Homework 2.1 Solutions
Math 5110/6830

1. (a) The variables, parameters & terms:

$$\begin{aligned} a_n &= \text{whale population after n years} \\ k &= \text{growth rate} \\ M &= \text{carrying capacity} \\ m &= \text{minimum survival level} \end{aligned}$$

- If $0 < a_n < m$, then $k(M - a_n)(a_n - m) < 0$ and the population will decline.
- If $m < a_n < M$, then $k(M - a_n)(a_n - m) > 0$ and the population will grow.
- If $a_n > M$, then $k(M - a_n)(a_n - m) < 0$ and the population will decline.
- If $a_n = m$ or $a_n = M$, then $k(M - a_n)(a_n - m) = 0$ and the population will remain the same (ie. $a_n = m$ and $a_n = M$ are fixed points).

(b) To find fixed points, we set $a_{n+1} = a_n = a^*$:

$$\begin{aligned} a^* &= a^* + k(M - a^*)(a^* - m) \\ 0 &= k(M - a^*)(a^* - m) \\ a^* = M \quad &\text{OR} \quad a^* = m \end{aligned}$$

With $M = 5000$, $m = 100$, and $k = 0.0001$, we have fixed points $a^* = 5000$ and $a^* = 100$. To find the stability, let

$$\begin{aligned} f(a) &= a + 0.0001(5000 - a)(a - 100) \\ &= a + 0.0001(5100a - 500000 - a^2) \end{aligned}$$

Then,

$$f'(a) = 1 + 0.0001(5100 - 2a)$$

For $a^* = 100$:

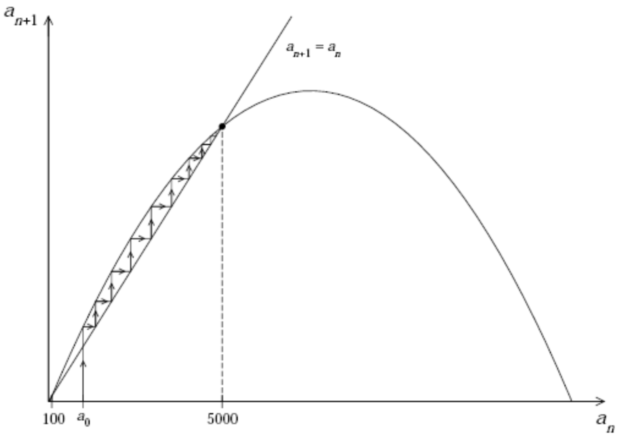
$$\begin{aligned} |f'(100)| &= |1 + 0.0001(5100 - 200)| \\ &= 1.49 > 1 \end{aligned}$$

So, $a^* = 100$ is **unstable**.
For $a^* = 5000$:

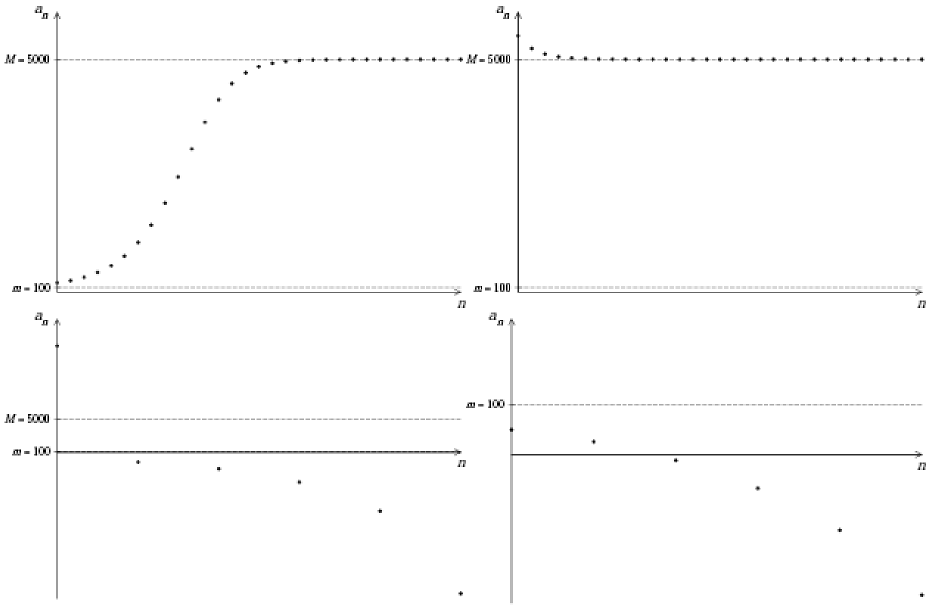
$$\begin{aligned} |f'(100)| &= |1 + 0.0001(5100 - 10000)| \\ &= 0.51 < 1 \end{aligned}$$

So, $a^* = 5000$ is **stable**.

(c) The following cobweb graphically says what we just found: $a^* = 100$ is unstable and $a^* = 5000$ is stable.



(d) The following represents solutions for different initial conditions:



(e) When $a_0 < m$, the population declines and eventually becomes negative. When $a_0 >> M$, then $a_1 < 0$ and the population continues to decline. Both of these cases are problematic since we can't have a negative number of whales. I don't think the whales would appreciate that!!

2. (a) The system we have is:

$$M_{t+1} = M_t - \frac{M_t^p}{K^p + M_t^p} M_t + S$$

With $K = 2$ and $S = 1$:

$$M_{t+1} = M_t - \frac{M_t^p}{2^p + M_t^p} M_t + 1$$

A fixed point would satisfy:

$$M^* = M^* - \frac{M^{*p}}{2^p + M^{*p}} M^* + 1$$

With $M^* = 2$:

$$\begin{aligned} 2 - \frac{2^p}{2^p + 2^p} 2 + 1 &= 3 - \frac{2^p}{2(2^p)} 2 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

This is true for all values of p.

(b) To find the stability of this system for the fixed point $M^* = 2$, we need to first calculate a derivative. Let

$$g(M) = M - \frac{M^p}{2^p + M^p} M + 1$$

Then,

$$\begin{aligned} g'(M) &= 1 - \frac{pM^{p-1}(2^p + M^p) - M^p p M^{p-1}}{(2^p + M^p)^2} M - \frac{M^p}{2^p + M^p} \\ &= 1 - \frac{pM^{p-1}(2^p)}{(2^p + M^p)^2} M - \frac{M^p}{2^p + M^p} \end{aligned}$$

With $M^* = 2$:

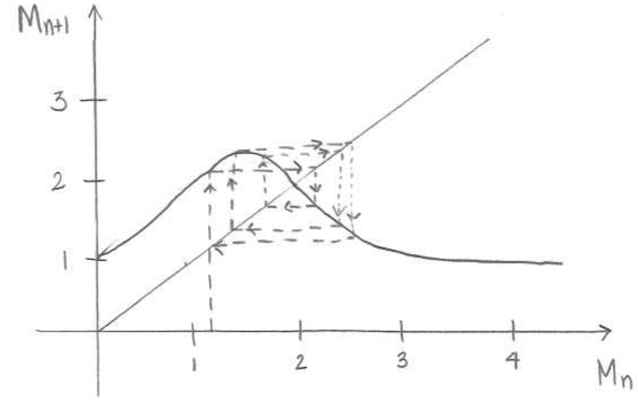
$$\begin{aligned} g'(2) &= 1 - \frac{p2^{p-1}(2^p)}{(2^p + 2^p)^2} 2 - \frac{2^p}{2^p + 2^p} \\ &= 1 - \frac{p2^{2p}}{4(2^{2p})} - \frac{2^p}{2(2^p)} \\ &= \frac{1}{2} - \frac{p}{4} \end{aligned}$$

For $M^* = 2$ to be stable, we need $|g'(2)| < 1$. So, this happens when $-1 < g'(2) < 1$:

$$\begin{aligned} -1 < & \quad \tfrac{1}{2} - \tfrac{p}{4} & < 1 \\ -\frac{3}{2} < & \quad -\tfrac{p}{4} & < \frac{1}{2} \\ \frac{3}{2} > & \quad \tfrac{p}{4} & > -\frac{1}{2} \\ 6 > & \quad p & > -2 \end{aligned}$$

Then, $M^* = 2$ is **stable** if $-2 < p < 6$ and **unstable** if $p < -2$ or $p > 6$. The solution will oscillate when $g'(2) < 0$, ie. when $p > 2$.

(c) Cobweb for when $M^* = 2$ is unstable:



Homework 2.2 Solutions
Math 5110/6830

1. (a)

$$\begin{aligned} f^2(x) &= r(rx(1-x))(1-(rx(1-x))) \\ &= r^2x(1-(r+1)x+2rx^2-rx^3) \end{aligned}$$

(b) First, we know that fixed points of $f(x)$ will also be fixed points of $f^2(x)$. So, from $f(x)$:

$$\begin{aligned} x^* &= r(1-x)x \\ 0 &= (r-rx-1)x \end{aligned}$$

Then, fixed points of both f and f^2 are $x^*=0$ and $x^*=\frac{r-1}{r}$. However, there are other fixed points of f^2. To find these, first write:

$$\begin{aligned} x^* &= r^2x^*(1-(r+1)x^*+2r(x^*)^2-r(x^*)^3) \\ 0 &= (r^2(1-(r+1)x^*+2r(x^*)^2-r(x^*)^3)-1)x^* \end{aligned}$$

We can already see that $x^*=0$ is a solution to this. Then,

$$\begin{aligned} 0 &= r^2(1-(r+1)x^*+2r(x^*)^2-r(x^*)^3)-1 \\ \frac{1}{r^2} &= 1-(r+1)x^*+2r(x^*)^2-r(x^*)^3 \\ 0 &= \left(1-\frac{1}{r^2}\right)-(r+1)x^*+2r(x^*)^2-r(x^*)^3 \\ 0 &= \frac{r^2-1}{r^3}-\left(1+\frac{1}{r}\right)x^*+2(x^*)^2-(x^*)^3 \\ 0 &= \frac{(r-1)(r+1)}{r^3}-\left(\frac{r+1}{r}\right)x^*+2(x^*)^2-(x^*)^3 \end{aligned}$$

And, since we also know that $x^*=\frac{r-1}{r}$ is a fixed point:

$$0 = \left(x^*-\frac{r-1}{r}\right)\left((x^*)^2-\left(\frac{r+1}{r}\right)x^*+\left(\frac{r+1}{r^2}\right)\right)$$

Then, with a little algebra

$$\begin{aligned} x^* &= 0 \\ x^* &= \frac{r-1}{r} \\ x^* &= \frac{r+1\pm\sqrt{(r-3)(r+1)}}{2r} \end{aligned}$$

The first two are trivial 2-cycles, so let's take a look at the third fixed point. If $r<3$, the roots are imaginary so there is no 2-cycle. If $r=3$ then, there is only one root so there is no 2-cycle. But, if $r>3$, then we get two distinct roots & a nontrival 2-cycle.

(c) Computing $\frac{d}{dx}f^2(x)$:

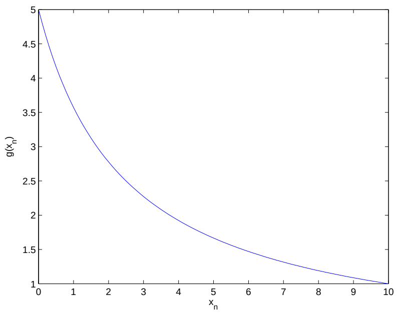
$$\frac{d}{dx}f^2(x) = -4r^3x^3+6r^3x^2-2(r^2+r^3)x+r^2$$

(d) To do this, we need to evaluate $|\frac{d}{dx}f^2(x^*)|$:

$$|\frac{d}{dx}f^2(x^*)| = |-r^2+2r+4|$$

Then, $|\frac{d}{dx}f^2(x^*)|<1$ (ie. stable) when $-1<-r^2+2r+4<1$. Solving each of these inequalities should yield that for the 2-cycle to be stable we need $r>3$ and $r<1+\sqrt{6}$. It is unstable elsewhere.

2. (a) Graph of $g(x_n)$ versus x_n:



(b) Fixed points satisfy:

$$\begin{aligned}
x^* &= \frac{r}{1+\frac{r-1}{K}x^*}x^* \\
1 &= \frac{r}{1+\frac{r-1}{K}x^*} \\
1+\frac{r-1}{K}x^* &= r \\
\frac{r-1}{K}x^* &= r-1 \\
x^*=0 \quad &\text{OR} \quad x^*=K
\end{aligned}$$

(c) Let

$$f(x) = \frac{r}{1+\frac{r-1}{K}x}x$$

Then,

$$f'(x) = \frac{r}{\left(1+\frac{r-1}{K}x\right)^2}$$

For $x^*=0$:

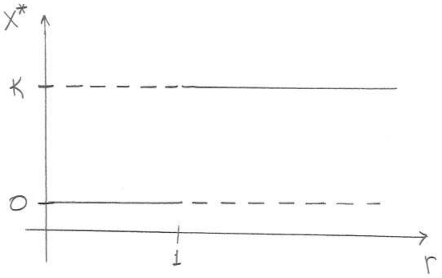
$$|f'(x)| = |r|$$

This is **stable** for $0<r<1$ and **unstable** for $r>1$.
For $x^*=K$:

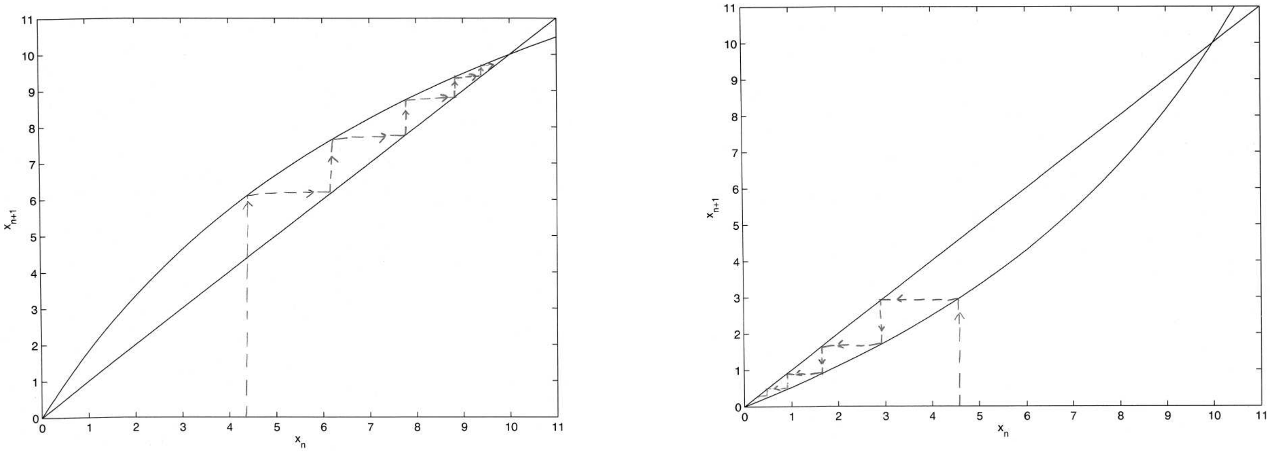
$$|f'(x)| = \left|\frac{1}{r}\right|$$

This is **stable** for $r>1$ and **unstable** for $0<r<1$.

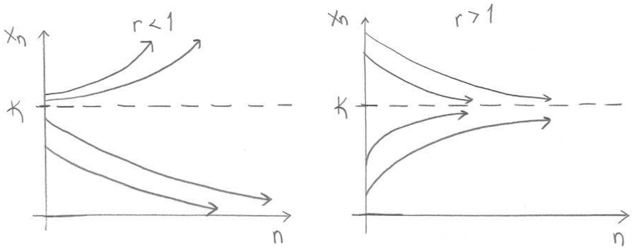
Bifurcation Diagram:



(d) Cobweb for $r > 1$ (left) and $r < 1$ (right):



(e) Solutions:



(f) So far this system doesn't look like it'll have cycles or chaos. We can tell this from the behavior of the cobweb plot, solutions, and bifurcation diagram.

(g) We have

$$x_{n+1} = \frac{r}{1 + \frac{r-1}{K} x_n} x_n$$

Let $u_n = \frac{1}{x_n}$. Then,

$$\begin{aligned} \frac{1}{u_{n+1}} &= \frac{r}{1 + \frac{r-1}{K}\frac{1}{x_n}} \frac{1}{x_n} \\ u_{n+1} &= \frac{1}{r}\left(1 + \frac{r-1}{K}\frac{1}{u_n}\right) u_n \\ &= \frac{1}{r} u_n + \frac{r-1}{rK} \end{aligned}$$

Rewrite this as $u_{n+1} = Au_n + B$, where $A = \frac{1}{r}$ and $B = \frac{r-1}{rK}$. Then, we can find a solution by doing the following:

$$\begin{aligned}
u_{n+1} &= Au_n + B \\
&= A(Au_{n-1} + B) + B = A^2 u_{n-1} + B(A+1) \\
&= A^2(Au_{n-2} + B) + B(A+1) = A^3 u_{n-1} + B(A^2 + A + 1) \\
&. \\
&. \\
&. \\
&= A^{n+1} u_0 + B(A^n + A^{n-1} + ... + A + 1) \\
&= A^{n+1} u_0 + B\frac{A^{n+1} - 1}{A - 1}
\end{aligned}$$

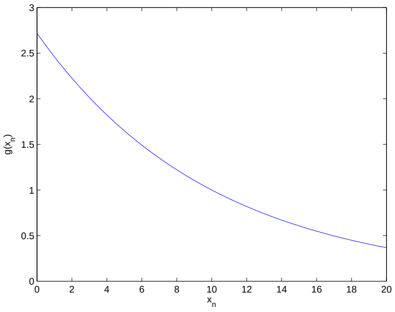
Now, return back to the original variables:

$$\begin{aligned}
\frac{1}{x_{n+1}} &= \frac{1}{r^{n+1}} \frac{1}{x_0} + \frac{r-1}{rK} \frac{\frac{1}{r^{n+1}} - 1}{\frac{1}{r} - 1} \\
x_{n+1} &= \frac{r^{n+1} x_0}{1 + \frac{r^{n+1}-1}{K} x_0}
\end{aligned}$$

3. (a) For this system we have:

$$x_{n+1} = exp\left[r\left(1 - \frac{x_n}{K}\right)\right] x_n$$

Plot of $g(x_n)$ for r=1, K=10:



(b) Fixed points satisfy:

$$\begin{aligned}
x^* &= exp\left[r\left(1 - \frac{x^*}{K}\right)\right] x^* \\
1 &= exp\left[r\left(1 - \frac{x^*}{K}\right)\right] \\
0 &= r\left(1 - \frac{x^*}{K}\right) \\
x^* = 0 \quad &\text{OR} \quad x^* = K
\end{aligned}$$

(c) Let

$$f(x) = exp\left[r\left(1 - \frac{x}{K}\right)\right] x$$

Then,

$$f'(x) \quad = \quad \left(1 - \frac{rx}{K}\right) exp\left[r\left(1 - \frac{x}{K}\right)\right]$$

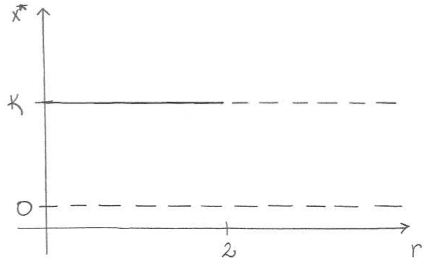
For $x^* = 0$:

$$|f'(0)| \quad = \quad |e^r|$$

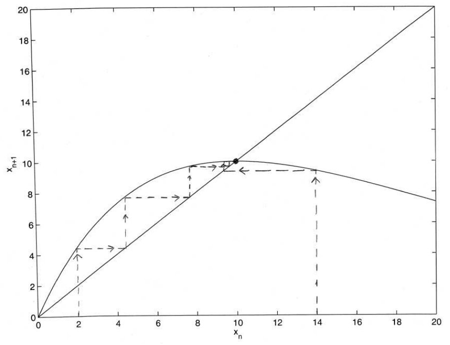
This is always **unstable**.
For $x^* = K$:

$$|f'(0)| \quad = \quad |1 - r|$$

This is **stable** if $0 < r < 2$ and **unstable** for $r > 2$.
Bifurcation Diagram:



(d) Cobweb:



Solutions: