# Homework 7.1 Solutions
Math 5110/6830

1. (a) For

$$\frac{dx}{dt} = 1 + rx + x^2$$

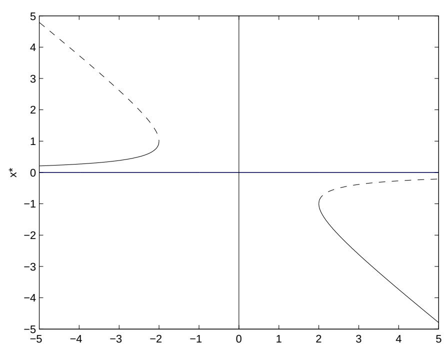
we have equilibria values

$$x^* = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

Then, our bifurcation values are $r = \pm 2$. Note that we also have complex values for $-2 < r < 2$. To check the stability, let

$$\begin{aligned} f(x) &= 1 + rx + x^2 \\ &\text{then} \\ f'(x) &= r + 2x \\ f'(x^*) &= \pm\sqrt{r^2 - 4} \end{aligned}$$

The negative root is stable, and the positive root is unstable.
Bifurcation diagram:




(b) For

$$\begin{aligned} \frac{dx}{dt} &= x - rx(1 - x) \\ &= rx^2 - (r - 1)x \end{aligned}$$

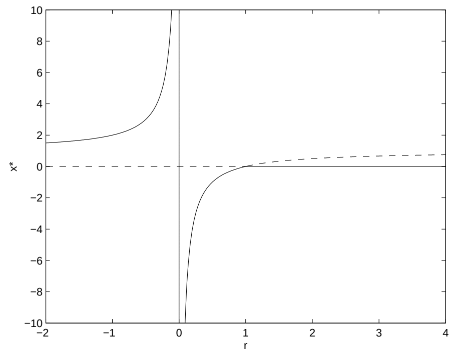
we have equilibria values

$$\begin{aligned} x^* &= 0 \\ x^* &= \frac{r - 1}{r} \end{aligned}$$

Then, our bifurcation values are $r = 1$. To check the stability, let

$$\begin{aligned} f(x) &= rx^2 - (r - 1)x \\ &\text{then} \\ f'(x) &= 2rx - r + 1 \\ f'(0) &= 1 - r \\ f'\left(\frac{r - 1}{r}\right) &= r - 1 \end{aligned}$$

So, $x^* = 0$ is stable for $r > 1$ and $\frac{r-1}{r}$ is stable for $r < 1$.
Bifurcation diagram:




(c) For

$$\frac{dx}{dt} = x(r - e^x)$$

we have equilibria values

$$x^* = 0$$
$$x^* = \ln(r)$$

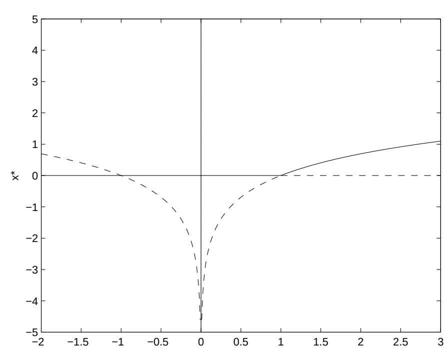
Then, our bifurcation values are $r = 1$. To check the stability, let

$$f(x) = x(r - e^x)$$

then

$$f'(x) = r - e^x - xe^x$$
$$f'(0) = r - 1$$
$$f'(\ln(r)) = -r\ln(r)$$

So, $x^* = 0$ is stable for $r < 1$ and $\ln(r)$ is stable for $r > 1$.
Bifurcation diagram:



(d) For

$$\frac{dx}{dt} = r + \frac{1}{2}x - \frac{x}{1+x}$$

we have equilibria values

$$x^* = [\pm\sqrt{}]$$

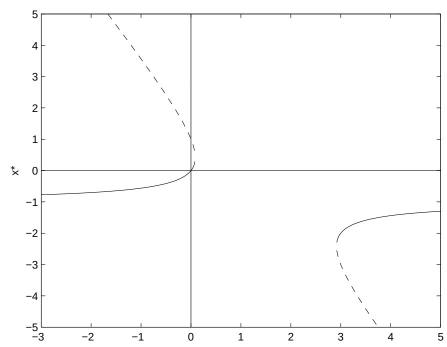
Our bifurcation value is $r = \frac{3\pm 2\sqrt{2}}{2}$. To check the stability, let

$$f(x) = r + \frac{1}{2}x - \frac{x}{1+x}$$

then

$$f'(x) = \frac{1}{2} - \frac{1}{(1+x)^2}$$

Above $r = \frac{3+2\sqrt{2}}{2}$, the positive root is stable, and below $r = \frac{3-2\sqrt{2}}{2}$ the negative root is stable.
Bifurcation diagram:




2. (a) $\frac{dP}{dt}$ represents the rate of change of the performance over time, ie. "how fast" someone picks up a new skill.

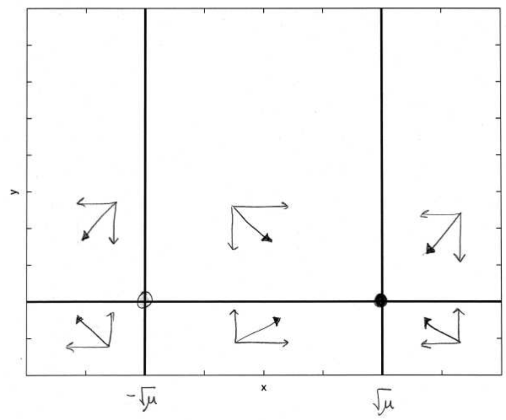
(b) When $M \geq P$, $\frac{dP}{dt} \geq 0$, so $P(t)$ is increasing or staying constant in time. When $M < P$, $\frac{dP}{dt} < 0$, which means that $P(t)$ is decreasing in time. We expect that with more and more training, a person will never have a decrease in performance. Notice that if we start with $P$ below $M$, $P$ can never get larger than $M$. If $P = M$, $P$ will remain constant. This model is reasonable. We interpret $M$ as the level when someone has mastered the skill.

(c) A reasonable initial condition would be $P(0) = 0$, ie. no previous knowledge.

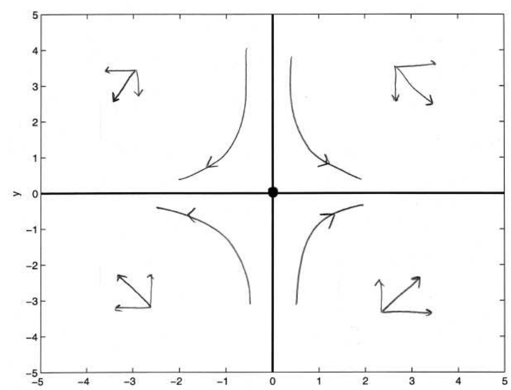
(d) Note that the equilibria point is $P^* = M$. It is stable for $k$ positive, and unstable otherwise.

(e) The bifurcation occurs at $k = 0$.

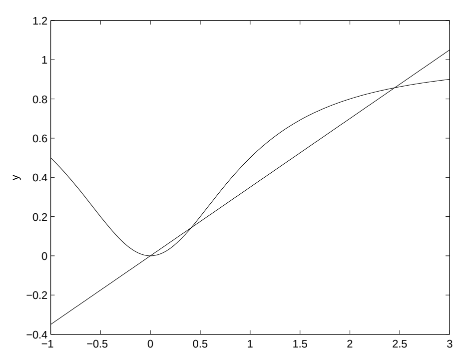
3. Phase portrait for $\mu > 0$:



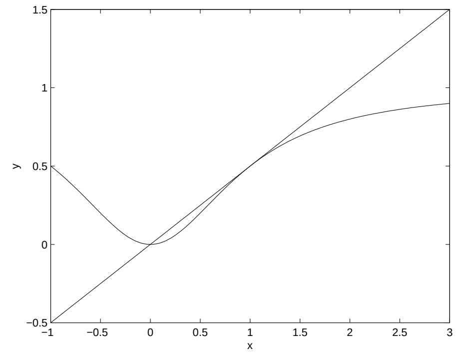

Phase portrait for $\mu = 0$:

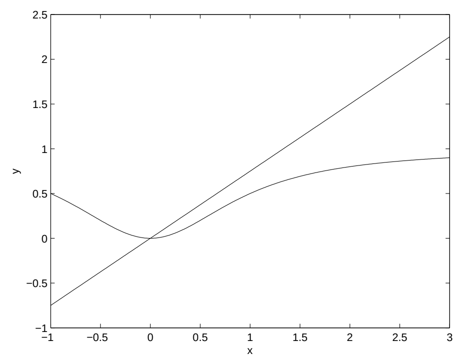



4. (a) For small $a$, there are three equilibria points:



(b) Note that as $a$ increases, we lose equilibria points:



Homework 7.2 Solutions
Math 5110/6830

1. (a) For

$$\frac{dx}{dt} = \mu x + 4x^3$$

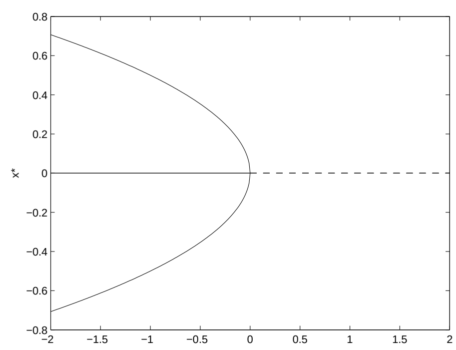
we have equilibria values

$$x^* = \pm\sqrt{\frac{-\mu}{4}}$$
$$x^* = 0$$

Then, our bifurcation value is $\mu = 0$. Note that we also have complex values for $\mu > 0$. To check the stability, let

$$f(x) = \mu x + 4x^3$$
then
$$f'(x) = \mu + 12x^2$$
$$f'(0) = \mu$$
$$f'\left(\pm\sqrt{\frac{-\mu}{4}}\right) = -2\mu$$

Then $x^* = 0$ is stable for negative $\mu$, and $x^* = \pm\sqrt{\frac{-\mu}{4}}$ is stable for positive $\mu$.
Bifurcation diagram:




(b) For

$$\frac{dx}{dt} = x + \frac{\mu x}{1 + x^2}$$

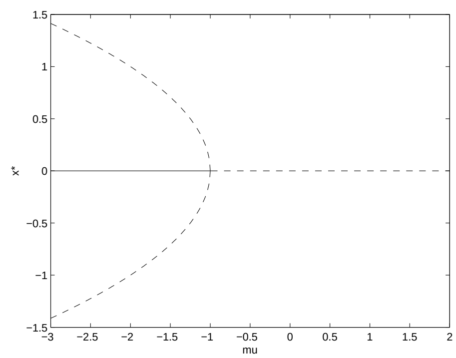
we have equilibria values

$$x^* = \pm\sqrt{-(\mu + 1)}$$
$$x^* = 0$$

Then, our bifurcation value is $\mu = -1$. Note that we also have complex values for $\mu > -1$. To check the stability, let

$$\begin{aligned} f(x) &= \mu x + \frac{\mu x}{1+x^2} \\ &\text{then} \\ f'(x) &= 1 + \frac{\mu(1-x^2)}{(1+x^2)^2} \\ f'(0) &= 1+\mu \\ f'\left(\pm\sqrt{-(\mu+1)}\right) &= \frac{2\mu+2}{\mu} \end{aligned}$$

Then $x^* = 0$ is stable for negative $\mu < -1$, and $x^* = \pm\sqrt{-(\mu+1)}$ is stable for positive $\mu > -1$. Bifurcation diagram:




(c) For

$$\frac{dx}{dt} = x - \frac{x}{1+x}$$

we have equilibria values

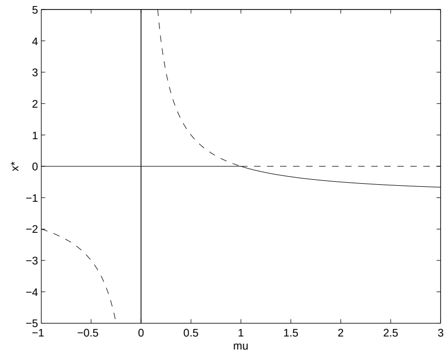
$$\begin{aligned} x^* &= \frac{1-\mu}{\mu} \\ x^* &= 0 \end{aligned}$$

Then, our bifurcation value is $\mu = 1$. To check the stability, let

$$\begin{aligned} f(x) &= \mu x - \frac{x}{1+x} \\ &\text{then} \\ f'(x) &= \mu - \frac{1}{(1+x)^2} \\ f'(0) &= \mu - 1 \\ f'\left(\frac{1-\mu}{\mu}\right) &= \mu - \mu^2 \end{aligned}$$

Then $x^* = 0$ is stable for negative $\mu < 1$, and $x^* = \frac{1-\mu}{\mu}$ is stable for positive $\mu > 1$.

Bifurcation diagram:




2. At the bifurcation value $\mu = 0$, we can show that the Jacobian at $(x^*, y^*) = (0, 0)$ has purely imaginary evals:

$$J(x^*, y^*) = \begin{bmatrix} \mu - (y^*)^2 & -1 + 2x^*y^* \\ 1 - 2x^* & \mu \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

The evals of this are

$$\lambda_{1,2} = \mu \pm i$$

So, when $\mu = 0$, the evals are purely imaginary.