.13. *For a convenience, we let the limit of an infinitely large sequence* $\{x_n\}$ *be the symbol* ∞*:*

$$\lim_{n\to\infty} x_n = \infty.$$

If $\{x_n\}$ *is infinitely large and bounded below (above) then we write*

$$\lim_{n\to\infty} x_n = +\infty, \quad (\lim_{n\to\infty} x_n = -\infty).$$

The statement "$\{x_n\}$ converges to a" means that the limit of x_n is a number (or a symbol) a, and is denoted as $\{x_n\} \to a$ as $n \to \infty$. When a sequence has a limit, it is called convergent. Otherwise, it is divergent.

We say that a sequence converges in $\overline{\mathbb{R}}$, if either it has a finite limit or it is infinitely large, i.e., if the sequence has a finite or infinite limit. A sequence converges in $\mathbb{R}$ if it has a finite limit, which may be denoted as

$$\lim_{n\to\infty} x_n < \infty.$$

Definition 1.14. *A sequence* $\{x_n\}$ *is called infinitely small if*

$$\lim_{n\to\infty} x_n = 0,$$

that is for any $\varepsilon > 0$ *there exists* n_ε *such that*

$$|x_n| < \varepsilon \quad \forall\, n \ge n_\varepsilon.$$

For example, the sequence $\{q^n\}$ for $|q| < 1$ is infinitely small. Indeed, for any $\varepsilon > 0$ let us find n_ε such that $|q^n| < \varepsilon \;\; \forall\, n \ge n_\varepsilon$. To this end, we solve the inequality $|q^n| < \varepsilon$, assuming that $0 < \varepsilon < 1$ (for $\varepsilon \ge 1$, the inequality is clearly true for any $n \in \mathbb{N}$):

$$n \ln|q| < \ln\varepsilon \quad\Rightarrow\quad n > \frac{\ln\varepsilon}{\ln|q|}, \quad n_\varepsilon = \left\lfloor \frac{\ln\varepsilon}{\ln|q|} \right\rfloor + 1,$$

where $\ln\varepsilon < 0$, and $\ln|q| < 0$, since $\varepsilon < 1$, and $|q| < 1$. Thus, for $n \ge n_\varepsilon$ we have

$$n \ge n_\varepsilon > \frac{\ln\varepsilon}{\ln|q|} < \ln\varepsilon \quad\Rightarrow\quad |q|^n < \varepsilon.$$

The fact that $\{x_n\}$ is not infinitely small means the following: there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$ there exists $k_n > n$ with $|x_{k_n}| > \varepsilon_0$.

§2. Geometric interpretations of the main concepts

We will give geometric interpretations of the concepts introduced in Sect. 1 arising from Definition 1.4, where sequence is defined as a function, see Fig. 1.

Fig. 1. Sequence as a function.

From the geometric point of view, a sequence $\{x_n\}$ is bounded above (below) if there exists a horizontal line $y = K$ such that all the points (n, x_n) are below (above) this line, see Fig. 2.

A sequence $\{x_n\}$ is bounded if there exist two horizontal lines $y = K$, and $y = -K$ such that all the points (n, x_n) are between these lines, see Fig. 3.

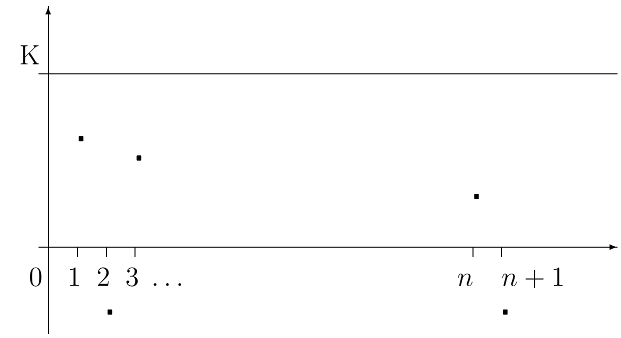

Fig. 2. Bounded above sequence.

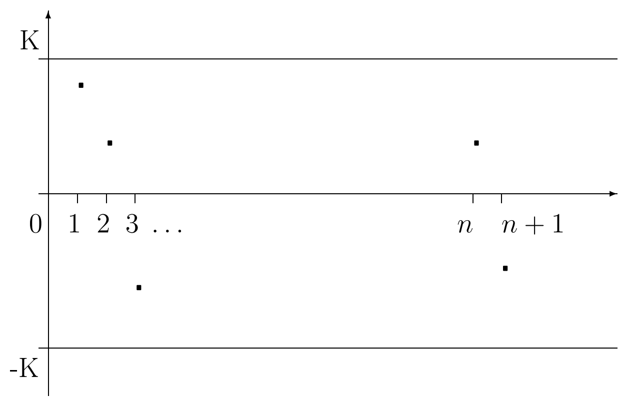

Fig. 3. Bounded sequence.

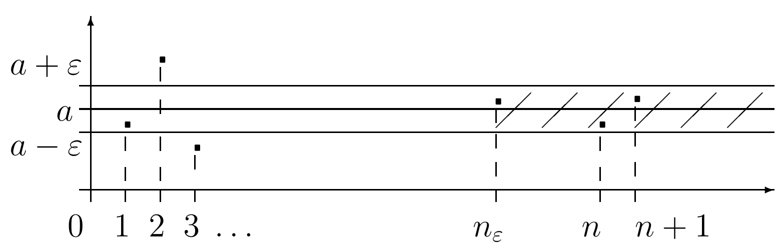

Fig. 4. Limit of a sequence.

A number a is a limit of the sequence $\{x_n\}$ if for any $\varepsilon > 0$, starting with some n_ε all the points (n, x_n) are between two horizontal lines $y = a - \varepsilon$, and $y = a + \varepsilon$, see the shaded area in Fig. 4. The points (n, x_n) tend to concentrate around the line $y = a$.

A sequence $\{x_n\}$ is infinitely small if for any $\varepsilon > 0$ starting with some n_ε all the points (n, x_n) are between two horizontal lines $y = -\varepsilon$, and $y = +\varepsilon$. The points (n, x_n) tend to concentrate around the line $y = 0$.

A sequence $\{x_n\}$ is infinitely large if for any $\varepsilon > 0$ only a finite number of points (n, x_n) are between the two horizontal lines $y = -\varepsilon$, and $y = +\varepsilon$.

§3. Properties of convergent sequences

In this section we consider sequences convergent in $\mathbb{R}$ (unless explicitly stated otherwise).

Theorem 3.1. *A convergent sequence has a unique limit.*

Proof: Assume that a sequence $\{x_n\}$ has two limits: a and b $(a \neq b)$. Then from Definition 1.12 with $\varepsilon = |b-a|/4$, it follows that there exists a number n_0 such that

$$|x_n - a| < |b-a|/4, \quad \text{and} \quad |x_n - b| < |b-a|/4 \ \ \forall n \geq n_0.$$

Then

$$|b-a| = |(x_n - a) + (b - x_n)| \leq |x_n - a| + |x_n - b| < |b-a|/2,$$

leading to a contradiction. The proof is complete. $\square$

Theorem 3.2. *A convergent sequence is bounded.*

Proof: Let $\{x_n\} \to a$. Then there exists $n_1 \in \mathbb{N}$ such that

$$|x_n - a| < 1 \ \ \forall\, n \geq n_1.$$

Then

$$|x_n| - |a| \leq |x_n - a| < 1 \quad \Rightarrow \quad |x_n| < |a| + 1 \ \ \forall\, n \geq n_1,$$

and

$$|x_n| \leq K = \max(|x_1|, \ldots, |x_{n_1 - 1}, |a| + 1) \ \ \forall\, n \in \mathbb{N}.$$

The proof is complete. $\square$

Theorem 3.3. *If* $\lim\limits_{n\to\infty} x_n = a$ *then for any number* $b > a$ $(b < a)$ *there exists* $n_b \in \mathbb{N}$ *such that*

$$x_n < b \ \ (x_n > b) \ \ \forall n \geq n_b.$$

Proof: *Let* $b > a$. *From Definition 3.4 it follows that there exists* n_b *such that*

$$|x_n - a| < b - a \ \ \forall n \geq n_b.$$

Rewriting this inequality yields

$$-(b-a) < x_n - a < b - a \quad \Rightarrow \quad x_n < b \ \ \forall n \geq n_b.$$

The proof is complete. □

Corollary 3.5. *Let* $\lim_{n\to\infty} x_n = a$.

1. *If* $a \neq 0$ *then there exists* n_0 *such that* $x_n \neq 0 \;\; \forall n \geq n_0$.
2. *If* $a > 0$ $(a < 0)$ *then there exists* n_0 *such that* $x_n > 0 \;\; (x_n < 0) \;\; \forall n \geq n_0$.

Theorem 3.6. *Let* $\lim_{n\to\infty} x_n = a$.

1. *If there exists* n_b *such that* $x_n \leq b$ $(x_n \geq b)$ $\forall n \geq n_b$, *then* $a \leq b$ $(a \geq b)$.
2. *If there exists* n_b *such that* $x_n < b$ $(x_n > b)$ $\forall n \geq n_b$, *then* $a \leq b$ $(a \geq b)$ *(where the strict inequality in general does not hold).*

Proof: Assume $x_n \leq b \; \forall n \geq n_b$. Suppose $a > b$. Then from Theorem 3.3 there exists $\overline{n}_b$ such that $x_n > b \; \forall n \geq \overline{n}_b$. This inequality for $n > \max\{n_b, \overline{n}_b\}$ leads to a contradiction. The following example shows that in general the strict inequality in part 2 does not hold.

$$\frac{1}{n} > 0 = b \;\; \forall n \in \mathbb{N}; \quad \lim_{n\to\infty} \frac{1}{n} = 0 = a; \quad a = b = 0.$$

The proof is complete. □

Theorem 3.7. *If starting from some* n_0, $x_n \leq y_n$, *and* $\{x_n\}$, $\{y_n\}$ *are both convergent, then*

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Proof: *Let*

$$\lim_{n\to\infty} x_n = a, \quad \lim_{n\to\infty} y_n = b, \quad c = (a+b)/2.$$

If $a > b$, *then* $a > c > b$, *and from Theorem 3.3 there exist* n_c', n_c'' *such that*

$$x_n > c \; \forall n \geq n_c', \quad y_n < c \; \forall n \geq n_c''.$$

Then for all $n \geq \max\{n_c', n_c''\}$ *both inequalities hold true, that is* $x_n > c > y_n$, *leading to a contradiction. The proof is complete.* □

Theorem 3.8. *Let sequences* $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *be such that there exists* $n_0 \in \mathbb{N}$ *with the property that*

$$\text{either} \;\; x_n \leq y_n \leq z_n, \quad \text{or} \;\; x_n \geq y_n \geq z_n \;\; \forall n \geq n_0.$$

If $\{x_n\}$ *and* $\{z_n\}$ *converge to the same limit (finite or infinite), then* $\{y_n\}$ *converges to the same limit as well.*

Proof: Let $\{x_n\} \to a$, $\{z_n\} \to a$, and $a \in \mathbb{R}$. By the assumption of the theorem,

$$x_n - a \leq y_n - a \leq z_n - a \;\; \forall n \geq n_0.$$

On the other hand, for any $\varepsilon > 0$ there exist $n'_\varepsilon, n''_\varepsilon \in \mathbb{N}$ such that

$$|x_n - a| < \varepsilon \quad \Leftrightarrow \quad -\varepsilon < x_n - a < \varepsilon \;\; \forall n \geq n'_\varepsilon,$$

$$|z_n - a| < \varepsilon \quad \Leftrightarrow \quad -\varepsilon < z_n - a < \varepsilon \;\; \forall n \geq n''_\varepsilon.$$

For $n \geq n_\varepsilon = \max(n_0,\; n'_\varepsilon,\; n''_\varepsilon)$ all the above inequalities hold true. Therefore,

$$-\varepsilon < x_n - a \leq y_n - a \leq z_n - a < \varepsilon \;\; \forall n \geq n_\varepsilon.$$

This implies that $\{y_n\} \to a$. For $a = \pm\infty$ the proof is very similar. □

Sometimes, Theorem 3.8 is called the "Squeeze" theorem.

Theorem 3.9.

1. *If sequences $\{x_n\}$ and $\{y_n\}$ converge then their sum, difference, and product converge as well. Moreover,*

$$\lim_{n\to\infty} (x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n,$$

$$\lim_{n\to\infty} (x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n.$$

2. *Additionally, if $y_n \neq 0 \;\; \forall n \in \mathbb{N}$, and $\lim_{n\to\infty} y_n \neq 0$ then*

$$\lim_{n\to\infty} (x_n / y_n) = \lim_{n\to\infty} x_n / \lim_{n\to\infty} y_n.$$

Proof: Let $\{x_n\} \to a$, and $\{y_n\} \to b$. By Theorem 3.2, the sequence $\{y_n\}$ is bounded:

$$\exists K \in \mathbb{R} : \quad |y_n| \leq K \quad \forall n \in \mathbb{N}.$$

Then, for $\varepsilon > 0$ there exist n'_ε and n''_ε such that

$$|x_n - a| < \frac{\varepsilon}{2M} \quad \forall n \geq n'_\varepsilon, \quad |y_n - b| < \frac{\varepsilon}{2M} \quad \forall n \geq n''_\varepsilon,$$

where $M = \max\{1, K, |a|\}$. For $n \geq n_\varepsilon = \max\{n'_\varepsilon, n''_\varepsilon\}$ both inequalities hold true, and

$$|(x_n \pm y_n) - (a \pm b)| \leq |x_n - a| + |y_n - b| < \varepsilon/M \leq \varepsilon,$$

$$|x_n y_n - ab| = |(x_n y_n - y_n a) + (y_n a - ab)| \leq$$

$$|y_n||x_n - a| + |a||y_n - b| < \frac{K\varepsilon}{2M} + \frac{|a|\varepsilon}{2M} \leq \varepsilon.$$

Therefore, $\{x_n \pm y_n\} \to (a \pm b)$, and $\{x_n y_n\} \to ab$.

We now prove part 2. First we show that the sequence $\{1/y_n\}$ is bounded, and $\{1/y_n\} \to 1/b$. Indeed, by Definition 3.4 there exists n_b such that

$$|y_n - b| < |b|/2 \;\; \forall n \geq n_b.$$

Then for all $n \geq n_b$

$$|b| - |y_n| \leq |b - y_n| < |b|/2 \ \Rightarrow\ |y_n| > |b|/2 \ \Rightarrow\ \Big|\frac{1}{y_n}\Big| < \frac{2}{|b|}.$$

Next, for $\varepsilon > 0$ there exits $n_\varepsilon \geq n_b$ such that

$$|y_n - b| < \varepsilon b^2/2 \ \ \forall n \geq n_\varepsilon.$$

Thus, for $n \geq n_\varepsilon$

$$\Big|\frac{1}{y_n} - \frac{1}{b}\Big| = \frac{|y_n - b|}{|y_n| \cdot |b|} < \frac{|y_n - b|}{|b| \cdot |b|/2} < \varepsilon,$$

and $\{\frac{1}{y_n}\} \to \frac{1}{b}$. Therefore, $\{x_n/y_n\} \to a \cdot \frac{1}{b}$. The proof is complete. □

Corollary 3.10. *The sum, difference and product of infinitely small sequences are infinitely small as well.*

Theorem 3.11. *If $\{x_n\}$ is infinitely small, and $\{y_n\}$ is bounded then $\{x_n \cdot y_n\}$ is infinitely small.*

Proof: Let $\{x_n\} \to 0$, and $|y_n| \leq K \ \ \forall n \in \mathbb{N}$. Then for $\varepsilon > 0$ there exists n_ε such that

$$|x_n| < \varepsilon/K \ \ \forall n \geq n_\varepsilon.$$

Then

$$|x_n \cdot y_n| = |x_n| \cdot |y_n| < \varepsilon/K \cdot K = \varepsilon.$$

The proof is complete. □

Theorem 3.12. *If $\{x_n\}$ is infinitely large, and $x_n \neq 0 \ \ \forall n$, then the reciprocal $\left\{\frac{1}{x_n}\right\}$ is infinitely small.*

Proof: By Definition 1.11, for any positive ε there exists n_ε such that

$$|x_n| > \frac{1}{\varepsilon} > 0 \ \ \forall n \geq n_\varepsilon.$$

Therefore, $\left\{\frac{1}{x_n}\right\}$ is defined and

$$\Big|\frac{1}{x_n}\Big| < \varepsilon \ \ \forall n \geq n_\varepsilon.$$

Hence, $\left\{\frac{1}{x_n}\right\}$ is infinitely small. □

The converse statement is also true, and the proof is very similar to that of Theorem 3.12.

Theorem 3.13. *If $\{x_n\}$ is infinitely small, and $x_n \neq 0 \ \ \forall n \in \mathbb{N}$, then the reciprocal $\left\{\frac{1}{x_n}\right\}$ is infinitely large.*

§4. Monotonic sequences

Definition 4.1. *A sequence $\{x_n\}$ is called non-decreasing (non-increasing) if*

$$x_n \le x_{n+1} \quad (x_n \ge x_{n+1}) \ \ \forall n \in \mathbb{N}.$$

A sequence $\{x_n\}$ is called strictly increasing (strictly decreasing) if

$$x_n < x_{n+1} \quad (x_n > x_{n+1}) \ \ \forall n \in \mathbb{N}.$$

Definition 4.2. *A sequence $\{x_n\}$ is called monotonic if it is either non-decreasing or non-increasing. A sequence $\{x_n\}$ is called strictly monotonic if it is either strictly decreasing or strictly increasing.*

Theorem 4.3. *If $\{x_n\}$ is monotonic then it converges in $\overline{\mathbb{R}}$. Moreover, if $\{x_n\}$ is non-decreasing then*

$$\lim_{x\to\infty} x_n = \sup\{x_n\},$$

if $\{x_n\}$ is non-increasing then

$$\lim_{x\to\infty} x_n = \inf\{x_n\}.$$

A monotonic sequence converges in $\mathbb{R}$ if and only if it is bounded.

Proof: Let $\{x_n\}$ be non-decreasing, and $\sup\{x_n\} = a < \infty$. Then every $x_n \le a$, and for $\varepsilon > 0$ there exists x_{n_ε} such that

$$x_{n_\varepsilon} > a - \varepsilon.$$

On the other hand, $x_n \ge x_{n_\varepsilon} \ \ \forall n \ge n_\varepsilon$ since $\{x_n\}$ is non-decreasing. Hence, for $n \ge n_\varepsilon$

$$a - \varepsilon < x_{n_\varepsilon} \le x_n \le a < a + \varepsilon \ \ \forall n \ge n_\varepsilon.$$

Therefore, $\{x_n\} \to a$.

If $\{x_n\}$ is non-decreasing and unbounded, then it is bounded below, and $\sup\{x_n\} = +\infty$. Moreover, $\{x_n\}$ is infinitely large. Indeed, since it is not bounded above, for any ε there exists x_{n_ε} such that $x_{n_\varepsilon} > \varepsilon$. Due to monotonicity, $x_n \ge x_{n_\varepsilon} > \varepsilon \ \ \forall n \ge n_\varepsilon$. Therefore, $\{x_n\} \to +\infty$ (see Remark 1.13), and the proof is complete. □

Example. We will show that the sequence $\{(1+1/n)^n\}$ is strictly increasing, bounded, and by Theorem 4.3 has a finite limit: we denote it by e:

$$\lim_{x\to\infty} (1+1/n)^n =: e.$$

Let $x_n = (1+1/n)^n$. We first show that $x_n < x_{n+1} \ \ \forall n \in \mathbb{N}$. From binomial expansion we obtain:

Another important application of Theorem 4.3 is given in the following lemma.

Lemma 4.4 (On nested intervals). *Let the sequence of the intervals* $\{[a_n, b_n]\}$ *be such that* $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ $\forall n \in \mathbb{N}$ *(such intervals are called nested), and their lengths converge to zero, i.e.,* $\lim_{n\to\infty}(b_n - a_n) = 0$. *Then there exists a unique point* ξ *that belongs to every interval, i.e.,* $\xi \in [a_n, b_n]$ $\forall n \in \mathbb{N}$, *and*

$$\xi = \lim_{n\to\infty} a_n = \sup\{a_n\} = \lim_{n\to\infty} b_n = \inf\{b_n\}.$$

Proof: The sequences $\{a_n\}$ and $\{b_n\}$ are monotonic and bounded:

$$a_1 \le a_n \le a_{n+1} \le b_{n+1} \le b_n \le b_1 \quad \forall n \in \mathbb{N}.$$

By Theorem 3.4, they are convergent, and

$$\lim_{n\to\infty} a_n = \sup\{a_n\} =: a, \quad \lim_{n\to\infty} b_n = \inf\{b_n\} =: b.$$

However, $\lim_{n\to\infty}(b_n - a_n) = 0$ implies that $a = b =: \xi$, and clearly

$$a_n \le a = \xi = b \le b_n \quad \forall n \in \mathbb{N}.$$

Next we show that there is no point other than ξ that belongs to all the intervals $[a_n, b_n]$. Assume the opposite: $\exists \eta \in [a_n, b_n]$ $\forall n$ and $\eta \ne \xi$. Then by Theorem 3.6

$$b_n - a_n \ge |\xi - \eta| > 0 \ \ \forall n \quad \Rightarrow \quad \lim_{n\to\infty}(b_n - a_n \ge |\xi - \eta| > 0.$$

This contradiction completes the proof of the lemma. □

§5. Partial limits

So far we have been studying convergent sequences. This raises a natural question: What about divergent ones? Can they be studied? What characteristics need to be defined for that? It turns out that divergent sequences also can be describes and thoroughly studied. In order to do that we first introduce the following important concept.

Definition 5.1. *Let* x_n *be a sequence, and let* n_k *be a strictly increasing sequence of natural numbers. Then the sequence* x_{n_k} *is called a subsequence of* x_n.

Clearly, every sequence has an infinite number of subsequences. Next we prove the following important theorem.

Theorem 5.2 (Bolzano-Weierstrass). *Every bounded sequence contains a convergent in* $\mathbb{R}$ *subsequence.*

Proof: Let $\{x_n\}$ be bounded. Then there exists an interval $[-K, K]$ containing all terms of $\{x_n\}$. Let $a_1 := -K$, $b_1 := K$. Let $c_1 = (a_1 + b_1)/2$. Then one of the intervals, either $[a_1, c_1]$ or $[c_1, b_1]$, contains infinitely many terms of $\{x_n\}$ (both can not contain only finite number of the terms, because the whole sequence is infinite). We denote that interval by $[a_2, b_2]$. If both $[a_1, c_1]$ and $[c_1, b_1]$ contain infinitely many terms, then by $[a_2, b_2]$ we denote either one.

Next, let $c_2 = (a_2+b_2)/2$. By $[a_3, b_3]$ we denote that one of the intervals $[a_2, c_2]$ and $[c_2, b_2]$ which contains infinitely many terms of $\{x_n\}$. Again, if both contain infinitely many terms, then we choose either one.

Continuing this process, let $c_n = (a_n + b_n)/2$. By $[a_{n+1}, b_{n+1}]$ we denote that one of the intervals $[a_n, c_n]$ and $[c_n, b_n]$ which contains infinitely many terms of $\{x_n\}$. If both contain infinitely many terms, then we choose either one.

By this process, we obtain a sequence of nested intervals $\{[a_n, b_n]\}$: since $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. The lengths of the intervals go to zero:

$$\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{4K}{2^n} = 0.$$

According to the lemma 4.4 on nested intervals there exists a point ξ such that $\xi \in [a_n, b_n] \;\; \forall n \in \mathbb{N}$ and

$$\xi = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Since every interval $[a_n, b_n]$ contains infinitely many terms of $\{x_n\}$, for any $k \in \mathbb{N}$ there exists a term $x_{n_k} \in [a_k, b_k]$ with $n_k > n_{k-1}$. By Theorem 1.6 the subsequence $\{x_{n_k}\}$ converges to ξ:

$$a_k \le x_{n_k} \le b_k \;\; \forall k, \quad \{a_k\} \to \xi, \quad \{b_k\} \to \xi.$$

The proof is complete. □

For unbounded sequences the following theorem holds.

Theorem 5.3. *Every sequence that is not bounded above (below) contains a subsequence that converges to $+\infty$ ($-\infty$). Moreover, if a sequence is unbounded and not infinitely large, then it contains a subsequence convergent in $\mathbb{R}$.*

Proof: □

Combining Theorem 5.2 and Theorem 5.3 we conclude the next theorem.

Theorem 5.4 . *Every sequence contains a convergent in $\overline{\mathbb{R}}$. subsequence. Moreover, if a sequence is not infinitely large, then it contains a subsequence convergent in $\mathbb{R}$.*

Therefore, every sequence can be studied by analyzing its convergent subsequences. There may be many of them, and they can converge to different numbers in $\overline{\mathbb{R}}$. These numbers characterize the sequence.

Definition 5.5. *A number a is called a partial limit of x_n if there exists $\{x_{n_k}\}$ – a subsequence of $\{x_n\}$ – such that $\{x_{n_k}\} \to a$.*

Thus, every sequence can be characterized by the set of its partial limits. In the following sections, we denote by $Lim\ x_n$ the set of partial limits of $\{x_n\}$.

§6. Upper and Lower limits

First we introduce the following important characteristics of sequences.

Definition 6.1. *The quantities*

$$\sup\ Lim\ x_n =: \overline{\lim}\ x_n, \quad \textit{and} \quad \inf\ Lim\ x_n =: \underline{\lim}\ x_n$$

are called the upper and the lower limits of $\{x_n\}$ *respectively.*

Now we establish some properties of the upper and the lower limits. First of all we note that every sequence has upper and lower limits, and they are unique.

If a sequence is not bounded above (below), then by Theorem 5.3

$$\overline{\lim}\ x_n = +\infty \quad (\underline{\lim}\ x_n = -\infty).$$

Clearly, for any sequence

$$\overline{\lim}\ x_n \geq \underline{\lim}\ x_n.$$

From the properties of supremum and infimum it follows that

$$\overline{\lim}\ (-x_n) = -\underline{\lim}\ x_n,\ \underline{\lim}\ (-x_n) = -\overline{\lim}\ x_n.$$

Moreover, for any $\{x_{n_k}\}$ subsequence of $\{x_n\}$

$$\overline{\lim}\ x_n \geq \overline{\lim}\ x_{n_k}, \quad \underline{\lim}\ x_n \leq \underline{\lim}\ x_{n_k}.$$

Theorem 6.2. *Let* $\{x_n\}$ *be convergent. Then*

$$Lim\ x_n = \{\lim_{n\to\infty} x_n\},$$

and the following equalities hold

$$\underline{\lim}\ x_n = \overline{\lim}\ x_n = \lim_{n\to\infty} x_n.$$

Proof: We claim that if $\{x_n\} \to a$, then $\{x_{n_k}\} \to a$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Indeed, let $\lim\limits_{n\to\infty} x_n = a$. Then for any positive ε there exists n_ε such that

$$|x_n - a| < \varepsilon \quad \forall n \geq n_\varepsilon.$$

In particular,

$$|x_{n_k} - a| < \varepsilon \quad \forall n_k \geq n_\varepsilon.$$

Therefore, $\{x_{n_k}\} \to a$ as well for any subsequence. This means that $Lim\ x_n = \{a\}$, and the proof is complete. □

Theorem 6.3. *The upper and the lower limits are partial limits, i.e.,*

$$\underline{\lim}\ x_n \in Lim\ x_n, \qquad \overline{\lim}\ x_n \in Lim\ x_n.$$

Proof: □

§7. Criteria of convergence

In this section we establish several important criteria of convergence for sequences. The first one is trivial.

Theorem 7.1. *A sequence is convergent (in $\overline{\mathbb{R}}$) if and only if its every subsequence is convergent (in $\overline{\mathbb{R}}$).*

Proof: ($\Rightarrow$) follows from Theorem 6.2. ($\Leftarrow$) follows from the fact that the sequence itself is its own subsequence. $\square$

Theorem 7.2. *A sequence is convergent (in $\overline{\mathbb{R}}$) if and only if*

$$\underline{\lim}\, x_n = \overline{\lim}\, x_n.$$

Their common value is the limit of the sequence.

Proof: ($\Rightarrow$) follows from Theorem 6.2. ($\Leftarrow$) follows from Theorem 7.1. $\square$

Theorem 7.3. *A sequence is convergent (in $\overline{\mathbb{R}}$) if and only if the set of its partial limits consists of one point.*

Proof: ($\Rightarrow$) follows from Theorem 6.2. ($\Leftarrow$) follows from Theorem 7.2. $\square$

To establish one more criterion of convergence, we need the following definition.

Definition 7.4. *$\{x_n\}$ is called a fundamental (or Cauchy) sequence if $\forall \varepsilon > 0\ \exists n_\varepsilon$ such that*

$$|x_{n+m} - x_n| < \varepsilon \quad \forall n \geq n_\varepsilon \ \ \forall m \in \mathbb{N}.$$

Lemma 7.5. *A fundamental sequence is bounded.*

Proof: By Definition 7.4, $\exists n_1$ such that

$$|x_{n+m} - x_n| < 1 \quad \forall n \geq n_1 \ \ \forall m \in \mathbb{N}.$$

In particular,

$$|x_{n_1+m} - x_{n_1}| < 1 \qquad \forall m \in \mathbb{N}.$$

Then

$$|x_n| \leq K = \max\{|x_1|, \ldots, |x_{n_1-1}|, |x_{n_1}| + 1\} \ \ \forall n \in \mathbb{N}.$$

The proof is complete. $\square$

Theorem 7.6 (Cauchy Criterion of Convergence). *A sequence is convergent (in $\mathbb{R}$) if and only if it is fundamental.*

Proof: ($\Rightarrow$). Let $\lim\limits_{n\to\infty} x_n = a < \infty$. Then for any positive ε there exists n_ε such that

$$|x_n - a| < \varepsilon/2 \quad \forall n \geq n_\varepsilon.$$

Therefore,

$$|x_{n+m} - x_n| = |(x_{n+m} - a) + (a - x_n)| \leq |x_{n+m} - a| +$$

$$|a - x_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall n \geq n_\varepsilon, \quad \forall m \in \mathbb{N},$$

since $n + m > n \geq n_\varepsilon \ \ \forall m \in \mathbb{N}$.

($\Leftarrow$). Let $\{x_n\}$ be fundamental. By Definition 7.4, for any positive ε there exists n_ε such that

$$|x_{n+m} - x_n| < \varepsilon/2 \quad \forall n \geq n_\varepsilon \ \ \forall m \in \mathbb{N}.$$

By Lemma 7.5, $\{x_n\}$ is bounded. By Bolzano-Weiestrass Theorem 5.2, $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$, i.e., $\lim_{k\to\infty} x_{n_k} = a < \infty$. we claim that $\lim_{n\to\infty} x_n = a$. Indeed by Definition 3.4, for any positive ε there exists $n'_\varepsilon > n_\varepsilon$ such that

$$|x_{n_k} - a| < \varepsilon/2 \quad \forall n_k \geq n'_\varepsilon.$$

Combining these inequalities, for any $n > n'_\varepsilon$ we obtain:

$$|x_n - a| = |(x_n - x_{n'_\varepsilon}) + (x_{n'_\varepsilon} - a)| \leq |x_n - x_{n'_\varepsilon}| + |x_{n'_\varepsilon} - a| < \varepsilon.$$

Thus, $\{x_n\} \to a$, and the proof is complete. □

§8. Structures of sequences

In this section we summarize the theory of sequences, and consider structures of sequences. The set of partial limits $Lim\ x_n$ is a very important characteristics of a sequence.

First we characterize bounded sequences.

1. $\{x_n\}$ is bounded if and only if its upper and lowere limits are finite:

$$-\infty < \underline{\lim}\ x_n \leq \overline{\lim}\ x_n < +\infty.$$

2. $\forall \varepsilon > 0$ all terms of a bounded sequence with the exception of a finitely many of them are located in the interval:

$$(\underline{\lim}\ x_n - \varepsilon, \quad \overline{\lim}\ x_n + \varepsilon),$$

   or, equivalently, all the points (n, x_n) with the exception of a finitely many of them are located between the lines

$$y = \underline{\lim}\ x_n - \varepsilon, \quad \text{and} \quad y = \overline{\lim}\ x_n + \varepsilon.$$

3. $\forall \varepsilon > 0$ there are infinitely many terms of x_n located above the line $y = \underline{\lim}\ x_n - \varepsilon$ and below the line $y = \overline{\lim}\ x_n + \varepsilon$.

For unbounded sequences the following is true:

4. If $\{x_n\}$ is infinitely large, then its set of partial limits consists of not more that two points: $Lim\ x_n \subseteq \{-\infty, +\infty\}$. More precisely, if $\{x_n\}$ is bounded above (below), then $Lim\ x_n = \{-\infty\}$ ($Lim\ x_n = \{+\infty\}$. If it is neither bounded above nor below, then $Lim\ x_n = \{-\infty, +\infty\}$

   Example: We characterize the sequence $\{x_n\}$ with $Lim\ x_n = \{-\infty, 0, 2\}$.

For this sequence we conclude that $\underline{\lim}\, x_n = -\infty$, $\overline{\lim}\, x_n = 2$. It is divergent, not bounded below, bounded above, not infinitely large. It contains an infinitely large and infinitely small subsequences, as well as a subsequence convergent to 2.

§9. Problems

1. Characterize $\{x_n\}$ if

(a) $Lim\ x_n = \{0, 2\}$,

(b) $Lim\ x_n = \{-\infty, -1, 0, 2\}$,

(c) $Lim\ x_n = \{-1, 1, +\infty\}$,

(d) $Lim\ x_n = \{-\infty, 2, +\infty\}$,

(e) $Lim\ x_n = \{0, 2, +\infty\}$.

2. Characterize $\{x_n\}$ and $\{\frac{1}{x_n}\}$ if

(a) $Lim\ x_n = \{2\}$,

(b) $Lim\ x_n = \{-1\}$,

(c) $Lim\ x_n = \{-1, 1\}$,

(d) $Lim\ x_n = \{-\infty, -2, 2\}$,

(e) $Lim\ x_n = \{2, 3, +\infty\}$.

3. Characterize $\{x_n \cdot y_n\}$ if

(a) $Lim\ x_n = \{0\}$, $Lim\ y_n = \{-1, 1\}$;

(b) $Lim\ x_n = \{2\}$, $Lim\ y_n = \{-2, 3\}$;

(c) $Lim\ x_n = \{-1, 2\}$, $Lim\ y_n = \{2, 3\}$.

4. Find $Lim\ x_n$ for

(a) $\{(-1)^n\}$,

(b) $\{-n\}$,

(c) $\{1 - (-1)^n\}$,

(d) $\{\sin \pi n/3\}$.

§10. Power Series

Definition 10.1. *Let $\{a_n\}$ be a sequence of real numbers. The series*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{10.1}$$

is called a power series.

Ratio Criterion 10.2. *Let $R := \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. Then (10.1) converges absolutely whenever $|x| < R$ and diverges whenever $|x| > R$.*

Proof: Follows from the ratio test applied to (10.1). □

Root Criterion 10.3. *Let $\widetilde{R} := \lim_{n\to\infty} |a_n|^{1/n}$. Then (10.1) converges absolutely whenever $|x| < \widetilde{R}$ and diverges whenever $|x| > \widetilde{R}$.*

Proof: Follows from the root test applied to (10.1). □

From theorems 10.2 and 10.3 we comclude that $R = \widetilde{R}$. This quantity is reffered to as the radius of convergence.

Theorem 10.4. *Let R be the radius of convergence of (10.1). Then (10.1) converges uniformly on $[-K, K]$ for any $0 < K < R$.*

Proof: Follows from the Weierstrass M-test. □

Theorem 10.5. *Let R be the radius of convergence of (10.1). If (10.1) converges at $x = R$, then it converges uniformly on $[-K, R]$ for any $0 < K < R$. If (10.1) converges at $x = -R$, then it converges uniformly on $[-R, K]$ for any $0 < K < R$.*

HW Example. Consider $\sum_{n=0}^{\infty} x^n/n!$. Its radius of convergence is $+\infty$. Therefore, it uniformly converges on any closed interval $[-K, K]$. However, it does NOT uniformly convergent on $\mathbb{R}$. Prove all the statements made above.

§11. Taylor Series

Theorem 11.1. *Let f have $n+1$ derivatives on (a, b), and its first n derivatives be continuous on $[a, b]$. Then for each $x \in [a, b]$ with $x \neq x_0$ there exists c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n +$$

$$+ \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof: Fix $x \in [a, b]$ with $x \neq x_0$, and let M be the unique solution of

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \\ + M(x - x_0)^{n+1}.$$

Our proof will consist of showing that $M = \frac{f^{(n+1)}(c)}{(n+1)!}$ for some c between x and x_0. Define

$$F(t) = f(t) + f'(t)(x - t) + \ldots + \frac{f^{(n)}(t)}{n!}(x - t)^n + M(x - t)^{n+1}.$$

Since $F(x) = f(x)$, and by our choice of M, $F(x_0) = f(x)$, from the Mean Value Theorem it follows that $\exists\ c$ between x and x_0 such that

$$F'(c) = \frac{F(x) - F(x_0)}{x - x_0} = 0.$$

After computing the derivative of F, we conclude that

$$0 = F'(c) = \frac{f^{(n+1)}(c)}{n!}(x - c)^n - M(n + 1)(x - c)^n,$$

and the result follows. □

Definition 11.2. *Let $f \in C^\infty$. The series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{11.1}$$

is called the Taylor's series about x_0 generated by f. To indicate that f generates this series, we write

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Does (11.1) converge for any x besides $x = x_0$? If so, is its sum equal to $f(x)$? In general, the answer to both questions is "No". From Theorem 11.2 it follows that a necessary and sufficient condition for the Taylor's series to converge to $f(x)$ is that

$$\lim_{n\to\infty} \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} = 0.$$

In practice it may be quite dificult to deal with this limit because of the unknown position of c. In some cases, however, the limit can be shown to be zero.

Theorem 11.3. *Let $f \in C^\infty$ on $[a, b]$, and $|f^i(x)| \le M\ \forall\ x \in [a, b]\ \ \forall i \in \mathbb{N}$. Then $\forall\ x \in [a, b]$ we have*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Proof: *HW* □

§12. Problems

For each of the following functions

(a) Write the Taylor's series about $x_0 = 0$;

(b) Find its radius of convergence;

(c) Determine whether the Taylor's series converges to the generating function.

1. $f(x) = \frac{1}{(1-x)}$

2. $f(x) = \ln(1-x)$

3. $f(x) = e^x$

4. $f(x) = \sin x$

5. $f(x) = \cos x$

6. $f(x) = \frac{1}{(1-x)^2}$

7. $f(x) = \sqrt{1+x}$

8. $f(x) = (1+x)^\alpha, \quad \alpha \in \mathbb{R}$