Homework 4.1 Solutions
Math 5110/6830

1. (a) For

$$p_{n+1} = \frac{\alpha p_n}{(\gamma-\alpha)p_n^2 - 2(\gamma-\alpha)p_n + \gamma}$$

Equilibria points satisfy:

$$\begin{aligned}
p^* &= \frac{\alpha p^*}{(\gamma-\alpha)(p^*)^2 - 2(\gamma-\alpha)p^* + \gamma} \\
p_1^* = 0 \quad & \text{OR} \\
1 &= \frac{\alpha}{(\gamma-\alpha)(p^*)^2 - 2(\gamma-\alpha)p^* + \gamma} \\
(\gamma-\alpha)(p^*)^2 - 2(\gamma-\alpha)p^* + \gamma &= \alpha \\
(\gamma-\alpha)(p^*)^2 - 2(\gamma-\alpha)p^* + \gamma - \alpha &= 0 \\
(p^*)^2 - 2p^* + 1 &= 0 \\
(p^*-1)^2 &= 0
\end{aligned}$$

Then, we have equilibria points $p_1^* = 0$ and $p_2^* = 1$.

(b) Let

$$f(p) = \frac{\alpha p}{(\gamma-\alpha)p^2 - 2(\gamma-\alpha)p + \gamma}$$

Then,

$$\begin{aligned}
f'(p) &= \frac{\alpha((\gamma-\alpha)p^2 - 2(\gamma-\alpha)p + \gamma) - \alpha p(2(\gamma-\alpha)p - 2(\gamma-\alpha))}{((\gamma-\alpha)p^2 - 2(\gamma-\alpha)p + \gamma)^2} \\
&= \frac{\alpha(\gamma-\alpha)p^2 - 2\alpha(\gamma-\alpha)p + \alpha\gamma - 2\alpha(\gamma-\alpha)p^2 + 2\alpha(\gamma-\alpha)p}{((\gamma-\alpha)p^2 - 2(\gamma-\alpha)p + \gamma)^2} \\
&= \frac{\alpha\gamma - \alpha(\gamma-\alpha)p^2}{((\gamma-\alpha)p^2 - 2(\gamma-\alpha)p + \gamma)^2}
\end{aligned}$$

At $p_1^* = 0$:

$$\begin{aligned}
|f'(0)| &= \left|\frac{\alpha}{\gamma}\right| \\
&> 1 \quad \text{when } \alpha > \gamma \text{ (unstable)} \\
&< 1 \quad \text{when } \alpha < \gamma \text{ (stable)}
\end{aligned}$$

At $p_2^* = 1$:

$$|f'(1)| = |1|$$

So, we can't be sure of the stability of $p_2^* = 1$ from this. Do a cobweb to determine that this point is **stable** when p_1^* is unstable, etc.

2. (a) First we have:

$$\begin{aligned}
p_{n+1} &= \frac{2\alpha p_n^2 + 2\beta(1-p_n)p_n}{2(\alpha p_n^2 + 2\beta(1-p_n)p_n + \gamma(1-p_n)^2)} \\
&= \frac{(\alpha-\beta)p_n^2 + \beta p_n}{(\alpha - 2\beta + \gamma)p_n^2 - 2(\gamma-\beta)p_n + \gamma}
\end{aligned}$$

(b) Fixed points satisfy:

$$\begin{aligned}
p^* &= \frac{(\alpha-\beta)(p^*)^2+\beta p^*}{(\alpha-2\beta+\gamma)(p^*)^2-2(\gamma-\beta)p^*+\gamma} \\
p_1^*=0 &\quad \text{OR} \\
1 &= \frac{(\alpha-\beta)p^*+\beta}{(\alpha-2\beta+\gamma)(p^*)^2-2(\gamma-\beta)p^*+\gamma} \\
(\alpha-2\beta+\gamma)(p^*)^2-2(\gamma-\beta)p^*+\gamma &= (\alpha-\beta)p^*+\beta \\
(\alpha-2\beta+\gamma)(p^*)^2-2(\gamma-\beta)p^*+\gamma-(\alpha-\beta)p^*-\beta &= 0 \\
(\alpha-2\beta+\gamma)(p^*)^2-(\alpha-3\beta+2\gamma)p^*+\gamma-\beta &= 0 \\
(\alpha-2\beta+\gamma)(p^*)^2-(\alpha-3\beta+2\gamma)p^*+\gamma-\beta &= 0 \\
(p^*)^2-\left(\frac{\alpha-3\beta+2\gamma}{\alpha-2\beta+\gamma}\right)p^*+\frac{\gamma-\beta}{\alpha-2\beta+\gamma} &= 0 \\
(p^*-1)\left(p^*-\frac{\gamma-\beta}{\alpha-2\beta+\gamma}\right) &= 0
\end{aligned}$$

So, the equilibria points are

$$\begin{aligned}
p_1^* &= 0 \\
p_2^* &= 1 \\
p_3^* &= \frac{\gamma-\beta}{\alpha-2\beta+\gamma}
\end{aligned}$$

p_1^* and p_2^* exist for all parameter values. However, p_3^* needs to be positive to exist:

$$\begin{aligned}
\frac{\gamma-\beta}{\alpha-2\beta+\gamma} &> 0 \\
&\text{then} \\
\gamma>\beta \quad &\text{and} \quad \alpha+\gamma>2\beta \\
&\text{or} \\
\gamma<\beta \quad &\text{and} \quad \alpha+\gamma<2\beta
\end{aligned}$$

(c) Let

$$f(p) = \frac{(\alpha-\beta)p^2+\beta p}{(\alpha-2\beta+\gamma)p^2-2(\gamma-\beta)p+\gamma}$$

Then,

$$\begin{aligned}
f'(p) &= \frac{[2(\alpha-\beta)p+\beta][(\alpha-2\beta+\gamma)p^2-2(\gamma-\beta)p+\gamma]-[(\alpha-\beta)p^2+\beta p][2(\alpha-2\beta+\gamma)p-2(\gamma-\beta)]}{[(\alpha-2\beta+\gamma)p^2-2(\gamma-\beta)p+\gamma]^2} \\
&= \frac{(-2\alpha\gamma+\alpha\beta+\beta\gamma)p^2+2(\alpha\gamma-\beta\gamma)p+\beta\gamma}{[(\alpha-2\beta+\gamma)p^2-2(\gamma-\beta)p+\gamma]^2}
\end{aligned}$$

For $p_1^*=0$:

$$\begin{aligned}
|f'(0)| &= \left|\frac{\beta}{\gamma}\right| \\
&> 1 \quad \text{when } \beta>\gamma \text{ (unstable)} \\
&< 1 \quad \text{when } \beta<\gamma \text{ (stable)}
\end{aligned}$$

For $p_2^*=1$:

$$\begin{aligned}
|f'(1)| &= \left|\frac{\beta}{\alpha}\right| \\
&> 1 \quad \text{when } \beta>\alpha \text{ (unstable)} \\
&< 1 \quad \text{when } \beta<\alpha \text{ (stable)}
\end{aligned}$$

For $p_3^* = \frac{\gamma-\beta}{\alpha-2\beta+\gamma}$:

$$\begin{aligned}
f'\left(\frac{\gamma-\beta}{\alpha-2\beta+\gamma}\right) &= \frac{(-2\alpha\gamma+\alpha\beta+\beta\gamma)\left(\frac{\gamma-\beta}{\alpha-2\beta+\gamma}\right)^2+2(\alpha\gamma-\beta\gamma)\left(\frac{\gamma-\beta}{\alpha-2\beta+\gamma}\right)+\beta\gamma}{\left[(\alpha-2\beta+\gamma)\left(\frac{\gamma-\beta}{\alpha-2\beta+\gamma}\right)^2-2(\gamma-\beta)\left(\frac{\gamma-\beta}{\alpha-2\beta+\gamma}\right)+\gamma\right]^2} \\
&= \frac{(-2\alpha\gamma+\alpha\beta+\beta\gamma)\frac{(\gamma-\beta)^2}{(\alpha-2\beta+\gamma)^2}+2(\alpha\gamma-\beta\gamma)\frac{\gamma-\beta}{\alpha-2\beta+\gamma}+\beta\gamma}{\left[-\frac{(\gamma-\beta)^2}{\alpha-2\beta+\gamma}+\gamma\right]^2} \\
&= \frac{(-2\alpha\gamma+\alpha\beta+\beta\gamma)\frac{(\gamma-\beta)^2}{(\alpha-2\beta+\gamma)^2}+2\frac{(\alpha\gamma-\beta\gamma)(\gamma-\beta)(\alpha-2\beta+\gamma)}{(\alpha-2\beta+\gamma)^2}+\beta\gamma\frac{(\alpha-2\beta+\gamma)^2}{(\alpha-2\beta+\gamma)^2}}{\left[-\frac{(\gamma-\beta)^2}{\alpha-2\beta+\gamma}+\gamma\frac{\alpha-2\beta+\gamma}{\alpha-2\beta+\gamma}\right]^2} \\
&= \frac{(-2\alpha\gamma+\alpha\beta+\beta\gamma)(\gamma-\beta)^2+2(\alpha\gamma-\beta\gamma)(\gamma-\beta)(\alpha-2\beta+\gamma)+\beta\gamma(\alpha-2\beta+\gamma)^2}{[-(\gamma-\beta)^2+\gamma(\alpha-2\beta+\gamma)]^2} \\
&= \frac{(2\alpha\gamma-\alpha\beta-\beta\gamma)(\alpha\gamma-beta^2)}{(\alpha\gamma-\beta^2)^2} \\
&= \frac{2\alpha\gamma-\alpha\beta-\beta\gamma}{\alpha\gamma-\beta^2}
\end{aligned}$$

For this to be stable we need $|f'(p)| < 1$

$$\begin{aligned}
-1 < & \ \frac{2\alpha\gamma-\alpha\beta-\beta\gamma}{\alpha\gamma-\beta^2} < 1 \\
& \text{for} \\
\frac{2\alpha\gamma-\alpha\beta-\beta\gamma}{\alpha\gamma-\beta^2} & < 1
\end{aligned}$$

Remember that for p_3^* to exist, we need $\gamma > \beta$ AND $\alpha+\gamma > 2\beta$ OR $\gamma < \beta$ AND $\alpha+\gamma < 2\beta$. Then,

$$\begin{aligned}
\frac{2\alpha\gamma-\beta(\alpha+\gamma)}{\alpha\gamma-\beta^2} &> \frac{2\alpha\gamma-\beta(2\beta)}{\alpha\gamma-\beta^2} \\
&> \frac{2(\alpha\gamma-\beta^2)}{\alpha\gamma-\beta^2} = 2
\end{aligned}$$

Then for $\gamma > \beta$ which means also means that $\alpha > \beta$, this point is **unstable**. But for $\beta > \gamma$ and $\beta > \alpha$, then this point is **stable**.

(d) We'll start with one of the conditions we had for p_3^* to exist: $\gamma > \beta$ AND $\alpha+\gamma > 2\beta$. In this environment, the black moths are favored over the gray moths since $\gamma > \beta$. However, we are also favoring the peppered moths over the gray moths since

$$\begin{aligned}
2\beta < \ & \alpha+\gamma < \alpha+\beta \\
& \text{so} \\
\beta \ & < \ \alpha
\end{aligned}$$

Now let's look at the other condition we could have for p_3^* to exist: $\gamma < \beta$ AND $\alpha+\gamma < 2\beta$. In this environment, the grey moths are favored over the black moths since $\gamma < \beta$. But they are also favored over the peppered moths since

$$\begin{aligned}
2\beta > \ & \alpha+\gamma > \alpha+\beta \\
& \text{so} \\
\beta \ & > \ \alpha
\end{aligned}$$

We can't tell anything from these conditions about the selective advantages between the peppered and black moths.

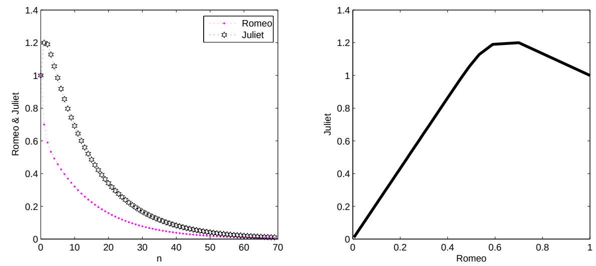
For the long term dynamics, we know that the W allele will either disappear ($p_1^* = 0$), take over ($p_2^* = 1$), or go to a coexistence p_3^*. Recall the conditions for each of these to be stable. For $p_1^* = 0$ to be stable we needed $\gamma > \beta$. So, the W allele is lost if the black moths are favored over the grey moths. And, for $p_2^* = 1$ to be stable we needed $\alpha > \beta$. So, the W allele is fixed if the peppered moths are favored over the grey moths. Both of these are true as in the case before. So, how could this happen? We have 2 stable equilibria! Remember that we still have an unstable equilibria inbetween the two stable ones. Mathematically this better be true!! We always need to alternate the stability of the points. This means that if we start to the left of p_3^*, then the W allele will be lost. But, if we start to the right of p_3^*, then the W allele wins everything! However, if both p_1^* and $p + 2^*$ are unstable (ie. for $\beta > \gamma$ & $\beta > \alpha$), then the coexistence equilibria p_3^* is stable. Therefore we get some peppered moths and some of the other 2 kinds depending on if $\gamma > \alpha$ or vice versa.

(e) We are now given that $\alpha < \beta < \gamma$ with the black (ww) moths at equilibrium. By our previous analysis, $p_1^* = 0$ is stable, $p_2^* = 1$ is unstable and p_3^* doesn't exist. However, the peppered moths (WW) are making their comeback & now have much higher levels. What happens? Well, since $p_1^* = 0$ is stable, then the peppered moths will eventually die out again. Poor peppered moths.

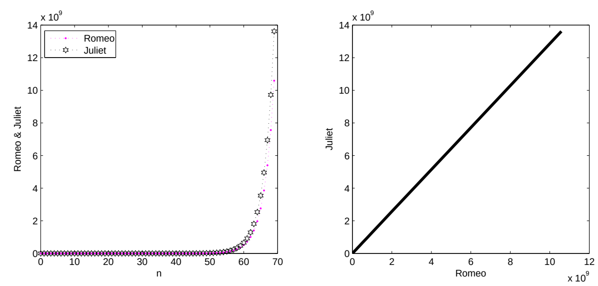
(f) For $\beta < \alpha < \gamma$, we again have three equilibria points. Recall that if we're above p_3^*, then the peppered moths population will survive. Otherwise, they will be lost again.

Homework 4.2 Solutions
Math 5110/6830

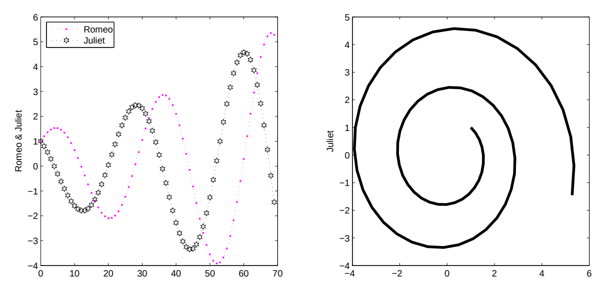
1. (a) with $a_R = 0.5$, $a_J = 0.7$, $p_R = 0.2$, and $p_J = 0.5$:



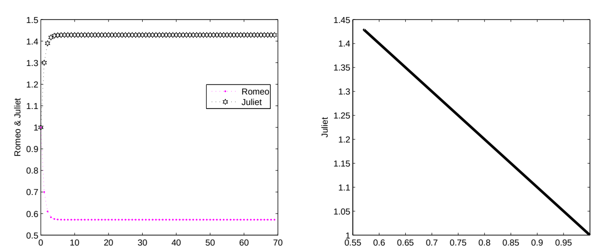
(b) with $a_R = 0.5$, $a_J = 0.7$, $p_R = 0.7$, and $p_J = 0.9$:



(c) with $a_R = 1.0$, $a_J = 1.0$, $p_R = 0.2$, and $p_J = -0.2$:



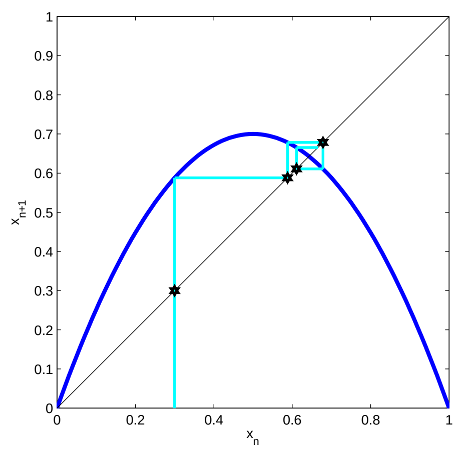
(d) with $a_R = 0.5$, $a_J = 0.8$, $p_R = 0.2$, and $p_J = 0.5$:



Matlab code:

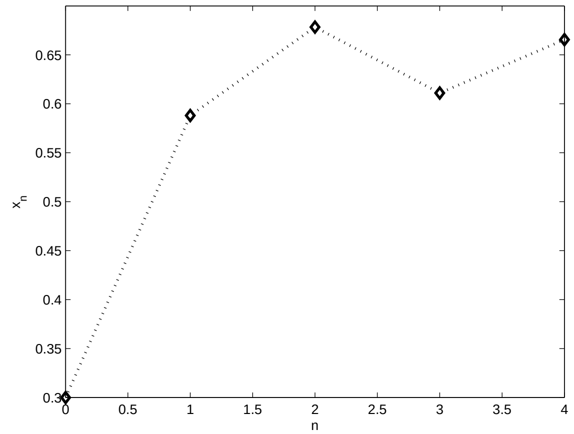
```
a=[.5;.7;.2;.5];
b=[.5;.7;.7;.9];
c=[1;1;.2;-.2];
d=[.5;.8;.2;.5];
A=[a b c d];
N=70;
for i=1:4
  ar=A(1,i);
  aj=A(2,i);
  pr=A(3,i);
  pj=A(4,i);
  R=zeros(1,N);
  J=zeros(1,N);
  R(1)=1;
  J(1)=1;
  for n=1:N-1
     R(n+1)=ar*R(n)+pr*J(n);
     J(n+1)=aj*J(n)+pj*R(n);
     figure(i)
     subplot(1,2,1)
     plot([0:(N-1)], R, 'm:.',[0:(N-1)],J,'k:h'); xlabel('n'); ylabel('Romeo & Juliet');
     legend('Romeo','Juliet',0);
     hold off;
     subplot(1,2,2)
     x=R;
     y=J;
     plot(x,y,'k','Linewidth',3); xlabel('Romeo'); ylabel('Juliet');
  end
end
```

2. (a) Cobweb diagram:



(b) The point $(0, 0)$ is **unstable**, and the nonzero equilibria is **stable**.

(c) Solution:



Matlab code for both part a) and c):

```
N=5;
x=zeros(1,N);
x(1)=.3;
figure(1);
for n=1:4
  x(n+1)=2.8*(1-x(n))*x(n);
  axis square; hold off;
  plot([0:(N-1)],x,'k:d','Linewidth',2); xlabel('n'); ylabel('x_n');
end

% for the cobwebbing:
t=0:0.01:1;
figure(2)
plot(t,2.8*(t.*(1-t)),'b','Linewidth',3); hold on; axis square;
fplot('1*y',[0 1],'Linewidth',3,'k');
line([x(1) x(1)],[0 x(2)],'Color','c','Linewidth',2);
plot(x(1),x(1),'kh','Linewidth',2);
for n=1:3
  line([x(n) x(n+1)],[x(n+1) x(n+1)],'Color','c','Linewidth',2)
  line([x(n+1) x(n+1)],[x(n+1) x(n+2)],'Color','c','Linewidth',2)
  plot(x(n+1), x(n+1),'kh','Linewidth',2); xlabel('x_n');ylabel('x_{n+1}');
end
line([x(4) x(4+1)],[x(4+1) x(4+1)],'Color','c','Linewidth',2)
```

(d) The fixed points are:

$$x_1^* = 0$$
$$x_2^* = \frac{K(r-1)}{r}$$

Stability of fixed points: let

$$f(x) = r\left(1 - \frac{x}{K}\right)x$$

Then,

$$\begin{aligned} f'(x) &= r\left(1-\frac{x}{K}\right) - rx\frac{1}{K} \\ &= r\left(1-\frac{2x}{K}\right) \end{aligned}$$

And,

$$\begin{aligned} f'(0) &= r \\ f'\left(\frac{K(r-1)}{r}\right) &= r\left(1-2\frac{(r-1)}{r}\right) \\ &= r-2(r-1) = 2-r \end{aligned}$$

So, $x_1^* = 0$ is **stable** for $0 < r < 1$, and **unstable** otherwise. And, $x_2^* = \frac{K(r-1)}{r}$ is **stable** for $1 < r < 3$, and unstable otherwise. We can also tell from this that there will be cycling (period doubling) that begins at $r = 3$. Your clue to know this should have been the fact that you can't have 2 unstable (or stable) equilibria points next to each other. They always have to alternate stability (ie. if one is stable, the other is unstable, and vice versa).
Bifurcation Diagram: