Marden’s theorem

Daniel Smolkin

Department of Mathematics
University of Utah

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Statement of the theorem

Theorem 1
Let $p$ be a degree-3 polynomial over $\mathbb{C}$. Suppose the roots of $p$ form a triangle in the complex plane. Then the roots of $p'$ are the foci of the Steiner inellipse of this triangle.
• Ellipse: \( \{ p : d(p, a) + d(p, b) = r \} \) for some \( a, b \) called foci and some \( r \) called the major axis length

• Steiner inellipse: the unique ellipse tangent to the three sides of a triangle at their midpoints
Ellipse properties

Optical property
Uniqueness property: given a pair of points and a line, there is at most one ellipse with foci at those points tangent to that line.
\[ \angle F_1 P G_1 = \angle F_2 P G_2 \]
Let $T$ be the triangle defined by the roots of $p$ and let $E$ be an ellipse with foci at the roots of $p'$. If $E$ intersects a side of $T$ at its midpoint, then...
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- $E$ is tangent to that side (at its midpoint)
- $E$ is tangent to the other two sides of $T$ as well
- $E$ is tangent to every side at its midpoint
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Outline in pictures

1.
Outline in pictures

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Outline in pictures

1. 

2. 

3. 

4.
Step 1

Let $T$ be the triangle defined by the roots of $p$ and let $E$ be an ellipse with foci at the roots of $p'$. If $E$ intersects a side of $T$ at its midpoint, then $E$ is tangent to that side.

Thus, the unique ellipse that is tangent to that side and has foci at the roots of $p'$ is tangent to that side at its midpoint (what we really need).
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Thus, the unique ellipse that is tangent to that side and has foci at the roots of $p'$ is tangent to that side at its midpoint (what we really need).

Proof:

WLOG, can rotate, scale, translate, reflect (exercise)
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- Thus, the *unique* ellipse that is tangent to that side and has foci at the roots of $p'$ is tangent to that side at its midpoint (what we really need).

Proof:
- WLOG, can rotate, scale, translate, reflect (exercise)
- So we can assume the following picture:
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Let $T$ be the triangle defined by the roots of $p$ and let $E$ be an ellipse with foci at the roots of $p'$. If $E$ intersects a side of $T$ at its midpoint, then $E$ is tangent to that side.

Thus, the unique ellipse that is tangent to that side and has foci at the roots of $p'$ is tangent to that side at its midpoint (what we really need).

Proof:

- WLOG, can rotate, scale, translate, reflect (exercise)
- So we can assume the following picture:
Step 1

- Roots = \{1, -1, w\}
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\[ \Rightarrow p(z) = z^3 - wz^2 - z, \quad p'(z) = 3z^2 - 2wz - 1 \]
Step 1

- Roots = \{1, -1, w\}
- \Rightarrow p(z) = z^3 - wz^2 - z, \quad p'(z) = 3z^2 - 2wz - 1
- Note:
  \[
  \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) + \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = -\frac{b}{a},
  \]
  \[
  \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) \cdot \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = \frac{c}{a}
  \]
\[ z_1 z_2 = -1/3 \Rightarrow \text{Arg } z_1 + \text{Arg } z_2 = \pi \pmod{2\pi\mathbb{Z}} \]
\[
\begin{align*}
\bullet \quad z_1 z_2 &= -1/3 \Rightarrow \text{Arg } z_1 + \text{Arg } z_2 = \pi \text{ (mod } 2\pi \mathbb{Z}) \\
\bullet \quad z_1 + z_2 &= 2w/3 \Rightarrow \text{Im } z_1 > 0 \text{ or } \text{Im } z_2 > 0
\end{align*}
\]
\[ z_1 z_2 = -1/3 \Rightarrow \text{Arg } z_1 + \text{Arg } z_2 = \pi \pmod{2\pi \mathbb{Z}} \]
\[ z_1 + z_2 = 2w/3 \Rightarrow \text{Im } z_1 > 0 \text{ or } \text{Im } z_2 > 0 \]
\[ \text{So } 0 < \text{Arg } z_1, \text{Arg } z_2 < \pi \text{ and } \text{Arg } z_1 + \text{Arg } z_2 = \pi \]
\[ z_1 z_2 = -1/3 \implies \text{Arg } z_1 + \text{Arg } z_2 = \pi \pmod{2\pi \mathbb{Z}} \]
\[ z_1 + z_2 = 2w/3 \implies \text{Im } z_1 > 0 \text{ or } \text{Im } z_2 > 0 \]
\[ \text{So } 0 < \text{Arg } z_1, \text{Arg } z_2 < \pi \text{ and } \text{Arg } z_1 + \text{Arg } z_2 = \pi \]

By the optical property of ellipses, \(x\)-axis is tangent to our ellipse.
Outline in pictures

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Step 2

Let $T$ be the triangle defined by the roots of $p$ and let $E$ be an ellipse with foci at the roots of $p'$. If $E$ is tangent to a side of $T$ at its midpoint, then $E$ is tangent to every side of $T$. 

Proof:
Step 2

Let $T$ be the triangle defined by the roots of $p$ and let $E$ be an ellipse with foci at the roots of $p'$. If $E$ is tangent to a side of $T$ at its midpoint, then $E$ is tangent to every side of $T$.

Proof:

- Assume the following picture:
Step 2

- \( p(z) = z^3 - (1 + w)z^2 + wz, \quad p'(z) = 3z^2 - 2(1 + w)z + w \)
Step 2

- $p(z) = z^3 - (1 + w)z^2 + wz$,  \quad p'(z) = 3z^2 - 2(1 + w)z + w$
- $z_1 + z_2 = \frac{2}{3}(1 + w)$, so one focus is above $x$-axis.
Step 2

- \( p(z) = z^3 - (1 + w)z^2 + wz, \quad p'(z) = 3z^2 - 2(1 + w)z + w \)
- \( z_1 + z_2 = \frac{2}{3}(1 + w) \), so one focus is above \( x \)-axis.
- Since ellipse tangent to \( x \)-axis, both foci on one side
Step 2

- \( z_1z_2 = \frac{w}{3} \)
Step 2

- \( z_1 z_2 = w/3 \Rightarrow \text{Arg } z_1 + \text{Arg } z_2 = \text{Arg } w. \)
Step 2

- $z_1 z_2 = w / 3 \implies \text{Arg } z_1 + \text{Arg } z_2 = \text{Arg } w$.
- The line between 0 and $w$ is tangent to the ellipse by third ellipse property.

$\angle F_1 PG_1 = \angle F_2 PG_2$
Outline in pictures

1. 

2. 

Marden's theorem
Step 3

\( E \) is tangent to each side at its midpoint

By step 1, there is some \( E' \) with same foci tangent to another side at its midpoint. By uniqueness property, \( E = E' \).
Step 3

*E is tangent to each side at its midpoint*

- By step 1, there is some $E'$ with same foci tangent to another side at its midpoint.
Step 3

$E$ is tangent to each side at its midpoint

- By step 1, there is some $E'$ with same foci tangent to another side at its midpoint
- By uniqueness property, $E = E'$
Empirical evidence
References

- My website (slides and python script)
  math.utah.edu/~smolkin/talks