Optimal Quasi-Interpolation by Quadratic C^1-Splines on Type-2 Triangulations

Tatyana Sorokina and Frank Zeilfelder

Abstract. We describe a new scheme based on quadratic C^1-splines on type-2 triangulations approximating gridded data. The quasi-interpolating splines are directly determined by setting the Bernstein-Bézier coefficients of the splines to appropriate combinations of the given data values. In this way, each polynomial piece of the approximating spline is immediately available from local portions of the data, without using prescribed derivatives at any point of the domain. Since the Bernstein-Bézier coefficients of the splines are computed directly, an intermediate step making use of certain locally supported splines spanning the space is not needed. We prove that the splines yield optimal approximation order for smooth functions, where we provide explicit constants in the corresponding error bounds.

§1. Introduction

The aim of this paper is to describe local methods which use quadratic C^1-splines on the type-2 triangulation to approximate data on a two-dimensional grid. By the nature of the new algorithm, we do not start from a particular basis or a set of locally supported spline functions spanning the spaces. In contrast, the basic idea here is to consider the piecewise Bernstein-Bézier representation and to set each B-coefficient of the approximating splines directly by applying natural and simple averaging rules to local portions of the given data. Hence, the approach falls into the class of quasi-interpolation methods for bivariate splines.

An essential difference to the current literature on quasi-interpolation and interpolation by bivariate splines based on the piecewise Bernstein-Bézier form (see [4 –11,13,15,17], the survey [16] and the references therein) is that here we do not make use of a local and stable minimal determining set or related concepts for the spaces. These concepts have been shown to

Proceedings Title
XXX (eds.), pp. 1–8.

ISBN 0-9728482-x-x

be very useful and successful, in particular if one is interested in finding interpolation and approximation operators which yield (nearly) optimal approximation order of smooth functions. Unfortunately, it is known that there are some cases where it is difficult - and sometimes impossible - to apply these concepts in a direct way. This concerns smooth spline spaces on general and particular classes of triangulations, where the polynomial degree of the splines is low or lowest possible. Such an example are quadratic C^1-splines on type-2 triangulations for which it seems difficult - if not impossible - to construct local and stable minimal determining sets or (completely) local interpolation sets (with optimal approximation properties), see [1,2,22,23,27]. On the other hand, using the uniform structure of these triangulations and a spanning set of locally supported spline functions, it has been shown that full approximation power is provided by these spaces [2,21]. Our approach deals with quadratic C^1-splines on type-2 triangulations given in its piecewise Bernstein-Bézier form without requiring intermediate constructions of this type. Hence, the method is related but different to the approach in [2,3,20,21] (see the discussion in the final section of this paper). The new idea of our approach is to (uniquely) set all the B-coefficients of a spline simultaneously using local portions of the data in such a way that all the smoothness conditions of the C^1-splines are automatically satisfied. This is possible since the uniform structure of the underlying triangulation implies that certain B-coefficients have to be simple averages of certain other B-coefficients in order to guarantee the overall C^1-smoothness. Since we only use the given data values and average them, (approximate) derivatives are not required at any point in the domain and the overall setting can be considered as a procedure of repeated averaging. Moreover, the setting of the B-coefficients is done carefully so that the resulting quasi-interpolating splines yield optimal approximation order of smooth functions. We show this by proving that the corresponding quasi-interpolation operator is uniformly bounded and reproduces bivariate quadratic polynomials. In addition, we provide some explicit constants in the corresponding error bounds of the approximating splines and its derivatives.

The paper is organized as follows. In Sect. 2, we give some preliminaries on the Bernstein-Bézier form of quadratic C^1-splines on the type-2 triangulation, describe the smoothness conditions of the spaces in a convenient form and prove a Markov type inequality with explicit constants for quadratic polynomial pieces on the triangles of this partition. The new quasi-interpolating scheme is described in Sect. 3, and some useful properties of the corresponding operator are discussed in Sect. 4. These results are used in Sect. 5, where we provide error bounds for the quasi-interpolating splines with explicit constants. We conclude with a discussion comparing the new scheme with previously existing quasi-interpolation methods for quadratic C^1-splines in Sect. 6.

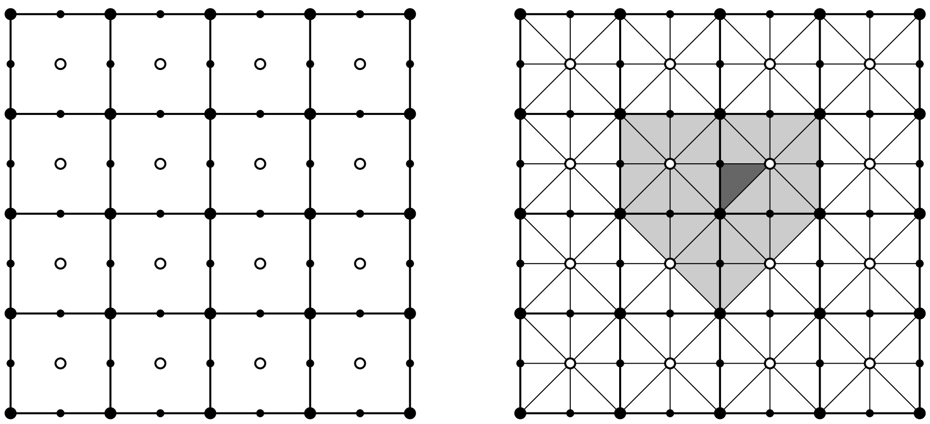

Fig. 1. Point sets associated with partition $\Diamond$ (left), triangulation $\diamondplus$ (right).

§2. B-form of Quadratic C^1-splines and Markov Inequality

Let $h > 0$, $n, m \geq 2$, and $V := \{v_{ij} = (ih, jh),\ i = 0, \ldots, n,\ j = 0, \ldots, m\}$ be the set of $(n+1) \times (m+1)$ points in the rectangular domain $\Omega := [0, nh] \times [0, mh]$. Then the collection of squares $Q_{ij} = [ih, (i+1)h] \times [jh, (j+1)h]$, where $i = 0, \ldots, n-1$, $j = 0, \ldots, m-1$, forms a partition $\Diamond$ of Ω. Let $X := \{x_{ij} = (v_{ij} + v_{i+1,j})/2,\ i \neq n\}$, $Y := \{y_{ij} = (v_{ij} + v_{i,j+1})/2,\ j \neq m\}$ and $W := \{w_{ij} = (v_{ij} + v_{i+1,j+1})/2,\ i \neq n,\ j \neq m\}$, be the sets of midpoints of horizontal and vertical edges of the squares in $\Diamond$, and the set of the centers of all squares in $\Diamond$, respectively. In Fig. 1 (left) the points from the set V are shown as big black dots, while the points from X and Y are small black dots and the points from W are open dots.

In each square Q_{ij}, see Fig. 2 on the left, we connect the center w_{ij} of Q_{ij} with the four vertices $v_{ij}, v_{i+1,j}, v_{i,j+1}, v_{i+1,j+1}$ of Q_{ij}, and with the four midpoints $x_{ij}, y_{ij}, x_{i,j+1}, y_{i+1,j}$ of the edges of Q_{ij}, to obtain a triangulation $\diamondplus$ of Ω, which is called the type-2 triangulation, see Fig. 1 (right). Note that each triangle T in $\diamondplus$ has one vertex from V, another from W, and the other one from either X or Y. In the following, we consider the space of quadratic C^1-splines on $\diamondplus$, defined by

$$\mathcal{S}_2^1(\diamondplus) = \{s \in C^1(\Omega) :\ s|_T \in \mathcal{P}_2,\ \text{for all } T \in \diamondplus\},$$

where $\mathcal{P}_2 = \operatorname{span}\{x^i y^j\ :\ i + j = 0, 1, 2\}$ denotes the space of bivariate polynomials of total degree two. We use the piecewise Bernstein-Bézier representation (B-form) of the splines (cf. [7,9,16]), i.e., for each spline $s \in \mathcal{S}_2^1(\diamondplus)$, the polynomial piece $p = s|_T \in \mathcal{P}_2$ on a triangle $T \in \diamondplus$ is given by

$$p = \sum_{i+j+k=2} c_{ijk}^T\ B_{ijk}, \tag{1}$$

where $B_{ijk} = (2/i!j!k!)\ \lambda_0^i \lambda_1^j \lambda_2^k$, $i + j + k = 2$, are the six quadratic Bernstein polynomials associated with $T = \langle v_0, v_1, v_2 \rangle$. Here, λ_ν are the linear

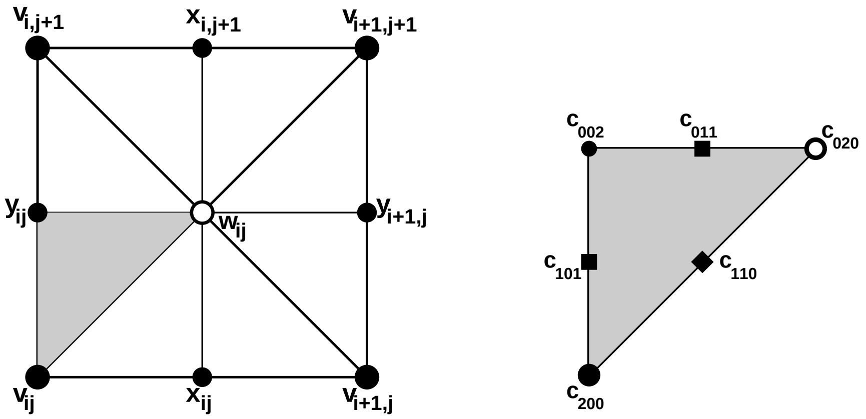

Fig. 2. A (triangulated) square of $\Diamond$ (left) and the triangle T (shaded grey) with the domain points (right) associated with the B-coefficients $c_{ijk} = c^T_{ijk}$.

polynomials determined by $\lambda_\nu(v_\mu) = \delta_{\nu\mu}$, or the so-called barycentric coordinates w.r.t. T ($\delta_{\nu\mu}$ denotes Kronecker's symbol). As usual, we associate the B-coefficients c^T_{ijk} of p with the domain points $\xi^T_{ijk} := (iv_0 + jv_1 + kv_2)/2$ in T (see Fig. 2 on the right, where the B-coefficients are related to the domain points for the triangle $T = \langle v_{ij}, w_{ij}, y_{ij} \rangle$). According to the uniform structure of $\diamondplus$, we can describe the union of the sets of domain points associated with each triangle of $\diamondplus$ denoted by $\mathcal{D}_{\diamondplus}$ in the following convenient form

$$\mathcal{D}_{\diamondplus} = \{\eta_{ij} := (ih, jh)/4,\ i = 0, \dots, 4n,\ j = 0, \dots, 4m\}.$$

By using the piecewise B-form, C^1-smoothness conditions of $\mathcal{S}^1_2(\diamondplus)$ can be described in a very simple way as well. Let c_{ij} be the B-coefficient associated with the domain point η_{ij}. Then, $s \in \mathcal{S}^1_2(\diamondplus)$ if and only if the following conditions are satisfied:

$$\begin{aligned}
c_{ij} &= (c_{i-1\,j} + c_{i+1\,j})/2, \qquad i \bmod 2 = 0,\\
c_{ij} &= (c_{i\,j-1} + c_{i\,j+1})/2, \qquad j \bmod 2 = 0,\\
c_{ij} &= (c_{i-1\,j-1} + c_{i+1\,j+1})/2 = (c_{i-1\,j+1} + c_{i+1\,j-1})/2,\\
&\qquad i \bmod 4 = j \bmod 4 \in \{0, 2\},\\
c_{ij} &= c_{i\,j-1} + c_{i-1\,j} - c_{i-1\,j-1} = c_{i\,j+1} + c_{i+1\,j} - c_{i+1\,j+1},\\
&\qquad i \bmod 4 = j \bmod 4 \in \{1, 3\},\\
c_{ij} &= c_{i\,j-1} + c_{i+1\,j} - c_{i+1\,j-1} = c_{i\,j+1} + c_{i-1\,j} - c_{i-1\,j+1},\\
&\qquad i \bmod 4 \neq j \bmod 4, \text{ and } i \bmod 4,\ j \bmod 4 \in \{1, 3\},
\end{aligned} \tag{2}$$

for all admissible values $i \in \{0, \dots, 4n\}$, $j \in \{0, \dots, 4m\}$. Note that the conditions (2) can be considered as averaging rules to be satisfied for the B-coefficients of a smooth quadratic spline s.

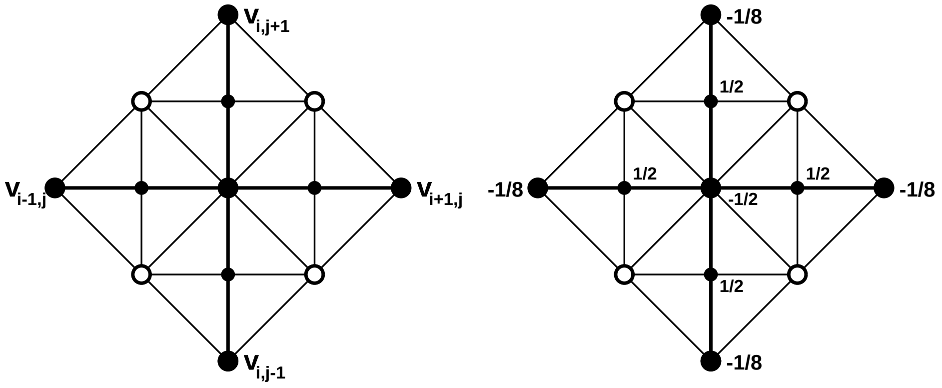

Fig. 3. Evaluation of the B-coefficient c_{200} associated with the domain point v_{ij}: fractions on the right build the mask of the nine data values at points from $V \cup X \cup Y$.

Throughout the paper, we let $D_x f$ and $D_y f$ be the first derivatives of a smooth function f in x and y direction, respectively, and denote higher derivatives by $D_x^\alpha D_y^\beta f$. Moreover, for $f \in C(\Omega)$ and any compact subset $B \subseteq \Omega$, we let $\|f\|_B := \sup\{|f(v)| : \ v \in B\}$ be the uniform norm. For later use, we give the following result which is an inequality of Markov type for quadratic polynomials on the triangles of $\diamondplus$ involving explicit constants.

Lemma 1. *Let $p \in \mathcal{P}_2$ and $T \in \diamondplus$. Then*

$$\|D_x^\alpha D_y^\beta \, p\|_T \le C_{\alpha,\beta} \, \|p\|_T \, h^{-\alpha-\beta}, \qquad \alpha+\beta = 1,2, \tag{3}$$

where $C_{\alpha,\beta} = \begin{cases} 20, & \text{if } \alpha+\beta=1, \\ 64, & \text{if } \alpha+\beta=2 \end{cases}$.

Proof: Let $T = \langle v_0, v_1, v_2 \rangle$ be a triangle from $\diamondplus$. Without loss of generality we assume that $v_0 \in V$, $v_1 \in W$, and $v_2 \in Y$, see Fig. 2. Let $z_{\nu\mu} := (v_\nu + v_\mu)/2$, $\nu \neq \mu$, be the midpoints on the edges of T. Using unique Lagrange interpolation at the six domain points in T, p in its representation (1) w.r.t. T can be written as

$$\begin{aligned} p = p_0 \, B_{200} + \big(2 \, p_{01} - (p_0 + p_1)/2\big) \, B_{110} + p_1 \, B_{020} + \\ \big(2 \, p_{12} - (p_1 + p_2)/2\big) \, B_{011} + p_2 \, B_{002} + \big(2 \, p_{02} - (p_0 + p_2)/2\big) \, B_{101}, \end{aligned}$$

where $p_\nu = p(v_\nu)$ and $p_{\nu\mu} = p(z_{\nu\mu})$. It follows that

$$\begin{aligned} D_x p &= 4/h \, \Big(\big(2 \, (p_{01} - p_{02}) + (p_2 - p_1)/2\big) \, \lambda_0 + \\ &\qquad \big(-2 \, p_{12} + (p_2 + 3p_1)/2\big) \, \lambda_1 + \big(2 \, p_{12} - (3p_2 + p_1)/2\big) \, \lambda_2 \Big), \\ D_x^2 p &= 16/h^2 \, (-2 \, p_{12} + p_1 + p_2), \\ D_x D_y p &= 16/h^2 \, (-p_2 + p_{12} + p_{02} - p_{01}). \end{aligned}$$

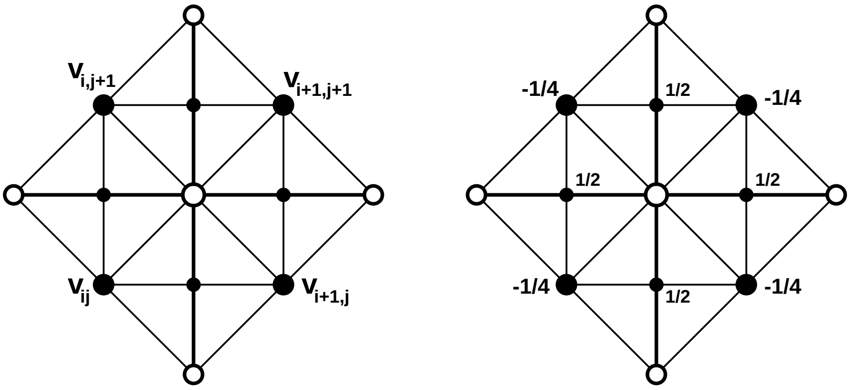

Fig. 4. Evaluation of the B-coefficient c_{020} associated with w_{ij}: the mask of eight data values is shown on the right.

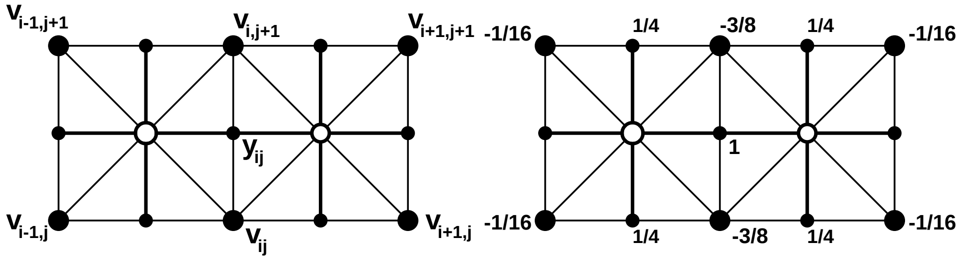

Fig. 5. Evaluation of c_{002} associated with y_{ij}: the mask of eleven data values is shown on the right.

Similar equations hold true for $D_y p$ and $D_y^2 p$, and the result follows from the triangle inequality and the fact that B-polynomials form a partition of unity. □

§3. Quasi-Interpolation Scheme

Let the $3\ nm + 2\ (n+m) + 1$ values

$$f_{ij} := f(v_{ij}),\ f_{ij}^x := f(x_{ij}),\ i \neq n,\ f_{ij}^y := f(y_{ij}),\ j \neq m, \qquad i = 0, \dots, n,\ j = 0, \dots, m, \tag{4}$$

of a function $f \in C(\Omega)$ at the points from $V \cup X \cup Y$ be given. Then, a quadratic spline s_f w.r.t. $\diamondplus$ in its piecewise B-representation (1) is directly determined from the data by setting the B-coefficients of s_f as appropriate averages of local portions of the data. We do this as follows.

It suffices to describe the setting of the B-coefficients c_{ijk}, $i+j+k = 2$, of $s_f|_T$ for the triangle $T := \langle v_{ij}, w_{ij}, y_{ij} \rangle$ as in Fig. 2. Since every triangle in $\diamondplus$ has one vertex at the point in V, another vertex at the point in W, and the third vertex at the point in either X or Y, the setting of the B-coefficients of s_f associated with the remaining domain points in Q_{ij} then follows from symmetry. We distinguish four cases, where we use the notation from Fig. 2 on the right.

Case 1. No vertex of T lies on the boundary of Ω. Then, we set

$$\begin{aligned}
c_{200} :=& 1/2\ (f^x_{ij} + f^y_{ij} + f^x_{i-1\,j} + f^y_{i\,j-1} - f_{ij}) \\
&- 1/8\ (f_{i-1\,j} + f_{i\,j-1} + f_{i+1\,j} + f_{i\,j+1}), \\
c_{020} :=& 1/2\ (f^x_{ij} + f^y_{ij} + f^x_{i\,j+1} + f^y_{i+1\,j}) \\
&- 1/4\ (f_{ij} + f_{i+1\,j} + f_{i\,j+1} + f_{i+1\,j+1}), \\
c_{002} :=& f^y_{ij} - 3/8\ (f_{ij} + f_{i\,j+1}) + 1/4\ (f^x_{ij} + f^x_{i-1\,j} + f^x_{i-1\,j+1} + f^x_{i\,j+1}) \\
&- 1/16\ (f_{i-1\,j} + f_{i-1\,j+1} + f_{i+1\,j} + f_{i+1\,j+1}), \\
c_{110} :=& f^x_{ij} + f^y_{ij} - 1/2\ f_{ij} - 1/4\ (f_{i+1\,j} + f_{i\,j+1}), \\
c_{101} :=& f^y_{ij} + 1/2\ (f^x_{ij} + f^x_{i-1\,j} - f_{ij}) - 1/4\ f_{i\,j+1} - 1/8\ (f_{i-1\,j} + f_{i+1\,j}), \\
c_{011} :=& f^y_{ij} + 1/2\ (f^x_{ij} + f^x_{i\,j+1}) - 1/8\ (3\ f_{ij} + 3\ f_{i\,j+1} + f_{i+1\,j} + f_{i+1\,j+1}).
\end{aligned} \tag{5}$$

Figs. 3–6 illustrate this setting. In these figures the fractions on the right build the mask of data values at points from $V \cup X \cup Y$ (shown as big and small black dots) which are used to determine the corresponding B-coefficient.

Case 2. Exactly one of the vertices of T lies on the boundary of Ω. For the choice of T this means that $\xi_{200} = v_{i0}$, where $i \in \{1, \ldots, n-1\}$. Then, we set $c_{200} := f_{i\,0}$, and the remaining five B-coefficients are set as in Case 1.

Case 3. Two of the vertices of T lie on the boundary of Ω, but not in the corners of Ω. For the choice of T this means that $\xi_{200} = v_{0j}$, where $j \in \{1, \ldots, m-1\}$. Then, we set $c_{200} := f_{0\,j}$,

$$\begin{aligned}
c_{002} :=& 1/2\ f^y_{0\,j} - 1/4\ (f^y_{0\,j-1} + f^y_{0\,j+1}) \\
&+ 1/16\ (7\ f_{0\,j} + 7\ f_{0\,j+1} + f_{0\,j+2} + f_{0\,j-1}), \\
c_{101} :=& f_{0\,j} + 1/2\ (f^y_{0\,j} - f^y_{0\,j-1}) + 1/8\ (f_{0\,j-1} - f_{0\,j+1}),
\end{aligned}$$

and the remaining three coefficients are set as in Case 1. Fig. 7 (left) illustrates this setting.

Case 4. Two of the vertices of T lie on the boundary of Ω, and one of them is a corner of Ω. For the choice of T this means that $\xi_{200} = v_{00}$ and

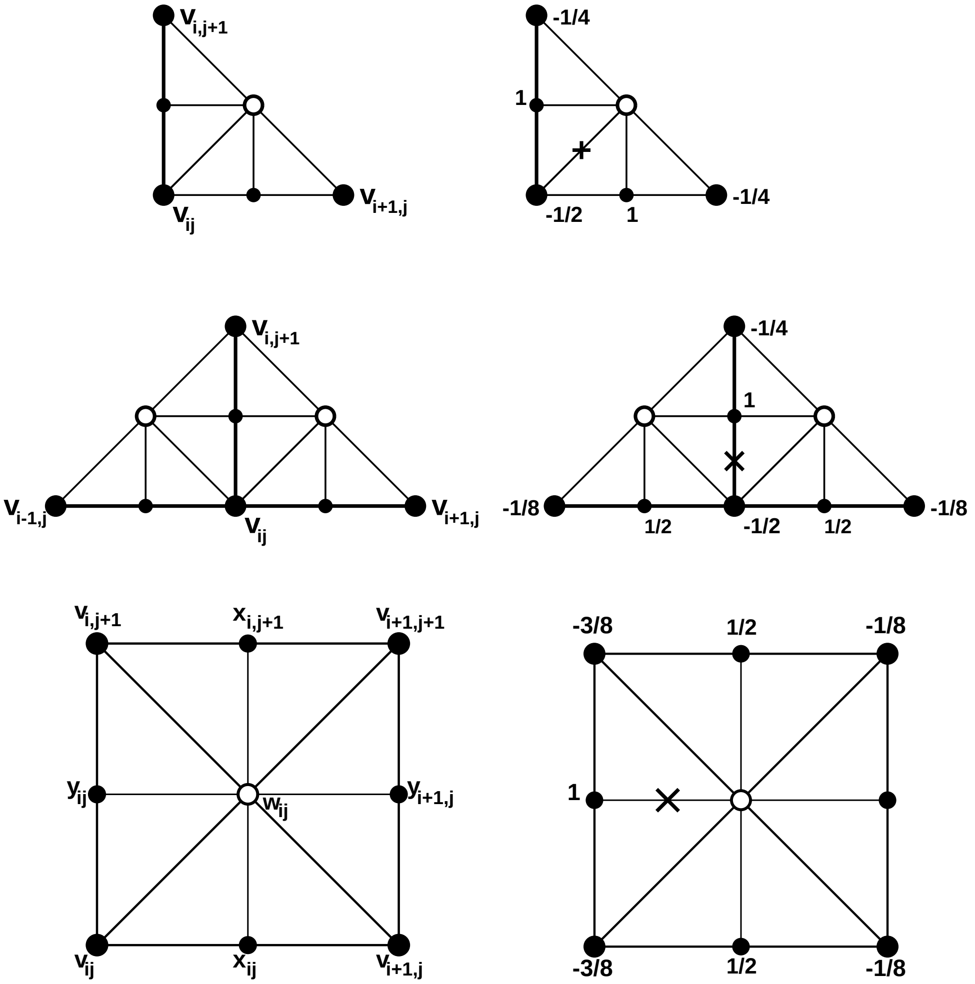

Fig. 6. Evaluation of c_{110} (top), c_{101} (middle), and c_{011} (bottom): the corresponding domain point is marked as a cross on the right.

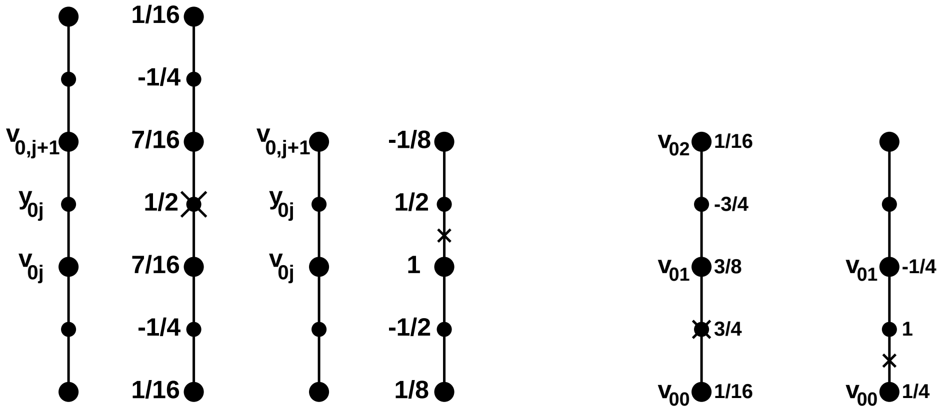

Fig. 7. Evaluation of c_{002} and c_{101} in Case 3 (left) and Case 4 (right).

$\xi_{002} = y_{00}$. Then, we set $c_{200} := f_{00}$, and

$$
\begin{aligned}
c_{002} :=& 1/4\ (3\ f^y_{00} - f^y_{01}) + 1/16\ (f_{00} + 6\ f_{01} + f_{02}),\\
c_{101} :=& f^y_{00} + 1/4\ (f_{00} - f_{01}).
\end{aligned}
$$

The remaining three coefficients are set as in Case 1. Fig. 7 (right) illustrates this setting.

The above setting uniquely determines every B-coefficient of the resulting quadratic, continuous spline s_f on $\diamondplus$, and it can be seen that for each triangle T in $\diamondplus$, the polynomial piece $s_f|_T$ is immediately obtained from small local portions of the given data in (4) by applying certain averaging rules. Moreover, the setting of B-coefficients of s_f is done carefully, as the next result shows.

Theorem 2. *The quadratic spline* s_f *is in* $C^1(\Omega)$, *i.e.*, $s_f \in \mathcal{S}^1_2(\diamondplus)$.

Proof: We have to check that all the C^1-smoothness conditions described in Sect. 2 are satisfied. This is straightforward, since we can take advantage of (2). Without loss of generality, we let $T_1 := \langle v_{ij}, w_{ij}, y_{ij}\rangle$, and check the C^1-conditions involving the B-coefficient associated with the point y_{ij}, $ij \neq 0$. The remaining C^1-conditions involving the B-coefficients associated with the points v_{ij}, w_{ij} and x_{ij}, as well as all boundary cases can be verified similarly.

There are four triangles T_ν, which share the vertex y_{ij}, see Fig. 5 (left). We assume that these triangles are arranged in counter-clockwise order such that for each of the triangles T_ν, $\nu = 2, 3, 4$, the first vertex is in V, while y_{ij} is the third vertex. First, we note that clearly $c^{T_1}_{002} = c^{T_\nu}_{002}$, $\nu = 2, 3, 4$, $c^{T_1}_{101} = c^{T_4}_{101}$, $c^{T_1}_{011} = c^{T_2}_{011}$, $c^{T_2}_{101} = c^{T_3}_{101}$, and $c^{T_3}_{011} = c^{T_4}_{011}$. Now, the setting in Case 1 implies that

$$
\begin{aligned}
c^{T_2}_{101} =& f^y_{ij} + 1/2\ (f^x_{i\,j+1} + f^x_{i-1\,j+1} - f_{i\,j+1}) - 1/4\ f_{ij}\\
&- 1/8\ (f_{i-1\,j+1} + f_{i+1\,j+1}),\\
c^{T_3}_{011} =& f^y_{ij} + 1/2\ (f^x_{i-1\,j} + f^x_{i-1\,j+1})\\
&- 1/8\ (3\ f_{ij} + 3\ f_{i\,j+1} + f_{i-1\,j} + f_{i-1\,j+1}),
\end{aligned}
$$

while the B-coefficients $c^{T_1}_{002}$, $c^{T_1}_{101}$, and $c^{T_1}_{011}$ are given exactly in the same form as c_{002}, c_{101}, and c_{011} in (5), respectively. An elementary calculation then shows that $c^{T_1}_{002} = 1/2\ (c^{T_1}_{101} + c^{T_2}_{101}) = 1/2\ (c^{T_1}_{011} + c^{T_3}_{011})$, and hence s_f is C^1-smooth at y_{ij}. □

The previous theorem shows that we can now define a linear operator $\mathcal{Q}$ mapping $C(\Omega)$ onto the space $\mathcal{S}^1_2(\diamondplus)$. More precisely, for each $f \in C(\Omega)$, we define $\mathcal{Q}(f) := s_f$, where $s_f \in \mathcal{S}^1_2(\diamondplus)$ is the quasi-interpolating spline which results from the setting of B-coefficients described above. In the following, we refer to $\mathcal{Q}$ as the quasi-interpolation operator associated with the above scheme.

§4. Properties of the Quasi-Interpolation Operator

We show that the quasi-interpolation operator $\mathcal{Q}$ associated with the scheme described in Sect. 3 is uniformly bounded and reproduces quadratic polynomials. These basic properties of $\mathcal{Q}$ are used in the next section, where we prove that s_f as well as its (piecewise) derivatives yield optimal approximation order for smooth functions.

Lemma 3. *Let T be in $\diamondplus$ and $Q_T \in \Diamond$ the square containing T. Then, for any $f \in C(\Omega)$,*

$$\|\mathcal{Q}(f)\|_T \leq \; 3 \; \|f\|_{\Omega_T},$$

where Ω_T is the union of the squares from $\Diamond$ which intersect Q_T.

Proof: First we note that according to our method all the B-coefficients c^T_{ijk} of the polynomial piece $p = s_f|_T = \mathcal{Q}(f)|_T$ on T in its B-form (1) are determined by using values of f at points from squares which intersect Q_T, only. Since the B-polynomials form a partition of unity, it follows that

$$\|p\|_T \leq \max\{|c^T_{ijk}| : \; i+j+k=2\},$$

and therefore some straightforward calculations for Cases 1–4 in Sect. 3 prove the assertion. □

Remark 4. A close inspection of Cases 1–4 in Sect. 3 shows that it is in fact possible to choose smaller subsets Ξ_T of Ω_T in Lemma 3. In Case 1, Ξ_T (shown as a shaded region in Fig. 1 on the right) is contained in four squares of $\Diamond$, while in Case 2 and 4, Ξ_T is contained in two squares of $\Diamond$, and in Case 3, data from three squares of $\Diamond$ are required for performing the method.

Lemma 5. *For any $p \in \mathcal{P}_2$, we have $\mathcal{Q}(p) = p$.*

Proof: We have to prove that $\mathcal{Q}$ reproduces the monomials from $\mathcal{P}_2$. The fact that $\mathcal{Q}$ reproduces constants is clear since according to our method in Sect. 3, each B-coefficient of $\mathcal{Q}(1)$ has the value 1, because the weights used to combine the values sum up to one. The reproduction of any other monomial q can be checked directly by comparing the B-coefficients of the spline $\mathcal{Q}(q)$ on an arbitrary triangle in $\diamondplus$ with the values of the B-coefficients of q on the same triangle. For example, for $q(x,y) = x$, and $T = \langle v_{ij}, w_{ij}, y_{ij} \rangle$, $ij \neq 0$, see Fig. 2 on the right, the evaluation of the B-coefficient c_{200} of $\mathcal{Q}(q)$, leads to

$$c_{200} = 1/2 \; (4ih) - 1/2 \; ih - 1/8 \; (4ih) = ih = q(v_{ij}),$$

and the explicit computation of the remaining B-coefficients can be done similarly. □

§5. Optimal Approximation

We now give an error bound for $f - \mathcal{Q}(f)$ and its derivatives in the supremum norm, which shows that the quasi-interpolating splines $s_f = \mathcal{Q}(f)$ yield optimal approximation order for smooth functions. In the next theorem, for any $f \in C^3(\Omega)$, we let

$$\|D^3 f\|_B := \max\{\|D_x^\alpha D_y^\beta f\|_B : \ \alpha + \beta = 3\}$$

where $B \subseteq \Omega$ is as in Sect. 2.

Theorem 6. *Let T be in $\diamondplus$. Then, for any $f \in C^3(\Omega)$,*

$$\|D_x^\alpha D_y^\beta (f - \mathcal{Q}(f))\|_T \le K_{\alpha,\beta} \ \|D^3 f\|_\Omega \ h^{3-\alpha-\beta}, \qquad \alpha + \beta = 0, 1, 2, \quad (6)$$

where $K_{\alpha,\beta} = \begin{cases} 18, & \textit{if } \alpha + \beta = 0 \\ 549/2, & \textit{if } \alpha + \beta = 1 \\ 867, & \textit{if } \alpha + \beta = 2 \end{cases}.$

Proof: Let $T \in \diamondplus$ be an arbitrary triangle in $\diamondplus$, and Ξ_T by chosen as in Remark 4. Then it can be immediately seen that

$$\rho_T := \min_{(x_1,y_1)\in\Xi_T} \ \max_{(x_2,y_2)\in\Xi_T} \max\{|x_1 - x_2| \ , |y_1 - y_2|\} \le 3h/2. \qquad (7)$$

Considering the Lagrange form of the remainder term in Taylor's formula for $p_f \in \mathcal{P}_2$ of f at the point $v_T \in \Xi_T$, where ρ_T is attained , as well as its derivatives, we obtain

$$\|D_x^\alpha D_y^\beta (f - p_f)\|_{\Xi_T} \le \kappa_{\alpha,\beta} \ \|D^3 f\|_{\Xi_T} \ h^{3-\alpha-\beta}, \qquad \alpha + \beta = 0, 1, 2, \quad (8)$$

where $\kappa_{\alpha,\beta} = \begin{cases} 9/2, & \text{if } \alpha + \beta \le 1 \\ 3, & \text{if } \alpha + \beta = 2 \end{cases}$. Lemma 5 implies that $\mathcal{Q}(p_f) = p_f$, and therefore

$$\|D_x^\alpha D_y^\beta (f - \mathcal{Q}(f))\|_T \le \|D_x^\alpha D_y^\beta (f - p_f)\|_T + \|D_x^\alpha D_y^\beta \mathcal{Q}(f - p_f)\|_T,$$

for $\alpha + \beta = 0, 1, 2$. In view of (8), it now suffices to estimate the second term of this inequality. Since $\mathcal{Q}(f - p_f)|_T \in \mathcal{P}_2$, it follows from Lemma 1, Lemma 3 and the error bound (8) that

$$\begin{aligned} \|D_x^\alpha D_y^\beta \mathcal{Q}(f - p_f)\|_T &\le C_{\alpha,\beta} \ \|\mathcal{Q}(f - p_f)\|_T \ h^{-\alpha-\beta} \\ &\le 3 \ C_{\alpha,\beta} \ \|f - p_f\|_{\Xi_T} \ \ h^{-\alpha-\beta} \\ &\le 3 \ C_{\alpha,\beta} \ \kappa_{0,0} \ \|D^3 f\|_{\Xi_T} \ h^{3-\alpha-\beta}, \end{aligned}$$

where $C_{\alpha,\beta},\ \alpha+\beta=1,2$, is the constant from Lemma 1, and $C_{0,0}:=1$. Combining these inequalities leads to

$$\|D_x^\alpha D_y^\beta(f-\mathcal{Q}(f))\|_T \le K_{\alpha,\beta}\ \|D^3 f\|_{\Omega_T}\ h^{3-\alpha-\beta}, \tag{9}$$

where $K_{\alpha,\beta}:=\kappa_{\alpha,\beta}+3\ C_{\alpha,\beta}\ \kappa_{0,0},\ \alpha+\beta=0,1,2$. Since this local error bound implies (6) the proof is complete. □

Remark 7. A close inspection of the proof of Theorem 6 shows that the constants are in fact a little bit smaller than in the statement of the theorem. Moreover, note that except for Case 3, we may choose a point v_T such that $\rho_T=h$ in (7), and hence for each triangle T in $\diamondplus$ which is sufficiently away from the boundary of Ω, we have (9), with the constant

$$K_{\alpha,\beta}=\left\{\begin{array}{rl} 16/3, & \mathit{if}\ \alpha+\beta=0\\ 82, & \mathit{if}\ \alpha+\beta=1\\ 258, & \mathit{if}\ \alpha+\beta=2 \end{array}\right. .$$

§6. Discussion

In this section, we discuss an extension of our approach as well as the connection to some known methods which are different but related to our scheme. First, we describe some quasi-interpolation schemes which differ in the locations and number of data points as well as in the choice of rules. For each scheme, the boundary treatment can be done similarly to Cases 2–4 in Sect. 3, and, therefore, we briefly discuss the setting of B-coefficients only for Case 1.

1.) Let $2\ nm+n+m$ values of $f\in C(\Omega)$ as in (4) be given at the points in X and Y, that is, at the midedges of $\Diamond$, only. Assuming that T is a triangle as in Fig. 2 on the right, we set

$$\begin{aligned}
c_{200}:=&1/4\ (f^x_{ij}+f^y_{ij}+f^x_{i-1\,j}+f^y_{i\,j-1}),\\
c_{020}:=&1/4\ (f^x_{ij}+f^y_{ij}+f^x_{i\,j+1}+f^y_{i+1\,j}),\\
c_{002}:=&1/2\ f^y_{ij}+1/4\ (f^x_{ij}+f^x_{i-1\,j}+f^x_{i-1\,j+1}+f^x_{i\,j+1}),\\
c_{110}:=&1/2\ (f^x_{ij}+f^y_{ij}),\\
c_{101}:=&1/2\ f^y_{ij}+1/4\ (f^x_{ij}+f^x_{i-1\,j}),\\
c_{011}:=&1/2\ f^y_{ij}+1/4\ (f^x_{ij}+f^x_{i\,j+1}).
\end{aligned} \tag{10}$$

2.) Let $nm+n+m+1$ values of $f\in C(\Omega)$ only be given at the points in V, that is, at the grid points of $\Diamond$, only. With the notation of (4), we first set $f^x_{ij}=(3\ f_{i+1,j}+6\ f_{ij}-f_{i-1,j})/8$, and $f^y_{ij}=(3\ f_{i,j+1}+6\ f_{ij}-f_{i,j-1})/8$, and then, in a second step, we use the approach described in Sect. 3, where we replace each appearance of f_{ij}^x and f_{ij}^y in (5) by these formulae.

3.) Let values of $f \in C(\Omega)$ be given as in (4), i.e., at the points from $V \cup X \cup Y$. In addition, assume that the values f_{ij}^w of f at the points w_{ij} in W (centers of the squares in $\Diamond$) are given. Thus, the complete number of data points is $4\ nm + 2\ (n + m) + 1$. Then, we first apply the approach described in Sect. 3 to obtain a quasi-interpolating spline $s_f \in \mathcal{S}_2^1(\diamondplus)$ where we use the data at points from $V \cup X \cup Y$. Secondly, we change the role of X and Y, and apply the method of Sect. 3 to the data at the points from $W \cup X \cup Y$ to obtain an other quasi-interpolating spline $s_g \in \mathcal{S}_2^1(\diamondplus)$. Building the average of all the B-coefficients of these two splines, we get the quasi-interpolating spline $\tilde{s}_f := (s_f + s_g)/2 \in \mathcal{S}_2^1(\diamondplus)$.

The scheme described in 1.) is associated with the well known quasi-interpolation operator based on the Powell-Zwart element. It is known that this operator reproduces bilinear polynomials, but not all quadratic polynomials. Using our approach and the scheme described in 1.), one can see that the remaining quadratic monomials are reproduced up to a constant, and hence a proof similar to the one for Theorem 6 shows that the piecewise derivatives of these quasi-interpolating splines approximate the derivatives of a smooth function optimally.

The remaining schemes, i.e., the schemes in 2.)–3.) basically have the same properties as the method described in Sect. 3. In particular, the corresponding quasi-interpolanting splines as well as their piecewise derivatives yield optimal approximation order for smooth functions. On the other hand, the data stencils and the constants involved in the corresponding error bounds are different. According to our knowledge, the operator described in 3.) was first developed in [2] (see also [3,20,21]) by using a completely different approach based on the so-called box-splines. These are locally supported not linearly independent splines, which span the $(2\ nm + 3\ (n + m + 1))$-dimensional spline space $\mathcal{S}_2^1(\diamondplus)$. It is easy to see that this operator requires data at the points of W, which is not necessary in the approach from Sect. 3. Moreover, since we derive the same operator by using our approach (see 3.)), it follows that the method presented here is more general, and provides some flexibility in designing classes of approximation operators. In particular, there is no need to consider or construct certain locally supported splines in advance, since the B-coefficients of splines are directly computed from the given data with linear complexity. Moreover, we remark that the approach of Sect. 3 as well as the methods 1.)–3.) can be extended to certain more general partitions $\Diamond$.

Recently, a first trivariate generalization of the quasi-interpolation scheme in 1.) has been developed [12,19]. In this context, quadratic splines play an important role since they provide useful tools for contouring volumetric data (on a grid), which is the purpose of efficient volume visualization. An alternative method to construct quasi-interpolation operators based on quadratic C^1-splines on appropriate (refined) partitions of three-dimensional domains with different properties would be to use the local and stable minimal determining sets connected with the interpolation schemes in [24–26] (see also the local Lagrange interpolation methods for smooth splines for given tetrahedral partitions described in [14]) in combination with the quasi-interpolation approach from [9] (see also [4]). In this context, we note that the present paper can be understood as a first important test, where we consider a bivariate spline space for which it seems difficult - if not impossible - to construct local and stable minimal determining sets or (completely) local (Hermite and) Lagrange interpolation sets, while on the other hand we have shown that a variety of operators with advantageous properties exist for these spaces. Currently, the authors are studying quasi-interpolation methods of the above type for different and more general settings. Finally, we note that according to our experience (quasi-) interpolating bivariate and trivariate splines can be computed efficiently, where the piecewise B-form of the splines turns out to be very useful for the various purposes in the different applications since the Bernstein-Bézier techniques known from Computer Aided Geometric Design can be fully exploited. Concerning numerical examples for the reconstruction of surfaces and volumetric objects involving data of various type, we refer the interested reader to [5,12,17–19].

Acknowledgments. The authors want to thank the editors for their kind co-operation in extending the deadline of our submission by some days.

References

1. Beatson, R. K. and Z. Ziegler, Monotonicity preserving surface interpolation, SIAM J. Numer. Anal. **22** (1985), 401–411.
2. Chui, C. K. and X. He, On the location of sample points for interpolation by C^1 quadratic splines, in *Numerical Methods in Approximation Theory, Vol. 8*, L. Collatz, G. Meinardus, and G. Nürnberger (eds.), Birkhäuser ISNM 81, Basel 1986, 30–43.
3. Dagnino, C. and P. Lamberti, Some performances of local bivariate quadratic C^1 quasi-interpolating splines on non-uniform type-2 triangulations, preprint, 2005.
4. Davydov, O. and L. L. Schumaker, Stable approximation and interpolation with C^1 quartic splines, SIAM J. Numer. Anal. **39** (2002), 1732–1748.

5. Davydov, O. and F. Zeilfelder, Scattered data fitting by direct extension of local polynomials to bivariate splines, Advances in Comp. Math. **21** (2004), 223–271.
6. Lai, M.-J. and L. L. Schumaker, Scattered data interpolation using C^2 supersplines of degree six, SIAM J. Numer. Anal. **34** (1997), 905–921.
7. Lai, M.-J. and L. L. Schumaker, On the approximation power of bivariate splines, Advances in Comp. Math. **9** (1998), 251–279.
8. Lai, M.-J. and L. L. Schumaker, Quadrilateral macro-elements, SIAM J. Numer. Anal. **39** (2002), 1732–1748.
9. Lai, M.-J. and L. L. Schumaker, *Splines on Triangulations,* preprint, manuscript, 2005.
10. Manni, C. and P. Sablonnière, Quasi-Interpolants on Powell-Sabin partitions, preprint, 2005.
11. Nürnberger, G. and Th. Rießinger, Bivariate spline interpolation at grid points, Numer. Math. **71** (1995), 91-119.
12. Nürnberger, G., C. Rössl, H.-P. Seidel, and F. Zeilfelder, Quasi-Interpolation by quadratic piecewise polynomials in three variables, 2005, to appear in *J.CAGD.*
13. Nürnberger, G., L. L. Schumaker, and F. Zeilfelder, Lagrange interpolation by C^1 cubic splines on triangulated quadrangulations, Advances in Comp. Math **21** (2004), 357–380.
14. Nürnberger, G., L. L. Schumaker, and F. Zeilfelder, Two Lagrange interpolation methods based on C^1 splines on tetrahedral partitions, in *Approximation Theory XI: Gatlinburg 2004*, C.K. Chui, M. Neamtu, and L.L. Schumaker (eds.), Nashboro Press, Brentwood, TN, 2005, 101–118.
15. Nürnberger, G. and G. Walz, Error analysis in interpolation by bivariate C^1-splines, IMA J. Numer. Anal. **18** (1998), 485–508.
16. Nürnberger, G. and F. Zeilfelder, Developments in bivariate spline interpolation, J. Comp. Appl. Math. **121** (2000), 125–152.
17. Nürnberger, G. and F. Zeilfelder, Lagrange interpolation by bivariate C^1-splines with optimal approximation order, Advances in Comp. Math. **21** (2004), 381–419.
18. Nürnberger, G. and F. Zeilfelder, Local Lagrange interpolation on Powell-Sabin triangulations and terrain modelling, in *Recent Progress in Multivariate Approximation*, W. Haußmann, K. Jetter, and M. Reimer (eds.), Birkhäuser ISNM 137, Basel, 2001, 227–244.
19. Rössl, C., F. Zeilfelder, G. Nürnberger, and H.-P. Seidel, Reconstruction of Volume Data with Quadratic Super Splines, in *Transactions on Visualization and Computer Graphics*, J. van Wijk, R. Moorhead, and G. Turk (eds.), IEEE Computer Society, 2004, 397–409.

20. Sablonnière, P., Quadratic spline quasi interpolants on bounded domain of $\mathbb{R}^d$, $d = 1, 2, 3$, in *Spline and radial functions*, Rend. Sem. Mat. Univ. Pol. Torino, Vol. 61, 3, 2003, 229–246.
21. Sablonnière, P., On some multivariate quadratic spline quasi-interpolants on bounded domains, in *Modern Developments in Multivariate Approximation*, W. Haussmann, M. Reimer, and J. Stöckler (eds.), Birkhäuser Verlag, Basel, ISNM 145, 2003, 262–278.
22. Sha, Z., On interpolation by $S_2^1(\Delta_{m,n}^2)$, Approx. Theory Appl. **1** (1985), 71–82.
23. Sibson, R., *A seamed quadratic element for contouring,* PhD-thesis, University of Bath, UK, 1980.
24. Sorokina, T., C^1 *Multivariate Macroelements,* PhD-thesis, Vanderbilt University, Nashville, USA, 2004.
25. Sorokina, T. and A. Worsey, A multivariate Powell-Sabin interpolant, 2005, in preparation.
26. Worsey, A. and B. Piper, A trivariate Powell-Sabin interpolant, Comput. Aided Geom. Design **5** (1988), 177–186.
27. Zedek, F., Interpolation de Lagrange par des splines quadratique sur un quadrilatere de $\mathbb{R}^2$, RAIRO Anal. Numér. **26** (1992), 575–593.

Tatyana Sorokina
Department of Mathematics
University of Georgia
Athens, GA 30602-7403, USA
`sorokina@math.uga.edu`

Frank Zeilfelder
Institut of Mathematics
University of Mannheim
68 131 Mannheim, Germany
`zeilfeld@euklid.math.uni-mannheim.de`