

Homework 1.1 Solutions  
Math 5110/6830

1. (a) There are 2 ways to do this problem. You can either plug the solution in to verify that it actually solves the equation or you can derive the solution.

- To show this, plug  $x_n = A_1 + 2^n A_2$  into  $x_{n+2} - 3x_{n+1} + 2x_n = 0$  & verify. Since

$$\begin{aligned}x_n &= A_1 + 2^n A_2 \\x_{n+1} &= A_1 + 2^{n+1} A_2 \\x_{n+2} &= A_1 + 2^{n+2} A_2\end{aligned}$$

Then,  $x_{n+2} - 3x_{n+1} + 2x_n$  becomes

$$\begin{aligned}A_1 + 2^{n+2} A_2 - 3(A_1 + 2^{n+1} A_2) + 2(A_1 + 2^n A_2) &= A_1 + 4(2^n) A_2 - 3A_1 - 6(2^n) A_2 + 2A_1 + 2(2^n) A_2 \\ &= 0\end{aligned}$$

- OR 'Guess' a solution of the form  $x_n = A\lambda^n$  & find the values of the  $\lambda$ s:

$$\begin{aligned}0 &= x_{n+2} - 3x_{n+1} + 2x_n \\ &= A\lambda^{n+2} - 3A\lambda^{n+1} + 2A\lambda^n \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 2)(\lambda - 1)\end{aligned}$$

So,  $\lambda_1 = 1$  and  $\lambda_2 = 2$ . Then, the solution takes the form:

$$\begin{aligned}x_n &= 1^n A_1 + 2^n A_2 \\ &= A_1 + 2^n A_2\end{aligned}$$

- (b) Use the initial conditions to solve for the constants  $A_1$  &  $A_2$ :

$$\begin{aligned}10 &= x_0 = A_1 + 2^0 A_2 = A_1 + A_2 \\ 20 &= x_1 = A_1 + 2^1 A_2 = A_1 + 2A_2\end{aligned}$$

Solve this system to get  $A_1 = 0$  and  $A_2 = 10$ :

$$x_n = (10)2^n$$

2. (a) Guess a solution of the form  $x_n = C\lambda^n$  to find the evals, then use the ICs to find the constants:

$$\begin{aligned}0 &= x_n - 5x_{n-1} + 6x_{n-2} \\ &= C\lambda^n - 5C\lambda^{n-1} + 6C\lambda^{n-2} \\ &= 1 - 5\lambda^{-1} + 6\lambda^{-2} \\ &= \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)\end{aligned}$$

The evals are  $\lambda_1 = 2$  and  $\lambda_2 = 3$ . Then the solution takes the form,

$$x_n = 2^n C_1 + 3^n C_2$$

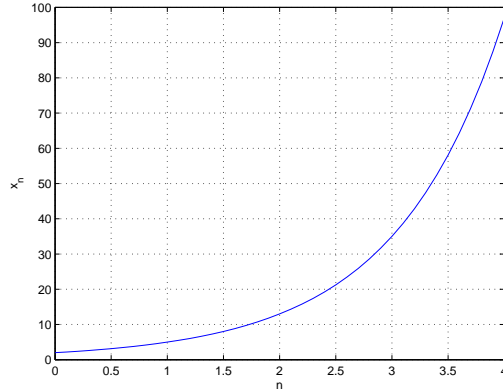
Now, use ICs to find  $C_1$  and  $C_2$ :

$$\begin{aligned}2 &= x_0 = C_1 + C_2 \\ 5 &= x_1 = 2C_1 + 3C_2\end{aligned}$$

And the solutions is:

$$x_n = 2^n + 3^n$$

Solution increases since both  $\lambda$ s are  $>1$ .



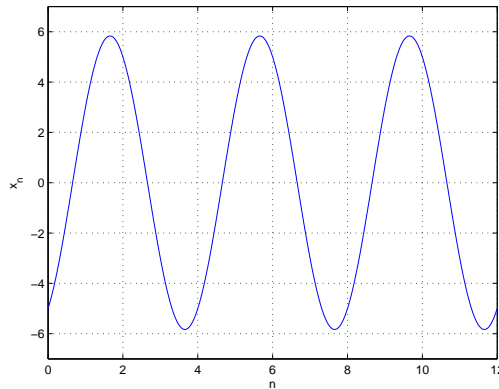
(b) Do the above procedure here too. The eigenvalues are complex:  $\lambda_{1,2} = \pm i$ . Solutions should be:

$$x_n = \frac{-3i - 5}{2}i^n + \frac{3i - 5}{2}(-i)^n$$

For less algebra, you can also write this as a combination of trig functions (see pages 23-24 in EK):

$$x_n = -5 \cos\left(\frac{n\pi}{2}\right) + 3 \sin\left(\frac{n\pi}{2}\right)$$

Since the eigenvalues are complex, the solution will oscillate.



3. (a) First reduce the system of 2 eqns down to 1 eqn. To do this, make an equation for  $x_{n+2}$ :

$$x_{n+2} = 3x_{n+1} + 2y_{n+1}$$

We know what  $y_{n+1}$  is from the original system. Plug that in to the above eqn:

$$\begin{aligned} x_{n+2} &= 3x_{n+1} + 2(x_n + 4y_n) \\ &= 3x_{n+1} + 2x_n + 8y_n \end{aligned}$$

Now, from the original equation for  $x_{n+1}$ , we can rearrange it to get an equation for  $y_n$ :

$$y_n = \frac{x_{n+1} - 3x_n}{2}$$

Then plug this into the  $x_{n+2}$  equation:

$$\begin{aligned} x_{n+2} &= 3x_{n+1} + 2x_n + 8\left(\frac{x_{n+1} - 3x_n}{2}\right) \\ &= 3x_{n+1} + 2x_n + 4x_{n+1} - 12x_n \\ &= -10x_n + 7x_{n+1} \end{aligned}$$

Now we have an equation with only  $x$  in it. 'Guess' a solution of  $x_n = C\lambda^n$  & use the same method you used in the previous problems to find that the eigenvalues are  $\lambda_1 = 2$  and  $\lambda_2 = 5$ . Then, using  $y_n = \frac{x_{n+1} - 3x_n}{2}$ , you know the solutions:

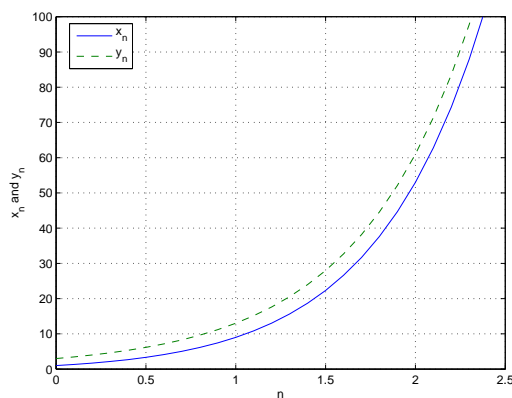
$$\begin{aligned} x_n &= C_1 2^n + C_2 5^n \\ y_n &= \frac{-C_1}{2} 2^n + C_2 5^n \end{aligned}$$

Now solve for the constants by using the ICs:

$$\begin{aligned} 1 &= x_0 = C_1 + C_2 \\ 3 &= y_0 = -\frac{C_1}{2} + C_2 \end{aligned}$$

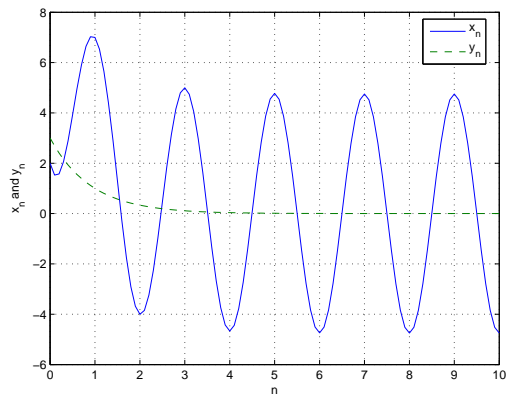
Solution:

$$\begin{aligned} x_n &= \frac{-4}{3} 2^n + \frac{7}{3} 5^n \\ y_n &= \frac{2}{3} 2^n + \frac{7}{3} 5^n \end{aligned}$$



(b) Do the same process here. Solution:

$$\begin{aligned} x_n &= \frac{-19}{4} (-1)^n + \frac{27}{4} \left(\frac{1}{3}\right)^n \\ y_n &= 3 \left(\frac{1}{3}\right)^n \end{aligned}$$



4. (a) Assume that each pair of rabbits can only produce one new pair of rabbits. Also, only 1 month olds and 2 month olds can reproduce. Then, the number of newborns in generation  $n+1$  is the number of rabbits produce by the 1 month olds & the number of rabbits produced by the 2 month olds. Since  $R_n^1 = R_{n-1}^0$  and  $R_n^2 = R_{n-2}^0 = R_{n-1}^1$ , then we can say that  $R_{n+1}^0 = R_n^0 + R_{n-1}^0$ .
- (b) 'Guess' a solution of  $R_n^0 = C\lambda^n$  & use the same methods as before to find:

$$R_n^0 = \frac{5 - \sqrt{5}}{10} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{5 + \sqrt{5}}{10} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

The total population of rabbits is  $R_n = R_n^0 + R_n^1 + R_n^2$  with  $R_n^1 = R_{n-1}^0$  and  $R_n^2 = R_{n-2}^0$ .

Homework 1.2 Solutions  
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1. (a) We know the following quantities:

$$S_n^0 = \gamma P_n \quad (1)$$

$$\bar{S}_n^1 = (1 - \alpha) S_n^1 \quad (2)$$

$$\bar{S}_n^2 = (1 - \beta) S_n^2 \quad (3)$$

$$S_{n+1}^1 = \sigma S_n^0 \quad (4)$$

$$S_{n+1}^2 = \sigma \bar{S}_n^1 \quad (5)$$

$$P_n = \alpha S_n^1 + \beta S_n^2 \quad (6)$$

Now combining eqn (2) & eqn (5):

$$S_{n+1}^2 = \sigma(1 - \alpha) S_n^1$$

Combining eqn (1) & eqn (4):

$$S_{n+2}^1 = \sigma \gamma P_{n+1}$$

Then using these and eqn (6), you can show that

$$S_{n+2}^1 = \alpha \sigma \gamma S_{n+1}^1 + \beta \sigma (1 - \alpha) \sigma \gamma S_n^1$$

- (b) From class we have:

$$P_{n+1} = \alpha \sigma \gamma P_n + \beta \sigma (1 - \alpha) \sigma \gamma P_{n-1}$$

The term  $\alpha \sigma \gamma P_n$  is the number of plants from the  $n^{\text{th}}$  generation that survived winter ( $\sigma$ ), germinated ( $\alpha$ ), and produced seeds ( $\gamma$ ).

- (c) For the population to grow we need  $\lambda > 1$ . First we need to find the evals. With  $\alpha = \beta = .001$  and  $\sigma = 1$ , we get the characteristic eqn:

$$\lambda^2 - (\alpha \sigma \gamma) \lambda - \gamma \beta \sigma^2 (1 - \alpha) = 0$$

Eigenvalues:

$$\lambda_{1,2} = \frac{.001\gamma \pm \sqrt{.000001\gamma^2 + .003996\gamma}}{2}$$

To find the gamma that allows our population to grow, set  $\lambda_+ > 1$ . You should find that  $\gamma > 500.25$ , so for the population to grow we need  $\gamma \geq 501$  seeds per plant.

- (d) First, use the hint to show that to have the eval  $> 1$  that we need  $a + b > 1$ . Then use this fact & that  $a = \alpha \sigma \gamma$  and  $b = \beta \gamma \sigma^2 (1 - \alpha)$  to find that  $\gamma > \frac{1}{\alpha \sigma + \beta \sigma^2 (1 - \alpha)}$ .

2. (a) For  $R_{t+1}$ , we have the number of RBCs that remain from the previous day ( $(1 - f)R_t$ ) plus the number of new RBCs that were made the previous day ( $M_t$ ). For  $M_{t+1}$ , we have new RBCs being made at a rate ( $\gamma$ ) proportional to the fraction of RBCs that were removed ( $f$ ) the previous day.  $\gamma > 1$  means that the number of RBCs lost today is less than the number gained tomorrow so we have an overall gain of RBCs.  $\gamma = 1$  means that the number of RBCs lost today is equal to the number gained tomorrow so nothing changes.  $\gamma < 1$  means that the number of RBCs lost today is greater than the number gained tomorrow so we are losing RBCs.

- (b)

$$\begin{bmatrix} R_{t+1} \\ M_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - f & 1 \\ \gamma f & 0 \end{bmatrix} \begin{bmatrix} R_t \\ M_t \end{bmatrix}$$

Evals:

$$\lambda_{1,2} = \frac{1 - f \pm \sqrt{(1 - f)^2 + 4\gamma f}}{2}$$

We know that  $\gamma > 0$  and  $0 < f < 1$ , we can say something about the signs and magnitudes of the evals. First,  $\sqrt{(1 - f)^2 + 4\gamma f} > \sqrt{(1 - f)^2} = 1 - f > 0$ . So, we have  $\lambda_+ > 0$  and  $\lambda_- < 0$ . Also,  $|\lambda_+| > |\lambda_-|$ .

- (c) Setting  $\lambda_+ = 1 = \frac{1 - f + \sqrt{(1 - f)^2 + 4\gamma f}}{2}$  and solving for  $\gamma$ , we find that  $\gamma = 1$ . This means that our system becomes:

$$\begin{aligned} R_{t+1} &= (1 - f)R_t + M_t \\ M_{t+1} &= fR_t \end{aligned}$$

Adding these together gives  $R_{t+1} + M_{t+1} = R_t + M_t$  so that the number of RBCs is the same each day.

- (d) With  $\gamma = 1$ , the other eval is  $\lambda_- = -f$ . The solution would then be of the form  $R_n = A(1)^n + B(-f)^n$  & it would have decreasing oscillations since  $\lambda_- = -f$  and  $0 < f < 1$ .
3. (a)

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} \gamma & m \\ \sigma & p \end{bmatrix} \begin{bmatrix} J_t \\ A_t \end{bmatrix}$$

with

- $m$  = # offspring produced by each adult
- $\sigma$  = prob of juv. reaching adulthood
- $p$  = prob of adult surviving
- $\gamma$  = fraction of juv. staying juv.

In the first year:

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ 0.9 & 0.5 \end{bmatrix} \begin{bmatrix} J_t \\ A_t \end{bmatrix}$$

Eigenvalues:  $\lambda_1 = 0.9659$  and  $\lambda_2 = -0.4659$ . So, the leading eval is  $\lambda_1 = 0.9659$ . Since  $|\lambda_1| < 1$ , then the solution will decay to 0.

In the second year:

$$\begin{bmatrix} J_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 2.0 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} J_t \\ A_t \end{bmatrix}$$

Eigenvalues:  $\lambda_1 = 0.93$  and  $\lambda_2 = -0.43$ . So, the leading eval is  $\lambda_1 = 0.93$ . Since  $|\lambda_1| < 1$ , then the solution will decay to 0.

- (b) For the two types of years to alternate, we'll take  $M = M_2M_1$ :

$$\begin{bmatrix} J_{t+2} \\ A_{t+2} \end{bmatrix} = \begin{bmatrix} 0 & 2.0 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.5 \\ 0.9 & 0.5 \end{bmatrix} \begin{bmatrix} J_t \\ A_t \end{bmatrix} = \begin{bmatrix} 1.8 & 1.0 \\ 0.45 & 0.35 \end{bmatrix} \begin{bmatrix} J_t \\ A_t \end{bmatrix}$$

- (c) Eigenvalues:  $\lambda_1 = 2.06$  and  $\lambda_2 = .087$ . So, the leading eval is  $\lambda_1 = 2.06$ . Since  $|\lambda_1| > 1$ , then the population grows.
- (d) Notice that in one population it tends to favor adults over juveniles in proportion. The other one conveniently favors juveniles over adults. When the two populations are swapped back and forth, it allows us to conserve individuals that would normally be eliminated by the opposite population dynamics. This allows the population to grow!