Exam 2 Review Solutions
Math 1100-005

1. Find $\frac{dy}{dx}$ for

   (a) $(x+y)^3 = x^3 + y^3$

$$\begin{aligned}
3(x+y)^2(1+\frac{dy}{dx}) &= 3x^2 + 3y^2\frac{dy}{dx} \\
3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} &= 3x^2 + 3y^2\frac{dy}{dx} \\
3(x+y)^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} &= 3x^2 - 3(x+y)^2 \\
\frac{dy}{dx}[3(x+y)^2 - 3y^2] &= 3x^2 - 3(x+y)^2 \\
\frac{dy}{dx} &= \frac{3x^2 - 3(x+y)^2}{3(x+y)^2 - 3y^2} \\
\frac{dy}{dx} &= \frac{-y(2x+y)}{x(x+2y)}
\end{aligned}$$

   (b) $y^2 = \frac{x^3}{4-x}$ at $(2, 2)$

$$\begin{aligned}
2y\frac{dy}{dx} &= \frac{3x^2(4-x) - x^3(-1)}{(4-x)^2} \\
2y\frac{dy}{dx} &= \frac{12x^2 - 3x^3 + x^3}{(4-x)^2} \\
\frac{dy}{dx} &= \frac{12x^2 - 2x^3}{2y(4-x)^2} \\
\frac{dy}{dx} &= \frac{x^2(6-x)}{y(4-x)^2} \\
\frac{dy}{dx}(2,2) &= \frac{2^2(6-2)}{2(4-2)^2} = \frac{16}{8} = 2
\end{aligned}$$

2. Find the critical values AND inflection points for:

   (a) $f(x) = (x+2)^{2/3}$
   Critical Values:

$$\begin{aligned}
f'(x) &= \frac{2}{3}(x+2)^{-1/3} = \frac{2}{3(x+2)^{1/3}} \\
f'(x) &= 0 \quad \Rightarrow \text{no critical values from this!} \\
f'(x) &= \text{undefined} \quad \Rightarrow x = -2
\end{aligned}$$

   Inflection Points:

$$\begin{aligned}
f''(x) &= \frac{-2}{9(x+2)^{4/3}} \\
f''(x) &= 0 \quad \Rightarrow \text{no inflection points from this!} \\
f''(x) &= \text{undefined} \quad \Rightarrow x = -2
\end{aligned}$$

(b) $f(x) = \sqrt{x^2-1}$
Critical Values:

$$\begin{aligned} f'(x) &= \frac{1}{2}(x^2-1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2-1}} \\ f'(x) &= 0 \quad \Rightarrow x = 0 \\ f'(x) &= \text{undefined} \quad \Rightarrow x = -1 \text{ and } x = 1 \end{aligned}$$

Inflection Points:

$$\begin{aligned} f''(x) &= \frac{-1}{(x^2-1)^{3/2}} \\ f''(x) &= 0 \quad \Rightarrow \text{no inflection points from this!} \\ f''(x) &= \text{undefined} \quad \Rightarrow x = -1 \text{ and } x = 1 \end{aligned}$$

3. For $f(x) = 6x^3 - 15x^2 + 12x$, find:

(a) critical values

$$\begin{aligned} f'(x) &= 18x^2 - 30x + 12 \\ f'(x) &= 0 \quad \Rightarrow x = 1 \text{ and } x = \frac{2}{3} \\ f'(x) &= \text{undefined} \quad \Rightarrow \text{no critical values from this!} \end{aligned}$$

(b) increasing and decreasing intervals

Intervals	(-∞,$\frac{2}{3}$)	($\frac{2}{3}$,1)	(1, ∞)
Test Value	$x = 0$	$x = \frac{5}{6}$	$x = 2$
Sign of f'(x)	+	-	+
Inc/Dec	INC	DEC	INC

(c) inflection points

$$\begin{aligned} f''(x) &= 36x - 30 \\ f''(x) &= 0 \quad \Rightarrow x = \frac{5}{6} \\ f''(x) &= \text{undefined} \quad \Rightarrow \text{no inflection points from this!} \end{aligned}$$

(d) concavity intervals

Intervals	(-∞,$\frac{5}{6}$)	($\frac{5}{6}$, ∞)
Test Value	$x = 0$	$x = 1$
Sign of f"(x)	-	+
Concave up/down	DOWN	UP

(e) all extrema
Relative Max: $x = \frac{2}{3}$
Relative Min: $x = 1$

4. For $f(x) = x^3 - 3x^2$ on the interval $[-1, 3]$, find:

(a) critical values

$$
\begin{aligned}
f'(x) &= 3x^2 - 6x \\
f'(x) &= 0 \quad \Rightarrow x = 0 \text{ and } x = 2 \\
f'(x) &= \text{undefined} \quad \Rightarrow \text{no critical values from this!}
\end{aligned}
$$

(b) increasing and decreasing intervals

Intervals	$[-1,0)$	$(0,2)$	$(2, 3]$
Test Value	$x = \frac{-1}{2}$	$x = 1$	$x = \frac{5}{2}$
Sign of f'(x)	+	-	+
Inc/Dec	INC	DEC	INC

(c) inflection points

$$
\begin{aligned}
f''(x) &= 6x - 6 \\
f''(x) &= 0 \quad \Rightarrow x = 1 \\
f''(x) &= \text{undefined} \quad \Rightarrow \text{no inflection points from this!}
\end{aligned}
$$

(d) concavity intervals

Intervals	$[-1,1)$	$(1, 3]$
Test Value	$x = 0$	$x = 2$
Sign of f"(x)	-	+
Concave up/down	DOWN	UP

(e) all extrema

To find absolute mins/maxs:

$$
\begin{aligned}
f(-1) &= -4 \\
f(3) &= 0 \\
f(0) &= 0 \\
f(2) &= -4
\end{aligned}
$$

Relative Max: x=0

Relative Min: x=2

Absolute Min: x=-1 and x=2

Absolute Max: x=0 and x=3

5. For $f(x) = x^3 - 3x$,

(a) find the relative extrema using the 1st derivative test.

Critical Values:

$$
\begin{aligned}
f'(x) &= 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1) \\
f'(x) &= 0 \quad \Rightarrow x = 1 \text{ and } x = -1 \\
f'(x) &= \text{undefined} \quad \Rightarrow \text{no critical values from this!}
\end{aligned}
$$

Intervals	(-∞,−1)	(−1, 1)	(1, ∞)
Test Value	$x = -2$	$x = 0$	$x = 1$
Sign of f"(x)	+	-	+
Inc/Dec	INC	DEC	INC

Relative Max: $x = -1$
Relative Min: $x = 1$

(b) verify what you found in part (a) with the 2nd derivative test.
Find the 2nd derivative & test your critical values

$$\begin{aligned} f''(x) &= 6x \\ f''(-1) &= -6 < 0 \quad \Rightarrow \quad \text{x=-1 is relative MAX} \\ f''(1) &= 6 > 0 \quad \Rightarrow \quad \text{x=1 is relative MIN} \end{aligned}$$

6. Find 2 positive numbers such that the sum of the first and twice the second is 100 and the product is a maximum.

- Equation from the given information: $x + 2y = 100$
- We want to *maximize* their product: P=xy
- Getting $x + 2y = 100$ in terms of x: $x = 100 - 2y$
- Plugging $x = 100 - 2y$ into $P = xy$: $P = (100 - 2y)y$
- Now maximize P by taking the derivative and setting it equal to zero to find your critical values:

$$\begin{aligned} P' &= 100 - 4y \\ P' &= 0 \quad \Rightarrow y = 25 \end{aligned}$$

- Check if $y = 25$ is a maximum by using the 1st or 2nd derivative test for relative extrema. I'll check with the 2nd derivative test:

$$\begin{aligned} P'' &= -4 \\ P'' &< 0 \quad \Rightarrow \text{y=25 is a MAX} \end{aligned}$$

- Now solve for the value of x: $x = 100 - 2(25) = 50$

7. The combined perimeter of a circle and a square is 16in. Find the dimensions of the circle and square that produce a minimum total area.

- Perimeter of a circle is the circumference: $2\pi r$
- Area of a circle: πr^2
- Let x be the length of each side of the square.
- Equation from the given information: $16 = 2\pi r + 4x$
- We need to *minimize* the total area: $A = \pi r^2 + x^2$
- Getting $16 = 2\pi r + 2x$ in terms of r: $r = \frac{16-4x}{2\pi} = \frac{8-2x}{\pi}$
- Plugging $r = \frac{8-2x}{\pi}$ into $A = \pi r^2 + x^2$:

$$\begin{aligned} A &= \pi \left(\frac{8-2x}{\pi} \right)^2 + x^2 \\ &= \pi \frac{(8-2x)^2}{\pi^2} + x^2 \\ &= \frac{(8-2x)^2}{\pi} + x^2 \end{aligned}$$

- Now minimize A:

$$\begin{aligned} A' &= \tfrac{2[(\pi+4)x-16]}{\pi} \\ A' &= 0 \quad \Rightarrow x = \frac{16}{4+\pi} = 2.24\text{in} \end{aligned}$$

- Check if $x = 2.24$ is a minimum:

$$\begin{aligned} A'' &= \frac{2(\pi+4)}{\pi} \\ A'' &> 0 \quad \Rightarrow \text{x=2.24 is a MIN} \end{aligned}$$

- Now solve for the value of r: $r = \frac{8-2\left(\frac{16}{4+\pi}\right)}{\pi} = 1.12\text{in}$

8. For $0.3x^2 + 6x + 600$, find the average cost for producing 25 units.

$$\begin{aligned} \text{Average Cost : } \overline{C(x)} &= \frac{0.3x^2 + 6x + 600}{x} \\ \overline{C(25)} &= \frac{0.3(25)^2 + 6(25) + 600}{25} = 37.50 \end{aligned}$$