

Exam 2 Practice Problems Solutions  
Math 5110/6830

1. (a) Immune cells eat strep for breakfast. They don't alter any rates, but they do make strep die when the two are in contact at a rate  $\alpha$ .
- (b) The birth of new strep bacteria is a type 2 functional response so it saturates; you should graph the function to see what it looks like. What this means is that at first the bacteria grow exponentially, but as they run out of space to occupy, their population size doesn't change much with each time step (ie. as to saturate). This is biologically correct since we'd all be eaten alive already if bacteria were allowed to grow exponentially forever.
- (c) This term is how immune cells are recruited by cytokines, which is a type 3 functional response. It is also known as a sigmoidal curve since it looks like an S in shape. This says that for small amounts of cytokines, immune cells are not recruited; however, as the amount of cytokine grows, the number of immune cells around grow exponentially. But, similarly to the type 2 response, it also saturates. This again is correct because when there is a lot of cytokine around, the number of immune cells recruited to the infection area won't change much (there just isn't room). The  $k_2$  term is the "half saturation" constant. It tells us when we are at the half way point to our saturation level.
- (d)  $p$  is in the term  $-apIS$  from the  $\frac{dI}{dt}$  equation (don't confuse this with  $\rho$ ). This term accounts for the fact that many immune cells will die (apoptose) after they engulf (phagocytize) a bacterium. Macrophages are a typical immune cell that eats bacteria, but they can only eat a few before they need to kill themselves. Yep, that's right, they are suicidal. All cells are - they are all pre-programmed to die. So, the  $p$  in this term can be thought of as the probability that an immune cell dies when it encounters a bacterium. The values of  $p$  will be between 0 and 1.
- (e) Cytokines are generated when an immune cell encounters a bacterium. This is the signal that your immune cells (and other cells too) secrete to alert other immune cells that there is a foreign invader. They are degraded naturally at rate  $\mu$ .
- (f) To assume that cytokine dynamics are fast comparatively to that of the immune cells and bacteria, we are assuming quasi-steady state dynamics. That is, we can assume that  $\gamma$  and/or  $\mu$  are large. Let's assume that  $\gamma$  is large so that  $\frac{1}{\gamma}$  is small, ie. we can set this equal to zero. Then, we'll have

$$0 = \frac{1}{\gamma} \frac{dC}{dt} = IS - \frac{\mu}{\gamma} C$$

$$C = \frac{\gamma IS}{\mu}$$

Plugging this into the other two equations so that we have a system of 2 ODEs:

$$\frac{dS}{dt} = \frac{\rho S}{k_1 + S} - \alpha IS$$

$$\frac{dI}{dt} = \frac{\beta \left( \frac{\gamma IS}{\mu} \right)^2}{k_2^2 \mu^2 + \left( \frac{\gamma IS}{\mu} \right)^2} - \alpha p IS - \delta I$$

$$= \frac{\beta (\gamma IS)^2}{(k_2 \mu)^2 + (\gamma IS)^2} - \alpha p IS - \delta I$$

2. (a) The only thing different about this model is that it doesn't explicitly include immune cell recruitment via cytokines. Therefore the recruitment term in the immune cell equation is only dependent on the number of bacteria present. This is again a type 2 functional response.
- (b) There are possibly three equilibria points. The easiest one to write down is  $(0, 0)$ . The other two are easier to see from the phase plane.  
S-nullclines:

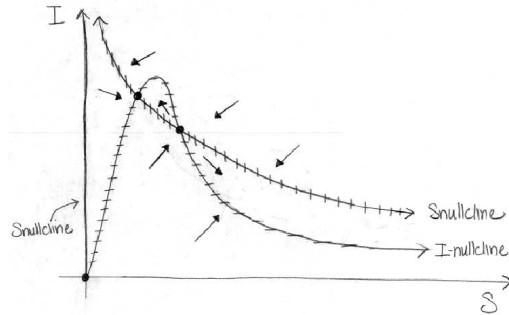
$$S = 0$$

$$I = \frac{\rho}{\alpha(k_1 + S)}$$

I-nullclines:

$$I = \frac{\beta S}{(k_2 + S)(\alpha p S + \delta)}$$

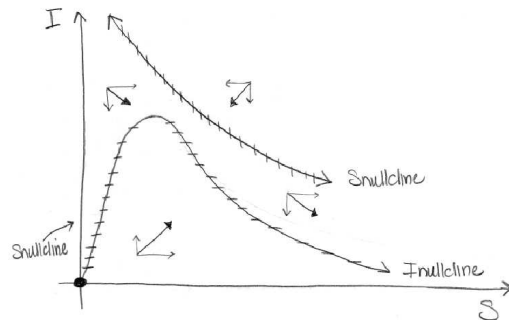
Phase-plane diagram:



The  $(0,0)$  point is unstable. The first non-zero point is stable, and the other point is unstable (looks to be a saddle). If the algebra isn't terrible, then you should check this analytically.

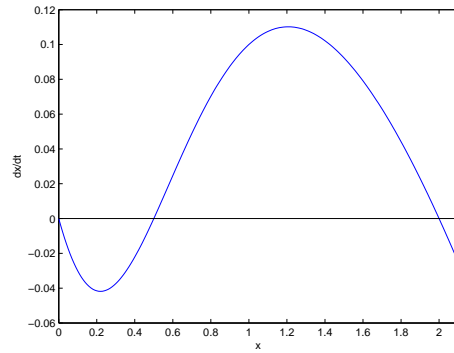
- (c) In this case, the only equilibria point is  $(0,0)$  which is unstable. The strep win! A fast growing bacteria coupled with a slow moving immune system will produce something such as this.

Phase plane diagram:

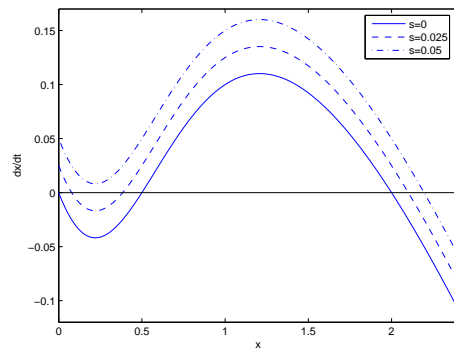


3. (a) The first term is a constant growth term. The growth of  $g$  depends linearly on the concentration of  $S$ , with a growth rate of  $k_1$ . In this case, it is a constant growth because  $s_0$  is constant. The second term is a gene's natural decay term. With nothing else, the gene would decay exponentially at a rate  $k_2$ . The third term is a self-production term, with a limited rate of reproduction of  $k_3$ . When  $g$  gets large, this term becomes approximately constant (this is a type 3 functional response). The parameter  $k_4$  determines how large  $g$  has to be before this term starts behaving as in the large  $g$  limit.

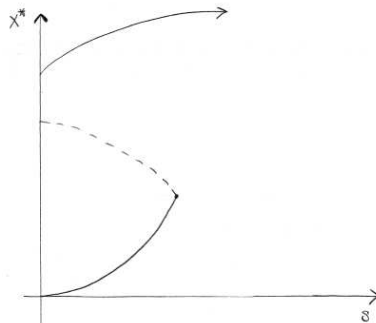
(b) Plot with  $s = 0$ :



(c) Plots for  $s \geq 0$ :



(d) Bifurcation diagram (you can obtain this from the above picture):



- (e) When  $s$  is slowly increased,  $x(\tau)$  will approach the lower stable equilibria point. However, once the value of  $s$  gets large enough, we lose two of the fixed points. So,  $x(\tau)$  will then move towards the upper stable point.
- (f) If it has been allowed to stabilize at the upper fixed point, then, it will stay there. With  $s = 0$ , there are again 3 fixed points but the middle one is unstable so trajectories will move away from it.