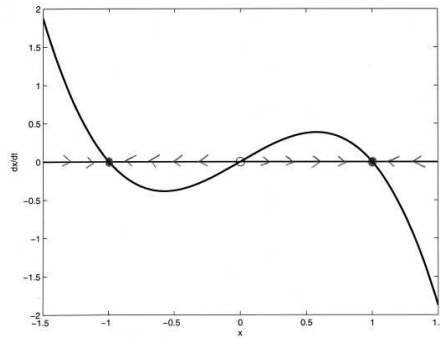


## Homework 5.1 Solutions Math 5110/6830

1. (a) For

$$\frac{dx}{dt} = x - x^3$$

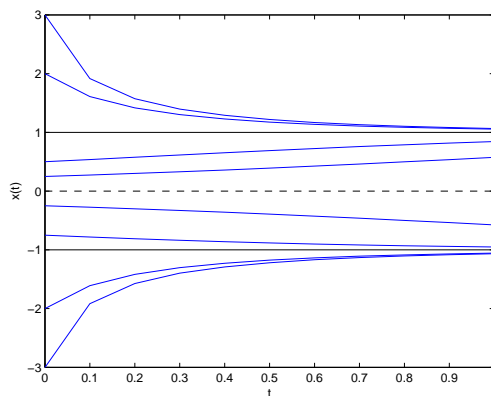
Phase-line diagram:



Fixed points are  $x^* = 0$  and  $x^* = \pm 1$ . Stability analytically (you can get this from your phase-line diagram too). Let

$$\begin{aligned} f(x) &= x - x^3 \\ f'(x) &= 1 - 3x^2 \\ f'(0) &= 1 \\ f'(1) &= -2 \\ f'(-1) &= -2 \end{aligned}$$

Then,  $x^* = 0$  is unstable and both  $x = 1$  and  $x = -1$  are stable. Sketch of  $x(t)$ :



To solve  $\dot{x} = x - x^3$ , we can use separation of variables.

$$\begin{aligned}\frac{1}{x - x^3} dx &= dt \\ \int_{x_0}^{x(t)} \frac{1}{\bar{x} - \bar{x}^3} d\bar{x} &= \int_0^t d\bar{t} \\ \int_{x_0}^{x(t)} \frac{1}{\bar{x}(1 - \bar{x})(1 + \bar{x})} d\bar{x} &= t\end{aligned}$$

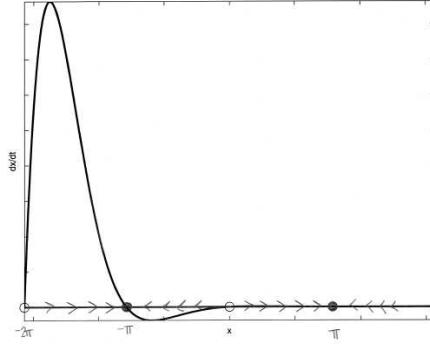
Using partial fractions:

$$\begin{aligned}\int_{x_0}^{x(t)} \left( \frac{1}{\bar{x}} + \frac{\frac{1}{2}}{1 - \bar{x}} - \frac{\frac{1}{2}}{1 + \bar{x}} \right) d\bar{x} &= t \\ \left( \ln(\bar{x}) - \frac{1}{2} \ln(1 - \bar{x}) - \frac{1}{2} \ln(1 + \bar{x}) \right)_{x_0}^{x(t)} &= t \\ \left[ \ln \left( \frac{\bar{x}}{\sqrt{1 - \bar{x}^2}} \right) \right]_{x_0}^{x(t)} &= t \\ \left[ \ln \left( \frac{\bar{x}}{\sqrt{1 - \bar{x}^2}} \right) \right]_{x_0}^{x(t)} &= t \\ \ln \left( \frac{x}{\sqrt{1 - x^2}} \right) - \ln \left( \frac{x_0}{\sqrt{1 - x_0^2}} \right) &= t \\ \ln \left( \frac{x^2}{1 - x^2} \right)^{1/2} &= t + \ln \left( \frac{x_0^2}{1 - x_0^2} \right)^{1/2} \\ \frac{1}{2} \ln \left( \frac{x^2}{1 - x^2} \right) &= t + \frac{1}{2} \ln \left( \frac{x_0^2}{1 - x_0^2} \right) \\ \ln \left( \frac{x^2}{1 - x^2} \right) &= 2t + \ln \left( \frac{x_0^2}{1 - x_0^2} \right) \\ \frac{x^2}{1 - x^2} &= \frac{x_0^2}{1 - x_0^2} e^{2t} \\ x^2 &= \frac{x_0^2}{1 - x_0^2} e^{2t} (1 - x^2) \\ x^2 \left( 1 + \frac{x_0^2}{1 - x_0^2} e^{2t} \right) &= \frac{x_0^2}{1 - x_0^2} e^{2t} \\ x^2 &= \frac{\frac{x_0^2}{1 - x_0^2} e^{2t}}{\left( 1 + \frac{x_0^2}{1 - x_0^2} e^{2t} \right)} \\ x(t) &= \sqrt{\frac{\frac{x_0^2}{1 - x_0^2} e^{2t}}{\left( 1 + \frac{x_0^2}{1 - x_0^2} e^{2t} \right)}} \\ x(t) &= \sqrt{\frac{x_0^2 e^{2t}}{(1 - x_0^2 + x_0^2 e^{2t})}} \\ x(t) &= \frac{x_0 e^t}{\sqrt{(1 - x_0^2 + x_0^2 e^{2t})}}\end{aligned}$$

(b) For

$$\frac{dx}{dt} = e^{-x} \sin(x)$$

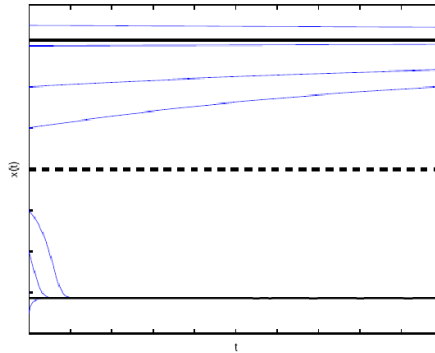
Phase-line diagram:



Fixed points are  $x^* = n\pi$ . Stability analytically (you can get this from your phase-line diagram too).  
Let

$$\begin{aligned} f(x) &= e^{-x} \sin(x) \\ f'(x) &= -e^{-x} \sin(x) + e^{-x} \cos(x) \end{aligned}$$

Then,  $x^* = n\pi$  is unstable for  $n$  even and stable for  $n$  odd.  
Sketch of  $x(t)$ :



This cannot be solve analytically.

- To find an equation for this phase-line diagram, we can notice that we will have roots  $x = -1$ ,  $x = 0$  and  $x = 2$ . So, the equation will take the form

$$\frac{dx}{dt} = x(x+1)(x-2)$$

But, to get the correct stability, we can check concavity. We need the function to be concave up at  $x = -1$ , concave down at  $x = 0$  and concave up at  $x = 2$ :

$$\begin{aligned} f(x) &= x(x+1)(x-2) \\ f'(x) &= 3x^2 - 2x - 2 \\ f''(x) &= 6x - 2 \\ f''(-1) &= -8 \end{aligned}$$

So, this fails. Lets think about it though. If we want this equation to be concave up at  $x = -1$  then the term  $(x + 1)$  should be squared. Let's try  $\frac{dx}{dt} = x(x + 1)^2(x - 2)$ :

$$\begin{aligned} f''(x) &= 12x^2 - 6 \\ f''(-1) &= 6 \\ f''(0) &= -6 \\ f''(2) &= 42 \end{aligned}$$

Then, this works! So, an equation to match the phase-line diagram is:

$$\frac{dx}{dt} = x(x + 1)^2(x - 2)$$

3. We can use separation of variables for this equation:

$$\begin{aligned} \frac{dN}{dt} &= K(t)N \\ \frac{1}{N}dN &= K(t)dt \\ \int_{N_0}^{N(t)} \frac{1}{\bar{N}}d\bar{N} &= \int_0^t K(\bar{t})d\bar{t} \\ \ln(\bar{N})_{N_0}^{N(t)} &= \int_0^t K(\bar{t})d\bar{t} \\ \ln(N(t)) - \ln(N_0) &= \int_0^t K(\bar{t})d\bar{t} \\ \ln\left(\frac{N(t)}{N_0}\right) &= \int_0^t K(\bar{t})d\bar{t} \\ \frac{N(t)}{N_0} &= e^{(\int_0^t K(\bar{t})d\bar{t})} \\ N(t) &= N_0e^{(\int_0^t K(\bar{t})d\bar{t})} \end{aligned}$$

4. **Solution using separation of variables:**

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) \\ \frac{1}{N\left(1 - \frac{N}{K}\right)}dN &= rdt \\ \int_{N_0}^{N(t)} \frac{1}{\bar{N}\left(1 - \frac{\bar{N}}{K}\right)}d\bar{N} &= \int_0^t rdt \end{aligned}$$

Using partial fractions:

$$\int_{N_0}^{N(t)} \left[ \frac{1}{\bar{N}} + \frac{\frac{1}{K}}{1 - \frac{\bar{N}}{K}} \right] d\bar{N} = \int_0^t rdt$$

Now integrate:

$$\begin{aligned}
 \left[ \ln \bar{N} - \ln \left( 1 - \frac{\bar{N}}{K} \right) \right]_{N_0}^{N(t)} &= rt \\
 \ln \left( \frac{\bar{N}}{1 - \frac{\bar{N}}{K}} \right)_{N_0}^{N(t)} &= rt \\
 \ln \left( \frac{N(t)}{1 - \frac{N(t)}{K}} \right) - \ln \left( \frac{N_0}{1 - \frac{N_0}{K}} \right) &= rt \\
 \ln \left( \frac{\left( 1 - \frac{N_0}{K} \right) N(t)}{\left( 1 - \frac{N(t)}{K} \right) N_0} \right) &= rt \\
 \frac{K - N_0}{K - N(t)} N(t) &= N_0 e^{rt} \\
 (K - N_0) N(t) &= N_0 e^{rt} (K - N(t)) \\
 (K - N_0 + N_0 e^{rt}) N(t) &= N_0 K e^{rt} \\
 N(t) &= \frac{N_0 K e^{rt}}{K - N_0 + N_0 e^{rt}} \\
 N(t) &= \frac{N_0 e^{rt}}{1 + \frac{N_0}{K} (e^{rt} - 1)}
 \end{aligned}$$

**Solution using the substitution**  $x = \frac{1}{N}$ :

With  $x = \frac{1}{N}$ , or equivalently  $N = \frac{1}{x}$ , we have:

$$\frac{dN}{dt} = -\frac{1}{x^2} \frac{dx}{dt}$$

Substituting this into the logistic eqn:

$$\begin{aligned}
 -\frac{1}{x^2} \frac{dx}{dt} &= r \frac{1}{x} \left( 1 - \frac{1}{K} \frac{1}{x} \right) \\
 \frac{dx}{dt} &= r \left( \frac{1}{K} - x \right)
 \end{aligned}$$

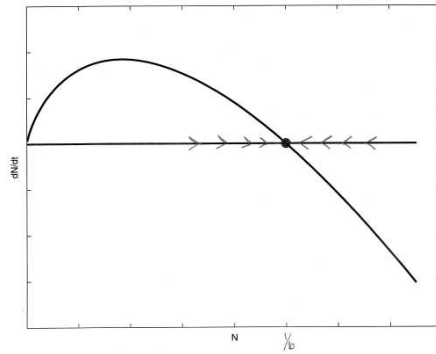
Now separate variables and integrate:

$$\begin{aligned}
 \int_{x_0}^{x(t)} \frac{1}{\frac{1}{K} - \bar{x}} d\bar{x} &= \int_0^t r d\bar{t} \\
 -\ln \left( \frac{1}{K} - \bar{x} \right)_{x_0}^{x(t)} &= rt \\
 -\ln \left( \frac{1}{K} - x(t) \right) + \ln \left( \frac{1}{K} - x_0 \right) &= rt \\
 \ln \left( \frac{\frac{1}{K} - x_0}{\frac{1}{K} - x(t)} \right) &= rt \\
 \frac{1 - Kx_0}{1 - Kx(t)} &= e^{rt} \\
 1 - Kx(t) &= (1 - Kx_0) e^{-rt} \\
 x(t) &= \frac{1}{K} [1 - (1 - Kx_0) e^{-rt}]
 \end{aligned}$$

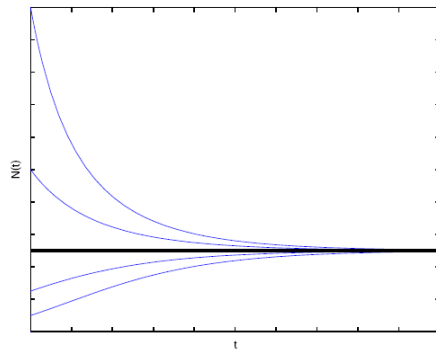
Returning to the original variables:

$$\begin{aligned} \frac{1}{N(t)} &= \frac{1}{K} \left[ 1 - \left( 1 - \frac{K}{N_0} \right) e^{-rt} \right] \\ N(t) &= \frac{K}{1 - \left( 1 - \frac{K}{N_0} \right) e^{-rt}} \\ &= \frac{K e^{rt}}{e^{rt} - 1 + \frac{K}{N_0}} \\ &= \frac{N_0 e^{rt}}{\frac{N_0}{K} e^{rt} - \frac{N_0}{K} + 1} \\ &= \frac{N_0 e^{rt}}{1 + \frac{N_0}{K} (e^{rt} - 1)} \end{aligned}$$

5. Phase-line diagram:



Solution:



Homework 5.2 Solutions  
Math 5110/6830

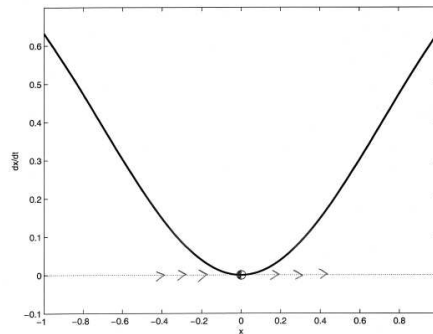
1. (a) Fixed points satisfy:

$$\frac{dx}{dt} = 0 = 1 - e^{-x^2}$$

The only fixed point is  $x^* = 0$ . To determine its stability:

$$\begin{aligned} f(x) &= 1 - e^{-x^2} \\ f'(x) &= 2xe^{-x^2} \\ f'(0) &= 0 \end{aligned}$$

From this, we cannot determine the stability of  $x^* = 0$ . Therefore, we will need to do this graphically: As you can see, this point is half-stable.



(b) Fixed points satisfy:

$$\frac{dx}{dt} = 0 = \ln(x)$$

The only fixed point is  $x^* = 1$ . To determine its stability:

$$\begin{aligned} f(x) &= \ln(x) \\ f'(x) &= \frac{1}{x} \\ f'(1) &= 1 \end{aligned}$$

And,  $x^* = 1$  is unstable.

(c) Fixed points satisfy:

$$\frac{dx}{dt} = 0 = x(1-x)(2-x)$$

The fixed points are  $x^* = 0$ ,  $x^* = 1$ , and  $x^* = 2$ . To determine their stability:

$$\begin{aligned} f(x) &= x(1-x)(2-x) \\ f'(x) &= (1-x)(2-x) - x(2-x) - x(1-x) \\ f'(0) &= 2 \\ f'(1) &= -1 \\ f'(2) &= 2 \end{aligned}$$

And,  $x^* = 0$  is unstable,  $x^* = 1$  is stable and  $x^* = 2$  is unstable.

(d) Fixed points satisfy:

$$\frac{dx}{dt} = 0 = ax - x^3 = x(a - x^2)$$

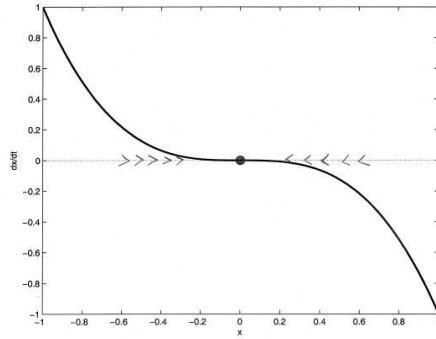
The fixed points are  $x^* = 0$ ,  $x^* = -\sqrt{a}$ , and  $x^* = \sqrt{a}$ . To determine their stability, let

$$f(x) = ax - x^3$$

Then, for  $a > 0$

$$\begin{aligned} f'(x) &= a - 3x^2 \\ f'(0) &= a \\ f'(-\sqrt{a}) &= -2a \\ f'(\sqrt{a}) &= -2a \end{aligned}$$

For  $a > 0$ ,  $x^* = 0$  is unstable,  $x^* = -\sqrt{a}$  is stable and  $x^* = \sqrt{a}$  is stable. But for  $a < 0$ ,  $x^* = 0$  is stable,  $x^* = -\sqrt{a}$  is unstable and  $x^* = \sqrt{a}$  is unstable. And, for  $a = 0$ , there is only one fixed point  $x^* = 0$  which we need to do a graphical stability analysis for:



Looks stable to me.

2. (a) To determine stability, let

$$\begin{aligned} f(x) &= -x^c \\ &\text{and} \\ f'(x) &= -cx^{c-1} \end{aligned}$$

Then, for  $c = 1$  this is stable. However, we need to do this graphically for other values of  $c$ : ????

(b) To find an equation for time here, we can separate variables and solve this DE:

$$\begin{aligned} \frac{dx}{dt} &= -x^c \\ \int_{x_0}^{x(t)} \frac{1}{\bar{x}^c} d\bar{x} &= \int_0^t -d\bar{t} \\ \int_{x_0}^{x(t)} \bar{x}^{-c} d\bar{x} &= \int_0^t -d\bar{t} \\ \left( \frac{\bar{x}^{-c+1}}{-c+1} \right)_{x_0}^{x(t)} &= -t \\ \frac{x(t)^{-c+1}}{-c+1} - \frac{x_0^{-c+1}}{-c+1} &= -t \\ x(t)^{-c+1} &= x_0^{-c+1} - t(-c+1) \\ x(t) &= [x_0^{-c+1} - t(-c+1)]^{\frac{1}{-c+1}} \end{aligned}$$

Letting  $x_0 = 1$ :

$$x(t) = [1 - t(-c + 1)]^{\frac{1}{-c+1}}$$

Now, to find the time it takes til  $x(t) = 0$ :

$$\begin{aligned}x(t) = 0 &= [1 - t(-c + 1)]^{\frac{1}{-c+1}} \\0 &= 1 - t(-c + 1) \\t &= \frac{1}{1 - c}\end{aligned}$$

3. To show that there are infinite solutions to  $\dot{x} = x^{1/3}$ , we can find at least 2 solutions that pass through the same point. Notice that  $x(t) = 0$  is one solution that passes through the point  $(0, 0)$ . The other solution, found by separation of variables, is  $x(t) = \frac{2}{3}x^{3/2}$ . This solution also passes through the point  $(0, 0)$ .