

Exam 1 Solutions
Math 5110/6830

1. (a) The fixed points satisfy:

$$\begin{aligned}V^* &= -w^*(V^* - V_0) + V^* \\w^* &= \alpha f(V^*) + (1 - \alpha)w^* \\&= \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} + (1 - \alpha)w^*\end{aligned}$$

From the first equation:

$$V^* = V_0$$

Then, from the second eqn:

$$\begin{aligned}w^* &= \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} + (1 - \alpha)w^* \\w^* - (1 - \alpha)w^* &= \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} \\ \alpha w^* &= \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} \\w^* &= \frac{1}{1 + e^{-\beta(V^* - V_0)}}\end{aligned}$$

Plugging $V^* = V_0$ into the above eqn:

$$w^* = \frac{1}{2}$$

So, the fixed points are:

$$\begin{aligned}V^* &= V_0 \\w^* &= \frac{1}{2}\end{aligned}$$

(b) To study the fixed points of this system, we need to find a Jacobian:

$$J(V^*, w^*) = \begin{pmatrix} -w^* + 1 & -V^* + V_0 \\ \alpha f'(V^*) & 1 - \alpha \end{pmatrix}$$

Then,

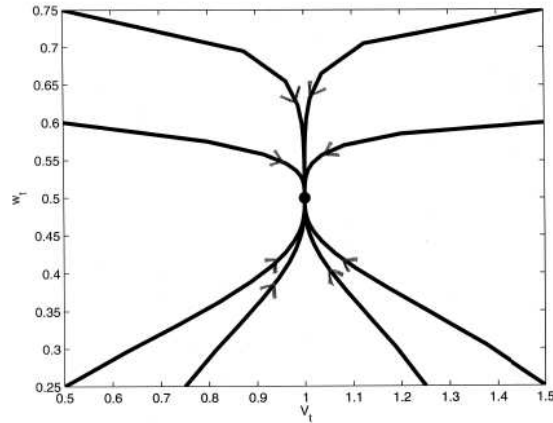
$$J\left(V_0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{\alpha}{2} & 1 - \alpha \end{pmatrix}$$

To check the stability of this point, you can either use the Jury conditions ($|Tr(J)| < 1 + Det(J) < 2$) or look at the evals. With $\alpha = 0.2$:

$$\begin{aligned}Tr(J) &= 1.3 \\Det(J) &= 0.4\end{aligned}$$

Since $1.3 < 1.4 < 2$, then, $(V_0, \frac{1}{2})$ is **stable**. Looking at the evals, we find that $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$. Since these are less than 1, then this point is **stable**.

(c) Phase-plane solutions (for $V_0=1$):



2. (a) New knowledgeable insects in the next generation (K_{t+1}) is the number of knowledgeable insects in this generation (K_t) plus insects that become knowledgeable by interacting with ones that are already knowledgeable ($p(N - K_t)K_t$). In this last term ($p(N - K_t)K_t$), p is the probability of the transmission of knowledge between knowledgeable (K_t) and naive ($N - K_t$) insects.
- (b) Fixed points satisfy

$$\begin{aligned} K^* &= K^* + p(N - K^*)K^* \\ &= K^*(1 + p(N - K^*)) \end{aligned}$$

Then,

$$\begin{aligned} K^* &= 0 \\ K^* &= N \end{aligned}$$

(c) To check the stability of these points, we need to take a derivative and find when $|f'(K^*)| < 1$. Let

$$f(K) = K + p(N - K)K$$

Then,

$$\begin{aligned} f'(K) &= 1 + p(N - K) - pK \\ &= 1 + p(N - 2K) \end{aligned}$$

And,

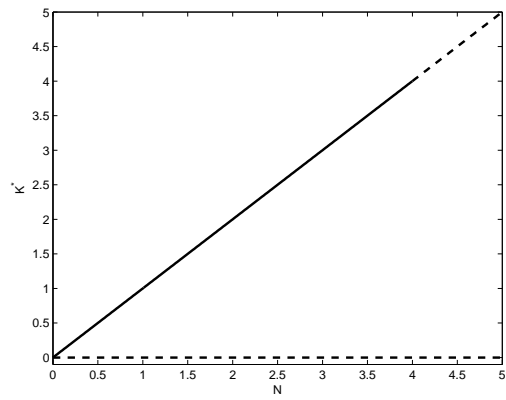
$$\begin{aligned} f'(0) &= 1 + pN \\ f'(N) &= 1 - pN \end{aligned}$$

For $p = 0.5$:

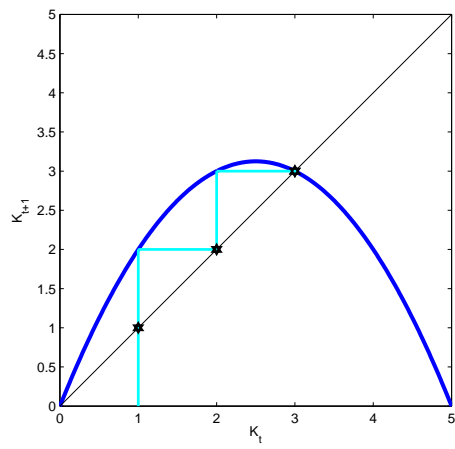
$$\begin{aligned} f'(0) &= 1 + \frac{N}{2} \\ f'(N) &= 1 - \frac{N}{2} \end{aligned}$$

Then, $K^* = 0$ is always **unstable**. And, $K^* = N$ is **stable** for $0 < N < 4$ and **unstable** for $N > 4$.

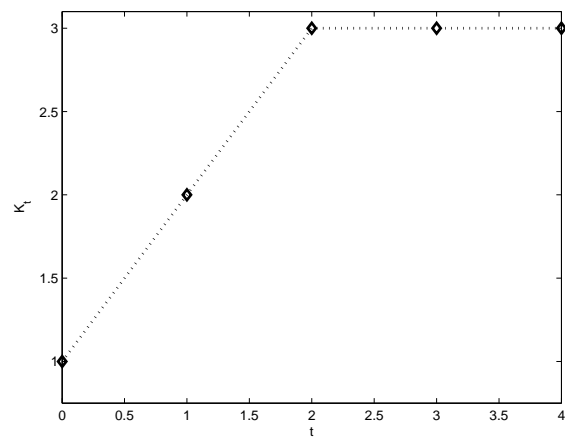
(d) Bifurcation Diagram:



(e) Cobwebbing:



(f) Solution:



- (g) When $N > 4$, cycles will start to appear. Remember that we cannot have 2 unstable fixed points next to each other (you can see this from the bifurcation diagram). Therefore, we will start to see period doubling. We may see cycles of longer period than 2 as well as chaos.
- (h) When K_0 is large, the model fails. It predicts that $K_1 < 0$ which is not biologically possible.
3. (a)

$$\begin{aligned} R_{n+1} &= (1 - \sigma)R_n + \frac{r}{2}G_n \\ G_{n+1} &= \left(1 - \sigma + \frac{g}{2}\right)G_n \end{aligned}$$

- (b) To determine the conditions for the population of gray birds to grow, we need to find the evals. For the population to grow, the magnitude of the leading eval needs to be > 1 :

$$\begin{pmatrix} R_{n+1} \\ G_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - \sigma & \frac{r}{2} \\ 0 & 1 - \sigma + \frac{g}{2} \end{pmatrix} \begin{pmatrix} R_n \\ G_n \end{pmatrix}$$

Evals are

$$\begin{aligned} \lambda_1 &= 1 - \sigma \\ \lambda_2 &= 1 - \sigma + \frac{g}{2} \end{aligned}$$

The larger eval is $\lambda_2 = 1 - \sigma + \frac{g}{2}$. We now need to look for when $|\lambda_2| > 1$:

$$\begin{aligned} \left|1 - \sigma + \frac{g}{2}\right| &> 1 \\ &\text{since } 0 < \sigma < 1 \\ 1 - \sigma + \frac{g}{2} &> 0 \end{aligned}$$

We can then drop the absolute values:

$$\begin{aligned} 1 - \sigma + \frac{g}{2} &> 1 \\ &\text{then} \\ \frac{g}{2} &> \sigma \end{aligned}$$

So, for the population to grow, we need $\frac{g}{2} > \sigma$. This means that the number of offspring per bird has to exceed the number of deaths per bird. Biologically, this makes sense!!

- (c) If the condition in part (b) is not satisfied, then the population will go extinct.
- (d) Let's start by writing the system in a matrix. Then, we will find the evals, use the ICs to find the constants, & write the solution. Our system (notice this is the same as you already found in part (b)):

$$\begin{pmatrix} R_{n+1} \\ G_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - \sigma & \frac{r}{2} \\ 0 & 1 - \sigma + \frac{g}{2} \end{pmatrix} \begin{pmatrix} R_n \\ G_n \end{pmatrix}$$

We already found the evals in part(b):

$$\begin{aligned} \lambda_1 &= 1 - \sigma \\ \lambda_2 &= 1 - \sigma + \frac{g}{2} \end{aligned}$$

With $\sigma = 0.5$ and $g = 3$:

$$\begin{aligned} \lambda_1 &= 0.5 \\ \lambda_2 &= 2 \end{aligned}$$

Then, we can write a general solution (with arbitrary constants) as:

$$\begin{aligned} R_n &= A_1(0.5)^n + A_22^n \\ G_n &= B_1(0.5)^n + B_22^n \end{aligned}$$

Notice that we have 2 equations but 4 unknowns. The way to take care of this is to rewrite the G_n equation in terms of the R_n equation. From our original system, we can rearrange terms to have

$$G_n = \frac{2}{r}(R_{n+1} - (1 - \sigma)R_n)$$

Plugging the R_n solution into this:

$$\begin{aligned} G_n &= \frac{2}{r}(A_1(0.5)^{n+1} + A_22^{n+1} - (1 - \sigma)(A_1(0.5)^n + A_22^n)) \\ &= \frac{2}{r}(A_1(0.5)^{n+1} + A_22^{n+1} - 0.5(A_1(0.5)^n + A_22^n)) \\ &= \frac{2}{r}(0.5A_1(0.5)^n + 2A_22^n - 0.5A_1(0.5)^n - 0.5A_22^n) \\ &= \frac{3}{r}A_22^n \end{aligned}$$

We now have 2 eqns and 2 unknowns:

$$\begin{aligned} R_n &= A_1(0.5)^n + A_22^n \\ G_n &= \frac{3}{r}A_22^n \end{aligned}$$

Lets use our ICs to find the constants A_1 and A_2 :

$$\begin{aligned} R_0 = 3 &= A_1 + A_2 \\ G_0 = 5 &= \frac{3}{r}A_2 \end{aligned}$$

Then,

$$\begin{aligned} A_1 &= \frac{9 - 5r}{3} \\ A_2 &= \frac{5r}{3} \end{aligned}$$

And, the solution is:

$$\begin{aligned} R_n &= \frac{9 - 5r}{3}(0.5)^n + \frac{5r}{3}(2)^n \\ G_n &= 5(2)^n \end{aligned}$$

(e) No. If G_n survives, then $G_n \uparrow \infty$ and $R_n \rightarrow \infty$ (for $r > 0$).