Worksheet #3 SOLUTIONS
Math 1100-005
04/04/06

1. For the following functions, find both vertical and horizontal asymptotes:

   (a) $f(x) = \frac{1}{x}$
   Vertical asymptote (where $f(x)$ is undefined) at $x = 0$ since we can't divide by 0.
   To find horizontal asymptotes:

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{x} &= \frac{1}{\infty} \\ &= 0 \\ \lim_{x\to-\infty} \frac{1}{x} &= \frac{1}{-\infty} \\ &= 0 \end{aligned}$$

   Therefore, f(x) has a horizontal asymptote at $y = 0$.

   (b) $g(x) = \ln(x-2)$
   Vertical asymptote (where $g(x)$ is undefined) at $x = 2$ since we can't have 0 or negative numbers in logs.
   To find horizontal asymptotes:

$$\begin{aligned} \lim_{x\to\infty} \ln(x-2) &= \infty \\ \lim_{x\to-\infty} \ln(x-2) &= \text{undefined (can't have negatives in logs)} \end{aligned}$$

   Therefore, g(x) has no horizontal asymptotes.

   (c) $h(x) = \frac{1}{2-e^{-x}}$
   Vertical asymptote (where $h(x)$ is undefined) at $x = -\ln(2) \approx -0.693$ since we can't divide by 0.
   To find horizontal asymptotes:

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{2-e^{-x}} &= \frac{1}{2-e^{-\infty}} \\ &= \frac{1}{2-0} \\ &= \frac{1}{2} \\ \lim_{x\to-\infty} \frac{1}{2-e^{-x}} &= \frac{1}{2-e^{\infty}} \\ &= \frac{1}{2-\infty} \\ &= 0 \end{aligned}$$

   Therefore, h(x) has 2 horizontal asymptotes. As $x \to \infty$, h(x) has a horizontal asymptote at $y = \frac{1}{2}$. And, as $x \to -\infty$, h(x) has a horizontal asymptote at $y = 0$.

2. Find the derivative of the following functions.

   (a) $f(x) = xe^x - e^{x^2+3x}$

$$\begin{aligned} f'(x) &= e^x + xe^x - e^{x^2+3x}(2x+3) \\ &= (x+1)e^x - (2x+3)e^{x^2+3x} \end{aligned}$$

   (b) $f(x) = \ln((2x - x^3)\sqrt{x})$
   Rewrite first:

$$\begin{aligned} f(x) &= \ln((2x - x^3)\sqrt{x}) \\ &= \ln(2x - x^3) + \ln(sqrtx) \\ &= \ln(2x - x^3) + \frac{1}{2}\ln(x) \end{aligned}$$

   Derivative:

$$f'(x) = \frac{2 - 3x^2}{2x - x^3} + \frac{1}{2x}$$

   (c) $f(x) = \frac{1}{x^3} + \frac{x}{e^{3x}} - \ln\left(\frac{x}{x+1}\right)$
   Rewrite first:

$$\begin{aligned} f(x) &= \frac{1}{x^3} + \frac{x}{e^{3x}} - \ln\left(\frac{x}{x+1}\right) \\ &= \frac{1}{x^3} + xe^{-3x} - (\ln(x) - \ln(x+1)) \end{aligned}$$

   Derivative:

$$\begin{aligned} f'(x) &= \frac{-3}{x^4} + e^{-3x} - 3xe^{-3x} - \frac{1}{x} + \frac{1}{x+1} \\ &= (1-3x)e^{-3x} - \frac{x^3 + 3x + 3}{x^4(x+1)} \end{aligned}$$

3. Evaluate the following integrals.

   (a) $\int 2x^3 - \frac{1}{x^2} dx$

$$\begin{aligned} \int 2x^3 - \frac{1}{x^2} dx &= \frac{2x^4}{4} - \frac{-1}{x} + C \\ &= \frac{x^4}{2} + \frac{1}{x} + C \end{aligned}$$

   (b) $\int \sqrt{x} + 5x^3 + 4dx$

$$\begin{aligned} \int \sqrt{x} + 5x^3 + 4dx &= \frac{x^{3/2}}{3/2} + \frac{5x^4}{4} + 4x + C \\ &= \frac{2x^{3/2}}{3} + \frac{5x^4}{4} + 4x + C \end{aligned}$$

(c) $\int 2x\sqrt{3x^2-1}dx$
Need u-substitution:

$$\begin{aligned} u &= 3x^2-1 \\ \frac{du}{dx} &= 6x \quad \rightarrow \quad \frac{du}{3} = 2xdx \end{aligned}$$

Then, substituting in u and $\frac{du}{3}$

$$\begin{aligned} \int 2x\sqrt{3x^2-1}dx &= \int \frac{\sqrt{u}}{3}du \\ &= \frac{2u^{3/2}}{9} + C \end{aligned}$$

Now, substituting $u = 3x^2 - 1$ back in for u

$$\int 2x\sqrt{3x^2-1}dx = \frac{2(3x^2-1)^{3/2}}{9} + C$$

(d) $\int \frac{x+1}{(x^2+2x-3)^2}dx$
Need u-substitution:

$$\begin{aligned} u &= x^2+2x-3 \\ \frac{du}{dx} &= 2x^2+2 \quad \rightarrow \quad \frac{du}{2} = (x+1)dx \end{aligned}$$

Then, substituting in u and $\frac{du}{2}$

$$\begin{aligned} \int \frac{x+1}{(x^2+2x-3)^2}dx &= \int \frac{1}{2(u)^2}du \\ &= \frac{-1}{2u} + C \end{aligned}$$

Now, substituting $u = x^2 + 2x - 3$ back in for u

$$\int \frac{x+1}{(x^2+2x-3)^2}dx = \frac{-1}{2(x^2+2x-3)} + C$$

(e) $\int e^{5x}dx$
Need u-substitution:

$$\begin{aligned} u &= 5x \\ \frac{du}{dx} &= 5 \quad \rightarrow \quad \frac{du}{5} = dx \end{aligned}$$

Then, substituting in u and $\frac{du}{5}$

$$\begin{aligned} \int e^{5x}dx &= \int \frac{e^u}{5}du \\ &= \frac{e^u}{5} + C \end{aligned}$$

Now, substituting $u = 5x$ back in for u

$$\int e^{5x} dx = \frac{e^{5x}}{5} + C$$

(f) $\int \frac{1}{x-1} dx$
Can use u-substitution:

$$\begin{aligned} u &= x - 1 \\ \frac{du}{dx} &= 1 \rightarrow du = dx \end{aligned}$$

Then, substituting in u and du

$$\begin{aligned} \int \frac{1}{x-1} dx &= \int \frac{1}{u} du \\ &= \ln(u) + C \end{aligned}$$

Now, substituting $u = x - 1$ back in for u

$$\int \frac{1}{x-1} dx = \ln(x-1) + C$$

(g) $\int x e^{3x^2} dx$
Need u-substitution:

$$\begin{aligned} u &= 3x^2 \\ \frac{du}{dx} &= 6x \rightarrow \frac{du}{6} = x dx \end{aligned}$$

Then, substituting in u and $\frac{du}{6}$

$$\begin{aligned} \int x e^{3x^2} dx &= \int \frac{e^u}{6} du \\ &= \frac{e^u}{6} + C \end{aligned}$$

Now, substituting $u = 3x^2$ back in for u

$$\int x e^{3x^2} dx = \frac{e^{3x^2}}{6} + C$$

(h) $\int \frac{x^2}{x^3+1} dx$
Need u-substitution:

$$\begin{aligned} u &= x^3 + 1 \\ \frac{du}{dx} &= 3x^2 \rightarrow \frac{du}{3} = x^2 dx \end{aligned}$$

Then, substituting in u and $\frac{du}{3}$

$$\begin{aligned}\int \frac{x^2}{x^3+1}dx &= \int \frac{1}{3u}dx \\ &= \frac{1}{3}\ln(u) + C\end{aligned}$$

Now, substituting $u = x^3 + 1$ back in for u

$$\int \frac{x^2}{x^3+1}dx = \frac{\ln(x^3+1)}{3} + C$$

4. The marginal cost of a product is modeled by $\frac{dC}{dx} = \frac{4x^2}{\sqrt{x^3+1}}$ where $C = 10$ when $x = 2$. Find the cost function.
   To find the cost function, we need to integrate the marginal:

$$C(x) = \int \frac{4x^2}{\sqrt{x^3+1}}dx$$

Need u-substitution:

$$\begin{aligned}u &= x^3 + 1 \\ \frac{du}{dx} &= 3x^2 \quad \rightarrow \quad \frac{du}{3} = x^2 dx\end{aligned}$$

Then, substituting in u and $\frac{du}{3}$

$$\begin{aligned}C(x) = \int \frac{4x^2}{\sqrt{x^3+1}}dx &= \int \frac{4}{3\sqrt{u}}du \\ &= \frac{8\sqrt{u}}{3} + K\end{aligned}$$

Now, substituting $u = x^3 + 1$ back in for u

$$C(x) = \int \frac{4x^2}{\sqrt{x^3+1}}dx = \frac{8\sqrt{x^3+1}}{3} + K$$

Next step is to solve for K using the fact that $C = 10$ when $x = 2$:

$$\begin{aligned}C(2) &= \frac{8\sqrt{2^3+1}}{3} + K = 10 \\ &= 8 + K = 10 \\ K &= 2\end{aligned}$$

Then

$$C(x) = \frac{8\sqrt{x^3+1}}{3} + 2$$