Matlis duality is an important tool in commutative algebra. It states:

**Theorem 0.1** ([Hoc, Theorem 5.1, Theorem 5.4]). Let \((R, \mathfrak{m}, k)\) be a local ring, and let \(E = E_R(k)\) be the injective hull of the residue field of \(R\). If \(R\) is complete, then the functor, \((\cdot)^\vee\), defined by
\[ M^\vee := \text{Hom}_R(M, E), \]
induces an equivalence of categories from Noetherian \(R\)-modules to Artinian \(R\)-modules and vice-versa, and further \((M^\vee)^\vee = M\) for all Noetherian/Artinian \(M\).

Even if \(R\) is not complete, then \(\hat{R} = \text{Hom}_R(E, E)\).

If you’re not familiar with Matlis duality, then the first five chapters of [Hoc] are a great reference. The following two lemmas are used often in the older literature on test ideals, e.g. [Tak06]. Indeed, Takagi seems to refer to the first lemma itself as “Matlis duality.” I couldn’t find a reference for these results, so I’m putting them here:

**Lemma 0.2** (c.f. [BH98], exercise 3.2.15c). Let \((R, \mathfrak{m}, k)\) be a complete local ring, and let \(E = E_R(k)\). If \(M \subseteq E\) is a submodule, then
\[ \text{Ann}_E \text{Ann}_R M = M. \]
Similarly, if \(I \subseteq R\) is an ideal, then
\[ \text{Ann}_R \text{Ann}_E I = I. \]

**Proof.** I claim that \((M^\vee)^\vee = \text{Ann}_E \text{Ann}_R M\). Then the result follows by Matlis duality. First, note that by the universal property of injective modules, for any map \(M \to E\), we can fill in the diagram:

\[
\begin{array}{ccc}
0 & \to & M \\
& \downarrow & \downarrow \\
& & E \\
& \mathcal{E} & \searrow \\
& & (M : \mathcal{E}) \\
\end{array}
\]

In other words, \(\text{Hom}(E, E) \to \text{Hom}(M, E)\). But, since \(R\) is complete, any map \(\text{Hom}(E, E)\) is just given by multiplication by an element of \(R\). Thus \(M^\vee\) is a quotient of \(R\). Now we ask: what is the kernel of the quotient? The kernel of this quotient is the set of maps \(E \to E\) that restrict to 0 on \(M\). Since maps \(E \to E\) are given by multiplication by elements of \(R\), we see that the kernel is \(\text{Ann}_R(M)\). Thus \(M^\vee = R/\text{Ann}_R(M)\).

By the “first isomorphism theorem” (or, if you like, the universal property of quotient modules), an element of \(\text{Hom}_R(R/\text{Ann}_R(M), E)\) is the same as a map \(R \to E\) that restricts to 0 on \(\text{Ann}_R(M)\). A map \(R \to E\) is completely determined by where we send \(1 \in R\), so we see that \((M^\vee)^\vee = \text{Hom}_R(R/\text{Ann}_R(M), E) = \text{Ann}_E \text{Ann}_R M\).

The second statement is proved in the same way. \(\square\)

**Lemma 0.3.** Notation as in lemma 0.2. Let \(M \subseteq E\) be a submodule and \(I \subseteq R\) be an ideal. Then
\[ (0 : (M : I)_E)_R = I \cdot (0 : M)_R \]
In other words,
\[ \text{Ann}_R(M : I)_E = I \cdot \text{Ann}_R M \]
Proof. First I’ll show that \((M : I)_E = \text{Ann}_E (I \cdot \text{Ann}_R(M))\). To do so, we start by showing the left-hand side is smaller than the right-hand side. So let \(x \in (M : I)_E\). Then \(I \cdot x \subseteq M = \text{Ann}_E \text{Ann}_R(M)\). In other words, \(I \text{Ann}_R M \cdot x = 0\), as desired. To get the opposite inclusion, let \(y \in \text{Ann}_E (I \cdot \text{Ann}_R(M))\). By definition, \(\text{Ann}_R(M) \cdot (Iy) = 0\). In other words, \(Iy \subseteq \text{Ann}_E \text{Ann}_R M = M\), as desired.

From the above, it follows that

\[
\text{Ann}_R(M : I)_E = \text{Ann}_R \text{Ann}_E (I \cdot \text{Ann}_R(M)) = I \cdot \text{Ann}_R(M).
\]

\[\square\]

References

