Final Exam Review
Math 1100-005

1. Evaluate the following integrals:

   (a) $\int_0^2 x^2\sqrt{x^3+1}dx$
   Need u-substitution:

$$\begin{aligned} u &= x^3+1 \\ \frac{du}{dx} &= 3x^2 \quad \rightarrow \quad \frac{du}{3} = x^2dx \end{aligned}$$

   Changing Limits:

$$\begin{aligned} \text{at x=0} &: \quad u=1 \\ \text{at x=2} &: \quad u=9 \end{aligned}$$

   Then, substituting in u and $\frac{du}{3}$

$$\begin{aligned} \int_0^2 x^2\sqrt{x^3+1}dx &= \int_1^9 \frac{\sqrt{u}}{3}du \\ &= \left[\frac{2u^{3/2}}{9}\right]_{u=1}^{u=9} \\ &= \frac{52}{9} \end{aligned}$$

   (b) $\int_{-2}^2 x^2-3x-1dx$

$$\begin{aligned} \int_{-2}^2 x^2-3x-1dx &= \left[\frac{x^3}{3}-\frac{3x^2}{2}-x\right]_{x=-2}^{x=2} \\ &= \frac{4}{3} \end{aligned}$$

   (c) $\int_1^5 x(x+3)^2dx$
   Need u-substitution:

$$\begin{aligned} u &= x+3 \quad \rightarrow \quad x=u-3 \\ \frac{du}{dx} &= 1 \quad \rightarrow \quad du=dx \end{aligned}$$

   Changing Limits:

$$\begin{aligned} \text{at x=1} &: \quad u=4 \\ \text{at x=5} &: \quad u=8 \end{aligned}$$

Then, substituting in u and x and du

$$\begin{aligned}
\int_1^5 x(x+3)^2dx &= \int_4^8 (u-3)(u)^2du \\
&= \int_4^8 u^3-3u^2du \\
&= \left[\frac{u^4}{4}-\frac{3u^3}{3}\right]_{u=4}^{u=8} \\
&= 512
\end{aligned}$$

(d) $\int \ln(x-2)dx$
Need integration by parts:

$$\begin{aligned}
u &= \ln(x-2) & v &= x \\
du &= \frac{1}{x-2}dx & dv &= 1dx
\end{aligned}$$

Then, plugging into formula:

$$\int \ln(x-2)dx = \ln(x-2)x - \int \frac{x}{x-2}dx$$

Need u-substitution:

$$\begin{aligned}
u &= x-2 \rightarrow x = u+2 \\
\frac{du}{dx} &= 1 \rightarrow du = dx
\end{aligned}$$

Then, substituting in u and x and du

$$\begin{aligned}
\int \ln(x-2)dx &= \ln(x-2)x - \int \frac{x}{x-2}dx \\
&= \ln(x-2)x - \int \frac{u+2}{u}du \\
&= \ln(x-2)x - \int \frac{u}{u}+\frac{2}{u}du \\
&= \ln(x-2)x - u - 2\ln(u) + C \\
&= \ln(x-2)x - (x-2) - 2\ln(x-2) + C
\end{aligned}$$

(e) $\int x^2\ln(x)dx$
Need integration by parts:

$$\begin{aligned}
u &= \ln(x) & v &= \frac{x^3}{3} \\
du &= \frac{1}{x}dx & dv &= x^2dx
\end{aligned}$$

Then, plugging into formula:

$$\begin{aligned}
\int x^2 \ln(x)dx &= \ln(x)\frac{x^3}{3} - \int \frac{x^3}{3}\frac{1}{x}dx \\
&= \frac{x^3 \ln(x)}{3} - \int \frac{x^2}{3}dx \\
&= \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + C
\end{aligned}$$

(f) $\int_0^2 (x-1)e^{-x}dx$

Need integration by parts:

$$\begin{aligned}
u &= x-1 & v &= -e^{-x} \\
du &= dx & dv &= e^{-x}dx
\end{aligned}$$

Then, plugging into formula:

$$\begin{aligned}
\int_0^2 (x-1)e^{-x}dx &= [(x-1)(-e^{-x})]_{x=0}^{x=2} - \int (-e^{-x})dx \\
&= [(x-1)(-e^{-x})]_{x=0}^{x=2} + \int e^{-x}dx \\
&= [(x-1)(-e^{-x})]_{x=0}^{x=2} - [e^{-x}]_{x=0}^{x=2} \\
&= -.27067
\end{aligned}$$

2. Find the area of the regions bounded by the curves

(a) $y = x^2 + 3$, $y = x$, $x = -1$, and $x = 1$

$$\begin{aligned}
\int_{-1}^{1} [x^2 + 3 - x]dx &= \left[\frac{x^3}{3} + 3x - \frac{x^2}{2}\right]_{-1}^{1} \\
&= \frac{20}{3}
\end{aligned}$$

(b) $y = 2 - x$, $y = \sqrt{x}$, $x = 0$

First we need to find the intersection point:

$$\begin{aligned}
2 - x &= \sqrt{x} \\
x &= 1
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^1 [2 - x - \sqrt{x}]dx &= \left[2x - \frac{x^2}{2} - \frac{2x^{3/2}}{3}\right]_0^1 \\
&= \frac{5}{6}
\end{aligned}$$

3. Find the average value of $f(x) = \frac{x}{2x^2-1}$ on the interval $[1, 3]$.

$$\text{Average value} = \frac{1}{3-1}\int_1^3 \frac{x}{2x^2-1}dx$$

Need u-substitution:

$$\begin{aligned} u &= 2x^2 - 1 \\ \frac{du}{dx} &= 4x \quad \rightarrow \quad \frac{du}{4} = xdx \end{aligned}$$

Changing Limits:

$$\begin{aligned} \text{at x=1} &: \quad u = 1 \\ \text{at x=3} &: \quad u = 17 \end{aligned}$$

Then, substituting in u and x and du

$$\begin{aligned} \frac{1}{2}\int_1^3 \frac{x}{2x^2-1}dx &= \frac{1}{2}\int_1^{17} \frac{1}{4u}dx \\ &= \left[\frac{1}{8}\ln(u)\right]_1^{17} \\ &= .354 \end{aligned}$$

4. Find the amount of an annuity if the income is given by $c(t) = 3000$ with interest rate 5% for 10 years.

$$\text{Amount of Annuity} = e^{.05*10}\int_0^{10} 3000e^{-.05t}dt$$

Need u-substitution:

$$\begin{aligned} u &= -.05t \\ \frac{du}{dt} &= -.05 \quad \rightarrow \quad \frac{du}{-.05} = dt \end{aligned}$$

Change Limits:

$$\begin{aligned} \text{at t=0} &: \quad u = 0 \\ \text{at t=10} &: \quad u = -.5 \end{aligned}$$

Then, substituting in u and x and du

$$\begin{aligned} e^{.5}\int_0^{10} 3000e^{-.05t}dt &= e^{.5}\int_0^{-.5} 3000e^{u}du \\ &= e^{.5}[3000e^{u}]_0^{-.5} \\ &= \$14319.07 \end{aligned}$$

5. Find the producer and consumer surplus if the demand is given by $p_1(x) = 100 - 0.4x^2$ and the supply is given by $p_2(x) = 42x$.

First, we need to find the equilibrium point:

$$\begin{aligned} 100 - 0.4x^2 &= 42x \\ x &= 2.33 \end{aligned}$$

$$\begin{aligned} \text{Producer Surplus} &= \int_0^{2.33} [97.83 - 42x]dx \\ &= \left[97.83x - \frac{42x^2}{2}\right]_0^{2.33} \\ &= \$113.94 \end{aligned}$$

$$\begin{aligned} \text{Consumer Surplus} &= \int_0^{2.33} [100 - 0.4x^2 - 97.83]dx \\ &= \left[2.17x - \frac{0.4x^3}{3}\right]_0^{2.33} \\ &= \$3.37 \end{aligned}$$

6. Find the present value over 5 years of an income given by $c(t) = 500 + t$ with an annual inflation rate of 3%.

$$\text{Present Value} = \int_0^{5} (500 + t)e^{-.03t}dt$$

Need integration by parts:

$$\begin{aligned} u = 500 + t \qquad & v = \frac{e^{-.03t}}{-.03} \\ du = dt \qquad & dv = e^{-.03t}dt \end{aligned}$$

Then, plugging into formula:

$$\begin{aligned}\int_0^5 [(500+t)e^{-.03t}]dt &= \left[(500+t)\frac{e^{-.03t}}{-.03}\right]_0^5 - \int_0^5 \frac{e^{-.03t}}{-.03}dt \\ &= \left[(500+t)\frac{e^{-.03t}}{-.03}\right]_0^5 - \left[\frac{e^{-.03t}}{.0009}\right]_0^5 \\ &= \$2332.85\end{aligned}$$