I have very broad research interests, all of them focusing in the areas of algebraic geometry. More precisely, I use cohomological methods and tools coming from homological algebra in order to investigate many aspects of the geometry of projective varieties. These tools, like the use of derived categories, have nowadays extensive applications to other fields, such as high energy physics (see, for example, [PABC]).

In my Ph-D thesis, I employed the theory of $M$-regularity developed by Pareschi–Popa ([PP2, PP3, PP1]) in order to study the behavior of plurcanonical maps of varieties of maximal Albanese dimension. The principal outcome of this investigation, collected in the two papers [T1] and [JLT1], is an effective and sharp bound for a pluricanonical linear system to induce the Iitaka fibration.

Successively, in a joint project with Z. Jiang (University of Paris-Sud) and M. Lahoz (University of Paris 11) we used a mixture of tools from birational geometry and homological algebra to the study the birational classification of higher dimensional varieties with small invariants. One of our main results is the following theorem.

**Theorem 1** ([JLT2]). Let $X$ be a smooth complex projective variety of maximal Albanese dimension. Assume that $\chi(X, K_X) = 1$ and $h^1(X, O_X) = 2 \dim X - 1$. Then $X$ is birational to one of the following varieties:

1. a product of smooth curves of genus 2 with the 2-dimensional symmetric product of a curve of genus 3;
2. $(C_1 \times \tilde{Z})/(\tau)$, where $C_1$ is a bielliptic curve of genus 2, $\tilde{Z} \to C_1 \times \cdots \times C_{n-1}$ is an étale double cover of a product of smooth projective curves of genus 2, and $\tau$ is an involution acting diagonally on $C_1$ and $\tilde{Z}$ via the involutions corresponding respectively to the double covers.

More on this work and its possible developments will be explained in Section 3. However, the rough idea behind the above statement is that complex tori, i.e. quotients of $\mathbb{C}^n$ by $\Lambda$, a discrete subgroup of rank 2n, are, perhaps, among the most understood compact complex manifolds. They have an abelian Lie group structure. In addition, the line bundles over these manifolds can be described in terms of Hermitian forms and characters of $\mathbb{C}^*$, therefore it is remarkably easy to compute their cohomology.

To any complex Kähler manifold $X$ one can associate, in a canonical way, a complex torus $\text{Alb}(X)$, with a holomorphic map $\text{alb}_X : X \to \text{Alb}(X)$, obtained by integrating 1-forms over 1-cycles. The pair $(\text{Alb}(X), \text{alb}_X)$ satisfies the usual universal property (i.e. any map from $X$ to another complex torus $\mathbb{C}^n/\Lambda$ factors in a unique way through $\text{alb}_X$), and is called Albanese variety of $X$. If $X$ is a projective variety, also $\text{Alb}(X)$ is such, and therefore it is an abelian variety. It is, then, natural to ask whether it is possible to find information on the geometry of $X$, passing through $\text{Alb}(X)$.
When the map $\text{alb}_X$ is nice enough, this investigation has brought, during the years, many interesting results, starting with the work of Green–Lazarsfeld on generic vanishing theorems $[\text{GL1, GL2}]$, passing through the results of Hacon (who in $[H1]$ was the first to have the idea to use the derived categories in order to approach these problems), and Hacon–Pardini ($[\text{HP3, HP2}]$). Our work stems from these papers and generalizes in higher dimension what was known to hold for surfaces.

Lately I have been shifting my interests toward different problems. More precisely, together with L. Lombardi (Mathematics Institute of the University of Bonn), we approached an outstanding conjecture, due in it most generality to Debarre ($[D]$):

**Conjecture 2.** Let $(A, \Theta)$ be an indecomposable complex principally polarized abelian variety of dimension $g$ and let $X$ be a geometrically nondegenerate closed reduced subscheme of $A$ of pure dimension $d \leq g - 2$ such that it has minimal cohomology class $\left[\Theta\right]_{g-d}$. Then either one of the following holds:

1. $(A, \Theta)$ is a polarized Jacobian of a curve $C$ of genus $g$ and $X$ is an Abel-Jacobi embedded copy of the Brill-Noether loci $\pm W_d(C)$;
2. $g = 5$, $d = 2$, $(A, \Theta)$ is the intermediate Jacobian of a smooth cubic three-folds, and $X$ is a translate of $\pm S$, the Fano surface of lines.

We adopted two different strategies. On one hand, we studied the infinitesimal deformation of the inclusion map $\pm W_d(C) \rightarrow J(C)$ and showed that, if $C$ is not hyperelliptic, they are always induced by deformations of the curve $C$ itself. On the other hand, we focused on a cohomological condition introduced by Pareschi–Popa ($[PP4]$), closely related to the property of having minimal cohomology class, and we drew from it much interesting geometrical information. In Section 1 I will explain in more detail our work and I will outline further directions I intend to pursue.

Up to now, a lot has been said on how much geometry can be recovered by the study of the derived category and the employment of homological methods. However, in passing from a variety $X$ to its derived category $D(X)$ some information is lost and there are examples of not isomorphic varieties having the same derived categories. In a joint project with K. Honigs (UC Berkeley) and L. Lombardi, we plan to quantify this information loss in the case of algebraic surfaces defined over an arbitrary algebraically closed field. This venture stems from a recent preprint of Lieblich–Olsson ($[LO]$) in which the case of K3 surfaces is described, and aims to find a parallel with the analytic case studied by Bridgeland–Maciocia ($[B2, BM]$). In section 2 this project will be further explained.

Finally, the last topic I would like to mention her concerns syzygies of projective varieties and sit at the borderer between algebraic geometry and commutative algebra. Thanks to the work of Pareschi–Popa (see $[P2]$ for a nice survey on this matter), we have very powerful tools to study generation properties of sheaves on abelian varieties. In $[T2]$ I applied these methodology in an equivariant setting, finding results on generation properties on Kummer varieites, i. e. quotients of abelian varieties by the involution induced by the group structure. In the last section of this research
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(II) I collect my achievement in this direction and I point out some of my further goals.

1. On Debarre’s Conjecture

Conjecture 2 has been the focus of the work of many mathematicians. Matsusaka–Ran criterion tells us that it is true if dim $X = 1$. Moreover, it is known (cfr. [D, Thm 5.1]) that the only subschemes of a polarized Jacobian having minimal cohomology class are, indeed, the Brill–Noether loci appearing in the statement. Finally, in [H4], Höring proved that if $(A, \Theta)$ is a general intermediate Jacobian of a cubic three-fold, then the Fano surface of lines is essentially the only subscheme having minimal cohomology class, with the exception, of course, of the theta divisor.

In collaboration with L. Lombardi, we were able to make progresses in two different directions.

The starting point of our work [LT1] is a result of Debarre showing that the locus $C_{d,g}$ of ppav’s of dimension $g$ containing an effective cycle of dimension $d$ having minimal cohomology class is closed in $A_g$, the moduli space of ppav’s of dimension $g$, and it contains both the Jacobian locus $J_g$ and the locus of the 5-dimensional intermediate Jacobians $T_5$ as irreducible components (cfr. [D, §2]). Our main goal was to try to exclude the possible existence of other components of $C_{d,g}$ intersecting either the locus $J_g$ or $T_5$. Our approach towards this problem is to look at the simultaneous deformations of minimal cohomology classes $W_d(C)$ together with the Jacobian $J(C)$ of a non-hyperelliptic curve.

Our first main result in this direction regards the computation of first-order deformations of the pairs $(W_d(C), J(C))$ consisting of a Brill–Noether locus $W_d(C)$ and the Jacobian $J(C)$ of a non-hyperelliptic curve.

Theorem 3 ([LT1]). Let $C$ be a non-hyperelliptic smooth curve of genus $g$ and let $1 \leq d \leq g - 2$ be an integer. Then there is an isomorphism of functors $\text{Def}_C \simeq \text{Def}_{W_d(C)}$.

Going back to deformations of the inclusion $W_d(C) \hookrightarrow J(C)$, we prove the even these are induced by those of the curve itself in case of non-hyperelliptic curves, and therefore if the minimal cohomology class $W_d(C)$ deforms sideways with its Jacobian out the Jacobian locus, then $W_d(C)$ parametrizes line bundles on a hyperelliptic curve. More precisely, by denoting with $\text{Def}_{W_d(C) \hookrightarrow J(C)}$ the functor of infinitesimal deformations of the morphism $W_d(C) \hookrightarrow J(C)$, it follows:

Theorem 4. If $C$ is a non-hyperelliptic smooth curve of genus $g$ and $1 \leq d \leq g - 2$ is an integer, then there is an isomorphism of functors $\text{Def}_{W_d(C) \hookrightarrow J(C)} \simeq \text{Def}_C$. In particular, any irreducible component $Z$ of the locus $C_{g,d} \subset A_g$, parameterizing principally polarized abelian varieties with a $d$-dimensional scheme representing the minimal cohomology class, different from the Jacobian locus $J_g$, may intersect $J_g$ at most along the locus of hyperelliptic curves.

The previous theorem extends [R, Corollary III.1] reaching the same conclusion, but with $C_{g,d}$ replaced with the locus in $A_g$ of ppav’s carrying two subvarieties of complimentary dimension and representing minimal cohomology classes.
In order to prove this theorem we apply Ran’s theory of deformations of maps between (not necessarily smooth) compact complex spaces which has the great advantage of reducing the problem to compute first-order deformations of the locus $W_d(C)$ simply viewed as an abstract scheme.

On the other hand, we concentrated our attention on the paper [PP4], where the authors proposed to attack Conjecture 2 from another side, using derived category techniques. They introduced the notion of $GV$-subscheme of an abelian variety and they proved that these schemes have indeed minimal cohomology class. If Conjecture 2 were to be true, also the converse should hold. Using this notion they were able to prove (a weaker version of) the conjecture in codimension 2.

Starting from this work of Pareschi–Popa and using their Fourier–Mukai based techniques, we proved the following result.

**Theorem 5.** [LT2] Let $X$ be a non degenerate reduced $GV$-subscheme of an indecomposable principally polarized abelian variety $(A, \Theta)$ such that it generates $A$ as an abelian group. Then the inclusion map $i : X \hookrightarrow A$ does not factor through any isogeny.

Observe that if Debarre’s conjecture were to be true, a subscheme of $A$ with minimal cohomology class would be normal and $A$ would be its Albanese. The Albanese map of a normal variety does not factor through any isogeny. Therefore one could see our results as a further proof of the validity of the conjecture.

Furthermore, as an application of our Theorem 5 above, we were able to give a new proof of the celebrated result of Clemens–Griffiths ([CG]) stating that the intermediate Jacobian of a smooth cubic three-fold $Y$ is the Albanese variety of the Fano scheme of line of $Y$.

Another result we were able to obtain is the following:

**Theorem 6 ([LT2]).** Let $X$ be a reduced $GV$-subscheme of a principally polarized abelian variety $(A, \Theta)$ such that generates a proper abelian subvariety $J$. Then $J$ is principally polarized, $X$ is $GV$ in $J$, and $J$ is a direct factor of $A$.

Conjecture 2 is still open and it is deemed to be a very hard one. There are, however, some intermediate questions one might want to consider. For example, it would be really interesting to see that the inclusion map in Theorem 5 does not factor through any homomorphism and not just through isogenies. Are subschemes with minimal cohomology class irreducible? Is it possible to say anything about their singularities? (if the conjecture holds they should be irreducible, normal and with rational singularities). Is it possible to use integral functors to prove Matsusaka–Ran criterion? What is the extent of the conjecture in positive characteristic?

### 2. Derived Categories of Surfaces in Positive Characteristic

In this paragraph we will describe an embryonic project stemming from a recent work of Lieblich–Olsson ([LO]).

Let $X$ be a smooth projective variety defined over an algebraically closed field $k$. To it we can associate the abelian category of coherent sheaves on $X$, $\text{Coh}(X)$. While a celebrated result of Gabriel ([G1]) states that this latter object carries all the information of $X$ (i. e. two varieties share the same
category of coherent sheaves if, and only if, they are isomorphic), this is not the ideal environment in which to approach many problems. In fact many of the functors most common to algebraic geometers, such as, for example, taking cohomology, do not behave well with the abelian category structure on Coh(X).

A somewhat more natural tool to study the geometry of X is the bounded derived category of coherent sheaves on X, D^b(X), since the aforementioned functors can be extended to exact functors (functors preserving the triangulated structure proper of D^b(X)) of this category. However, the better behavior of functors comes at the price of loss of information.

It is well known, in fact, that there exist smooth, non-isomorphic, projective varieties with equivalent derived categories. The most famous example of this phenomenon is perhaps the one involving an abelian variety A and its dual abelian variety A^\vee ([M]).

Thus it becomes natural to ask to what extent D^b(X) determines the geometry of X. Thanks to the work of many mathematicians, such as Bridgeland, Bondal, Kawamata, Orlov and Toda, nowadays we know much in this direction.

If, for example, X has (anti)ample dualizing sheaf, then any Fourier–Mukai partner of X (that is any smooth projective variety Y whose derived category is equivalent to D^b(X)) is indeed isomorphic to X ([BO]).

We know also, thanks to Orlov ([O2]) for varieties defined over fields of characteristic 0, and to Honigs ([H3]) for an extension of this results in any characteristic, that the Fourier–Mukai partners of an abelian variety A are always abelian varieties isogenous to A.

Fourier–Mukai partners share many birational invariants, such as dimension, Kodaira dimension, or order of the canonical bundle. Furthermore Kawamata ([K1]) showed that, for varieties of general type X and Y (defined over the complex numbers), an equivalence of the derived categories implies that the two varieties are K-equivalent, that is that there exists a variety Z and two birational maps f : Z \to X and g : Z \to Y such that the pullbacks of the canonical divisors on X and Y through f and g respectively are linearly equivalent as divisors on Z.

It deserves a special mention the picture concerning complex projective surfaces, that it is now completely understood thanks to the work of Bridgeland and Bridgeland–Maciocia ([B3], [BM]). They, indeed, proved that the only complex surfaces admitting non-trivial Fourier–Mukai partners are, abelian, K3 and elliptic (but not bielliptic) surfaces. In any case, every Fourier–Mukai partner of a surface X is a fine moduli space of sheaves on X.

In addition, Sosna ([S]) proved that K3 surfaces and abelian surfaces arising as canonical covers of Enriques and bielliptic surfaces (the formal definition of canonical covers deferred to a later moment) do not admit non-trivial Fourier–Mukai partners.

Among the many questions that still await an answer, we choose to focus on the relatively unknown situation of varieties defined over fields of positive characteristic.

Some of the results stated above, like the ones concerning birational invariants, are known to hold in an algebraic setting too. Other, on the contrary, have weaker statements (or no statement at all) in positive characteristic. To give an example, Orlov in [O1] gives necessary and sufficient conditions that
grant that two abelian varieties are Fourier–Mukai partners. However, for abelian varieties defined over arbitrary fields, just one direction of the statement is known to hold.

The starting point of this investigation are two recent preprints.

In the first one ([LO]) the authors show that any Fourier–Mukai partner of a K3 surface $X$, defined over a field of odd characteristic, is again a K3 surface and it is isomorphic to a moduli space of sheaves on $X$.

Thus the positive characteristic situation seems to be not so much different from the complex one. Furthermore, after observing that the work of Bridgeland–Maciocia on quotient varieties, extends easily in positive characteristic, we can already say that

**Proposition 7.** If the characteristic of the base field is different from 2, then any Fourier–Mukai partner of an Enriques surface is again an Enriques surface and the corresponding K3 covering are derived equivalent.

If $\text{char}(k)$ is greater or equal 5, then the same is true if we consider abelian varieties covering bielliptic surfaces.

We propose to further investigate the picture of algebraic surfaces, starting from abelians and K3’s, and proceeding toward Enriques and bielliptic. In particular, in a joint project with L. Lombardi and K. Honigs, we formulate the following questions:

**Problem 1.** Do algebraic K3 surfaces and abelian surfaces that covers Enriques and bielliptic surfaces have non-trivial Fourier–Mukai partners?

**Problem 2.** Describe Fourier–Mukai partners of Enriques and bielliptic surfaces in positive characteristic.

We expect that odd characteristic, for Enriques, and different from 2 and 3, for bielliptic, the algebraic situation should looks exactly like the complex picture.

**Problem 3.** Look for examples where the algebraic setting might differ from the analytic one.

In another recent preprint ([H3]), Honigs proves that derived equivalent abelian varieties and algebraic surfaces share the same zeta-functions. In order to extends the results to three-folds, just the invariance of the irregularity under derived equivalence is missing. In [PS] the authors proved such invariance for complex varieties.

This lead us to formulate the following question:

**Problem 4.** Do smooth algebraic three-folds who are derived equivalent share the same zeta-functions?

3. **Classification of Varieties of Maximal Albanese dimension**

In [JLT2], together with my coauthors, we applied derived category techniques and made some progresses along two lines of research.

The first one, that brought us to the formulation of Theorem 1](1) concerns the problem of classifying maximal Albanese dimensional varieties with small invariants. These varieties behave, in many
as aspects, as surfaces of general type. For example, Hacon–Pardini \cite{HP3} showed, extending in higher dimension a results of Beauville \cite{B1} on surfaces, that the irregularity $q(X)$ of a smooth complex variety $X$ of maximal Albanese dimension and $\chi(X, K_X) = 1$ is bounded by $2 \dim X$. Furthermore, equality holds if, and only if, $X$ is birational to a product of curve of genus $2$.

In \cite{HP2} and \cite{P3} Hacon–Pardini and Pirola give a complete classification of surfaces with $\chi = 1$ and irregularity $3$. Our Theorem 1 generalizes such results in higher dimension.

The next obvious steps would be to attempt a classification of varieties with Euler characteristic $1$ and irregularity $2n - 2$. However, as the classification we provided relay on Hacon–Pardini and Pirola classification of surfaces with $q = p_g = 3$, such classification would relay on the classification of surfaces with $q = p_g = 2$ that is far to be completed.

The second line we explored was opened by Hacon \cite{H1, H2}, and later pursued by Chen–Hacon \cite{CH1, CH2}, Hacon–Pardini \cite{HP1} and is in part inspired by earlier the works of Kollár, Kawamata, Ein–Lazarsfeld. It concerns the cohomological characterization of theta divisors in principally polarized abelian varieties.

Theta divisors in principally polarized abelian varieties are the only known examples of ample divisor in abelian varieties whose smooth model has Euler characteristic $1$. In the same way, products of theta divisors in (decomposable) principally polarized abelian varieties are the only known example of varieties with maximal Albanese dimension, Albanese image not fibered by subtori and whose smooth models has Euler characteristic $1$. It is then natural to ask whether these might be the only examples.

With a mixture of tools from birational geometry, Green–Lazarsfeld generic vanishing theory \cite{GL1, GL2}, and, of course, the use of derived categories and integral functors, we were able to prove:

**Theorem 8.** A normal subvariety $X$ of a complex abelian variety (not assumed to be principally polarized) is a product of theta divisors if, and only if, given $\tilde{X} \to X$ a resolution of singularities of $X$, $\chi(\tilde{X}, K_{\tilde{X}}) = 1$.

**Theorem 9.** A smooth complex projective variety is birational to a product of theta divisors if and only if $\chi(X, K_X) = 1$, its Albanese map is generically finite, its Albanese image is of general type and smooth in codimension $1$.

It is not yet clear if the hypothesis on the singularities of the Albanese image can be removed, and this matter certainly constitutes one of my further research goals.

4. Syzygies of Kummer varieties

Let $X$ an algebraic variety over an algebraically closed field $k$ and let $A$ an ample invertible sheaf on $X$, generated by its global sections. With $R_a$ we will denote the ring associated to the sheaf $A$:

$$ R_a := \bigoplus_{n \in \mathbb{Z}} H^0(X, A^\otimes n) $$


while $S_{\mathcal{A}}$ will be the symmetric algebra of $H^0(X, \mathcal{A})$. The ring $R_{\mathcal{A}}$ is a graded $S_{\mathcal{A}}$ module and admits a minimal free resolution, that looks like:

$$0 \to \cdots \to E_p \to \cdots \to E_1 \to E_0 \to R_{\mathcal{A}} \to 0$$

where $E_i = \oplus_j S_{\mathcal{A}}(-a_{ij})$, $a_{ij} \in \mathbb{Z}$, $a_{ij} \geq 0$.

**Definition 10** (Property $N_p$ [L]). Let $p$ a given integer. The line bundle $\mathcal{A}$ satisfies property $N_p$ if

$$E_0 = S_{\mathcal{A}} \quad \text{when } p \geq 0$$

and

$$E_i = \oplus S(-i-1) \quad 1 \leq i \leq p.$$

More generally, it is possible to extend Green condition as follows (cfr [P1]): we say that, given a non negative integer $r$, property $N_p^r$ holds for $\mathcal{A}$ if, in the notation above, $a_{0j} \leq 1 + r$ for every $j$. Inductively we say that $\mathcal{A}$ satisfies property $N_p^r$ if $N_{p-1}^{r-1}$ holds for $\mathcal{A}$ and $a_{pj} \leq p + 1 + r$ for every $j$. We know that, given $X$ and $\mathcal{A}$ and $p$ as above, exist an integer $n_0 = n_0(\mathcal{A}, p)$ such that for all $n \geq n_0$ the sheaf $\mathcal{A}^\otimes n$ satisfies property $N_p^r$. The question that arises immediately is: can we find such $n_0$?

Green in [G2] addressed this issue in the case in which $X$ is a smooth curve of genus $g$ and $\mathcal{A}$ a divisor of degree $d \geq 2g + 1$, while in [G3] he challenged the problem of finding an upper bound for the syzygies of the projective space. The case when $X$ is an abelian variety was the object of attention of Pareschi-Popă in [PP3]. Their proof is a consequence of the powerful, Fourier-Mukai based, theory of $M$-regularity that they developed in [PP2].

I have been brooding over this problem when $X$ is a (singular) Kummer variety associated to an abelian variety $Y$ (i.e. $X$ is the quotient of $Y$ by the natural $\mathbb{Z}_2$ action on $Y$). The main results I obtained in this sense are the following.

**Theorem 11** ([T2]). Take $p \geq 1$ and $r \geq 0$ two integers such that $\text{char}(k)$ does not divide $p+1$, $p+2$. Let $A$ an ample line bundle on a Kummer variety $X$, such that its pullback $\pi^*_X A \simeq \mathcal{A}^\otimes 2$ with $\mathcal{A}$ an ample symmetric invertible sheaf on $X$ which does not have a base divisor. Then

(a) $A^\otimes n$ satisfies property $N_p$ for every $n \in \mathbb{Z}$ such that $n \geq p + 1$.

(b) more generally $A^\otimes n$ satisfies property $N_p^r$ for every $n$ such that $(r+1)n \geq p + 1$.

**Theorem 12** ([T2]). Fix two non negative integers $p$ and $r$ such that $\text{char}(k)$ does not divide $p + 1$, $p + 2$. Let $A$ an ample line bundle on a Kummer variety $X$, then

(a) $A^\otimes n$ satisfies property $N_p$ for every $n \in \mathbb{Z}$ such that $n \geq p + 2$.

(b) more generally $A^\otimes n$ satisfies property $N_p^r$ for every $n$ such that $(r+1)n \geq p + 2$.

My work generalizes and improves results of Khaled ([K3] and [K4]) and Kempf ([K2]) on projective normality and degree of the equations of Kummer varieties. The above two statements are sharp for $p = 0, 1$. It is not clear if for higher $p$’s they are so: as a matter of fact it is believed that the bound on property $N_p$ for Kummer variety should increase as $\frac{p}{2}$ rather than as $p$.

In addition to the further goal of improving the given bound, I reckon it will be interesting to use similar techniques in order to investigate syzygies of other quotients of abelian varieties. In [C1] the
authors study the étale case, but, again, it is not clear if the bound they provide is sharp. Syzygies of generalized Kummer varieties are at this moment totally unexplored.

**References**


